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Topologically protected edge gap solitons of interacting Bosons in one-dimensional superlattices*

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We comprehensively investigate the nontrivial states of an interacting Bose system in a cosine potential under the open boundary condition. Our results show that there exists a kind of stable localized state: edge gap solitons. We argue that the states originate from the eigenstates of independent edge parabolas. In particular, the edge gap solitons exhibit a nonzero topological-invariant behavior. The topological nature is due to the connection of the present model to the quantized adiabatic particle transport problem. In addition, the composition relations between the gap solitons and the extended states are also discussed.

Keywords: one-dimensional optical superlattices, interacting Boses, edge gap solitons, topological invariant

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1. Introduction

Recently, the topological properties in one-dimensional (1D) superlattices have attracted tremendous research attention. Besides the topological states of some well known models such as the Su–Schrieffer–Heeger model,[1,2] the Kitaev–Majorana chain[3] has also been investigated theoretically; topological states were also realized experimentally with the incommensurate superlattices in photonic crystals[4] and cold atoms.[5,6] In particular, the better tunability of particle hoping and interactions in cold atoms offer a clean and parameter-controllable platform to investigate 1D topological insulators.[7] It is well known that, in the mean field level, the interactions between Bose particles in low temperature can result in significant nonlinearities.[8–10] Nonlinearity in turn has significant effects on their stabilities.[11–13] For example, the instabilities are responsible for the formation of the train of localized filaments[8] and are closely related to the breakdown of the superfluidity.[14] The topological properties of an interacting Boson system has been studied in one-dimensional bichromatic superlattices.[15] The nontrivial edge solitions were found on the boundaries of the lattice and its topological properties were illuminated by Chern numbers of the nonlinear Bloch bands defined in the extended two dimensional parameter space. However, finding the topological states in a simple superlattice is still an important issue.

Here we show that the topological gap solitons can exist in a simple cosine potential. Their topologically nontrivial properties are due to the introduction of a modulation phase in the periodic potential well, which provides an additional dimension. We explain with numerical analysis how these states can be induced by the edge parabola. The stabilities and composition relations of the gap solitons are also discussed. The paper is organized as follows. In Section 2, we introduce the theoretical model and show how to obtain the eigenstates of the 1D interacting periodic Bose system. In Subsection 3.1, the edge states are obtained and verified to be edge gap solitons. The stabilities and topological invariant are studied in Subsections 3.2 and 3.3, respectively. Their composition relation between Gap solitons and the extended states are investigated in Subsection 3.4. Section 4 is a brief summary.

2. Theory

We consider a weakly interacting Bose gas loaded in a 1D periodic optical lattice with periodic potential $V(x) = v\cos(2\pi x/a+\delta)$. Here v is the potential strength, a is the period of the potential, and δ is an arbitrary phase. On the mean field level, the above system can be well described by the following nonlinear Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)+g\left|\Psi(x)\right|^2\right]\Psi(x)=\mu_n\Psi(x), \qquad (1)$$

where m is the mass of the Bose particle. g is the effective interaction between the particles. μ_n is the n-th single particle eigenvalue. The wave function is normalized under

$$\int_{-L/2}^{L/2}\left|\Psi(x)\right|^2\mathrm{d}x=1.$$

Despite the nonlinearity, equation (1) still admits the Bloch wave solutions under the periodic boundary condition $\Psi(x) =$

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$e^{ikx}\psi_k(x)$ where k is the Bloch wave vector. The Bloch wave state $\psi_k(x)$ is a periodic function $\psi_k(x) = \psi_k(x+a)$. The Bloch wave state $\psi_k(x)$ is also a periodic function of δ, i.e., $\psi_k(\delta) = \psi_k(\delta + 2\pi)$.

From the Schrödinger equation (1), we have the following equation for each Bloch wave state $\psi_k(x)$

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\mathrm{i}k\right)^2 + V(x) + g|\psi_k(x)|^2\right]\psi_k(x) = \mu_{kn}\psi_k(x). \tag{2}$$

However, under the open boundary condition (i.e., the Boson is confined in $[-L/2, L/2]$), the momentum k is no longer a good quantum number. It is expected that some nontrivial states exist in the periodic system.

Beside some special cases, there are no analytical solutions for the two equations [Eqs. (1) and (2)].[16–19] Several numerical methods have been proposed to solve them.[20] In this work, an effective method is used in the weak interactive case as follows. The equations are first solved in the linear case ($g = 0$) to obtain the eigenvalue and eigenstate. Then the eigenstate is brought back to the equation with the effective potential function $V(x) + g|\psi_k(x)|^2$, which produces a new set of eigenvalues and eigenstates. Iterating the above step several times, one can find the stable eigenvalues and eigenstates. For the nonlinear Schrödinger equation (1) under the open boundary condition, the different interval is taken to $[-L/2, L/2]$. For the Bloch equation (2) under the periodic condition, the different interval is taken to be $[0, a]$.

3. Results

The nonlinear equations [Eqs. (1) and (2)] are solved with $v = 16$, $\delta = 0.8\pi$, and $g = 1000$. The natural unit is used in the following discussion, i.e., $\hbar = m^* = 1$. For simplification, the lattice constant a is also taken to be 1. For the open boundary condition, the lattice length has been set to $L = 8$. This set of parameters is general and remains unchanged in the following discussions. The energy spectrum for the Bloch equation [Eq. (2)] is shown in Fig. 1(a). In the case of weak interaction, the nonlinearity changes the shapes of the Bloch bands slightly but lifts them into the linear band gaps. When g is increased, the nonlinear bands move up continuously without upper limits. Under the open boundary, the corresponding eigenvalues of the nonlinear Schrödinger equation (1) also form three energy bands as shown in Fig. 1(b). The nonlinear bands have been marked by cross symbols while the linear bands are marked by open circles. It is interesting to see that two eigenenergy levels μ_8 and μ_{16} appear in the first and the second band gaps, respectively. As is shown in Fig. 2 (blue solid lines), these two states are edge states and the corresponding wave functions Ψ_8 and Ψ_{16} are localized in the left and right boundaries, respectively. On the contrary, all the intra-band states are extended states. Three extended states [Ψ_1, Ψ_2, and Ψ_9] are shown in Fig. 5 as blue solid lines.

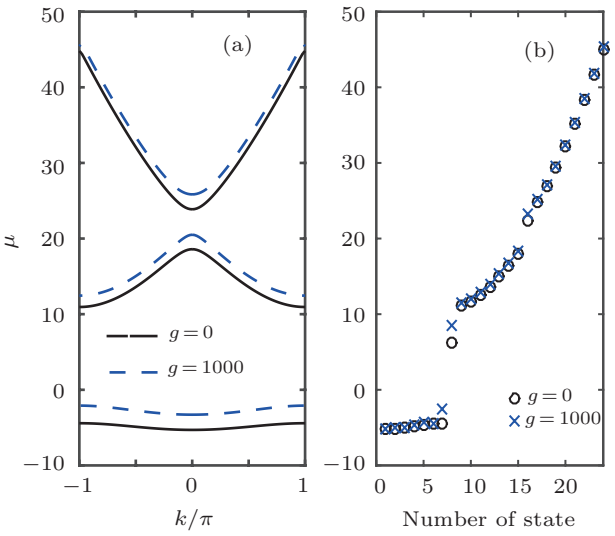


Fig. 1. (color online) (a) The linear Bloch energy bands (black solid lines) and the nonlinear Bloch energy bands (blue dashed lines) of the nonlinear Bloch Schrödinger equation (2). (b) The eigenvalues of the nonlinear Schrödinger equation (1) in ascending order under open boundary condition. The energy levels form three energy bands. One edge state appears in the first band gap and the other edge state appears in the second band gap. Their wave functions are plotted in Fig. 2 as blue solid lines.

3.1. Edge gap solitons

To understand the origin of the edge states, we have plotted the periodic potentials $V(x)$ (red dashed lines) in Fig. 2. Near the bottom of $V(x)$, the potential can be approximated by the parabolic potential

$$V(x) \sim v\left[(\pi x + n\pi + \delta/2)^2 - 1\right], \tag{3}$$

i.e., parabola or atom potential. Here, n can be taken to be an integer. When a condensed boson is confined in a parabola, the particles can be in a series of discrete eigenstates. The energy levels and wave functions of the parabolas remain unchanged provided the vertices of the parabola are confined in $[-L/2, L/2]$. A shift of the vertex along x simply shifts the parabola and wave function, and leaves its energy unaffected. This holds true until the parabola hits the wall. In such a case, the walls and the parabola provide the main confinement (thick red solid lines in Fig. 2). The particles now sit in a roughly triangular potential well. Due to the stronger confinement, the energy levels will be raised and be higher than the corresponding energy levels in the middle. The degeneracy is lost and

the states squeezed against the side of the wall. The edge state emerges in the inter band. On the contrary, if the degeneracy exists, the edge states will not appear. The edge states as obtained by solving the nonlinear equation [Eq. (1)] with a parabola near the boundaries (roughly triangular potential well) are shown in Fig. 2 as black lines with circles. Comparing the blue solid lines and the black lines with circle symbols in Fig. 2, we find that the edge states resulting from the two methods coincide well.

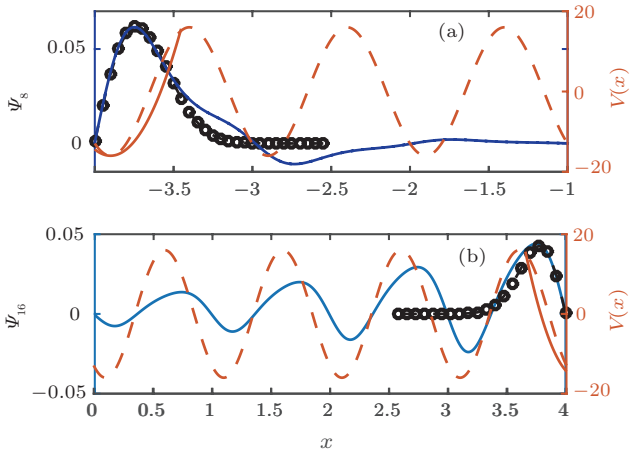


Fig. 2. (color online) The figures show the two edge states and explain how these states can be induced by the parabolas and the walls. From top to bottom, the blue solid lines represent two edge states of the nonlinear Schrödinger equation (1) (a): Ψ_8 and (b): Ψ_{16}. The black circle lines show the wave functions of the nonlinear Schrödinger equation (1) with the triangle potential conformed by the wall and the parabolas near the boundaries (yellow thick solid lines). The periodic potentials $V(x)$ are shown by the yellow dashed solid lines.

It has been pointed out that the nonlinear periodic system can host a special kind of waves, namely gap solitons, which are spatially localized wave packets with chemical potentials in the linear band gaps.[21] It has also been found that the gap solitons and the nonlinear Wannier functions match very well. The match gets even better as the periodic potential becomes stronger.[10,22] The excellent match between the gap solitons and the nonlinear Wannier functions suggests that the gap solitons can be approximated by the orbital wave functions of a unit cell since the orbital wave functions can be taken as Wannier functions when the periodic potential is stronger. Here, the formations of the stable edge state are due to the linear spreading from the kinetic energy and the repulsive interaction compensated by the confinement of a roughly triangular potential well. So the edge states can be called edge gap solitons. However, while the spatially localized wave packets form in the parabola away from the boundary, these states are called the gap solitons.

For the edge gap soliton, the wave function localizes in the boundary and has a long tail. This is due to the wave function overlap between the edge gap soliton and the near gap soliton. For the edge gap soliton Ψ_8 shown in Fig. 2, the chemical potential is in the first band gap. The wave function develops from the ground states of the parabola near the walls (thick blue solid line). As a result, the wave function shows only one peak.

For the edge gap solitons Ψ_{16}, its chemical potential lies in the second band gap. It is natural to estimate that the wave function develops from the first excited state of the right parabola and the wave function in a cell with a node. However, the wave function shows only one peak, which is the same as that of the state Ψ_8. This is due to the fact that the parabola for Ψ_{16} is closer to the right wall than the parabola for Ψ_8 in the left side. It gives a strong confinement of the particles. So the chemical potential of Ψ_{16} is higher than that of Ψ_8.

3.2. Stability of the edge gap solitons

The linear stability is studied according to the following standard procedure. Since the unstable solution is sensitive to a small perturbation, one can add a small perturbation $\Delta\Psi(x,t)$ to a known stationary solution $\Psi(x)$ of the nonlinear Schrödinger equation (1)

$$\Omega(x,t) = [\Psi(x) + \Delta\Psi(x,t)]\exp(-\mathrm{i}\mu t),$$

where the term $\Delta\Psi(x,t) = u(x)\exp(\mathrm{i}\lambda t) + w^*(x)\exp(-\mathrm{i}\lambda^* t)$. Inserting the perturbation into a time-dependent nonlinear Schrödinger equation and dropping the higher-order terms in (u,v), one then obtains the linear eigenfunction

$$\begin{pmatrix} \mathcal{L} & -g\Psi^2 \\ g\Psi^{*2} & -\mathcal{L} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \lambda \begin{pmatrix} u \\ w \end{pmatrix}, \tag{4}$$

where

$$\mathcal{L} \equiv \frac{1}{2}\frac{d^2}{dx^2} - V(x) - 2g|\Psi(x)|^2 + \mu.$$

Linear stability of a soliton is determined by the energy spectrum of the linear eigenfunction (4). Among all eigenvalues λ obtained, if there exists a finite imaginary part, the solution of $\Psi(x)$ would be unstable. Otherwise, the solution of $\Psi(x)$ is stable.

The stabilities of the edge gap solitons are investigated in Fig. 3. For the non-interaction case ($g = 0$), the two edge gap solitons are reduced to the stationary solutions of the linear Schrödinger equation and are also stable. When the interactive intensity g is increased, the two states will change from stable to unstable. The reason is that the confinement of the edge parabola is not sufficient for compensating the repulsive interaction and the kinetic energy. For the state Ψ_8 in Fig. 3(a), it is stable when $g < 6\times10^4$. However, Ψ_{16} becomes unstable when $g > 4000$ as shown in Fig. 3(b). This is due to the stronger confinement of the right parabola than that of the left parabola to the particles, which increases the kinetic energy and the repulsive interaction. In fact, there exists a little imaginary part in λ [Fig. 3(b)], and the edge gap soliton Ψ_{16} will decay for a long time. So Ψ_{16} is unstable when $1000 < g < 2000$.

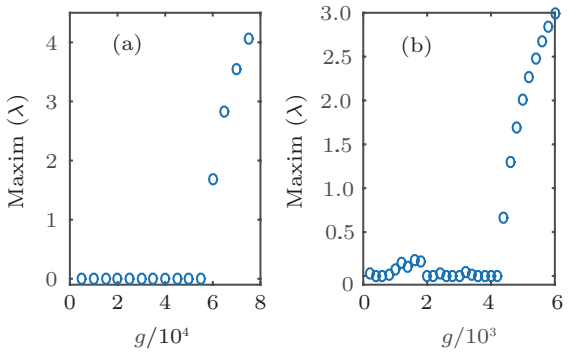


Fig. 3. (color online) Study of the stability of the edge gap solitons (a) Ψ_8 and (b) Ψ_{16}.

3.3. Topological invariant

The appearance of edge states in the current problem is attributed to the nontrivial topological properties of bulk systems. In Fig. 4(a), we show the spectrum of the periodic systems versus δ under the open boundary condition. The shade regimes correspond to the band regimes and the lines between bands are the spectra of the edge states. As the phase δ changes from 0 to 2π, the spectrum of the nonlinear Schrödinger equation (1) changes periodically. The position of the edge states in the gaps also varies continuously with the change of δ. In particular, the level continuously connects the upper and lower energy bands. A random potential $V_r(x)$ is constructed and added to the periodic potential $V(x)$ to study the topological properties. The random potential function $V_r(x)$ includes several Gauss functions $\left(v/\sqrt{2\pi\sigma}\right)\exp\left[-(x-x_0)^2/2\sigma\right]$. The randomicities come from v, x_0, and σ. As shown in Fig. 4(b), the spectrum of the edge states as functions of δ is still gapless although the three bands are widened due to the random potential. So the gapless edge states are robust against the weak disorder. When the amplitude of the disorder is increased, the band gaps are closed and the edge gap solitons are destroyed.

The gapless edge states are related to the topological property of the system. The topological structure is characterized by the topological invariant which can be characterized by the Chern numbers of the occupied energy bands.[23,24] For the current nonlinear periodic model, the introduction of the potential shift δ makes the potential $V(x)$ change periodically with δ. The eigenvector ψ_{nk} and eigenvalue E_{nk} of the Bloch equation [Eq. (2)] are periodic in δ and the lattice constant according to the Bloch theorem. The Hamiltonian

$$H(k,\delta)=\left[-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\mathrm{i}k\right)^2+\mathcal{V}(x,\delta)\right]$$

is also periodic in time and position over the lattice constant since the effective potential $\mathcal{V}(x,\delta)=V(x)+g|\psi_{nk}(\delta)|^2$ is periodic in time and position over the lattice constant in the mean-field level. The modulation phase in periodic potential and the momentum extend a 2D parameter space where a topological invariant is defined to distinguish different topological properties.

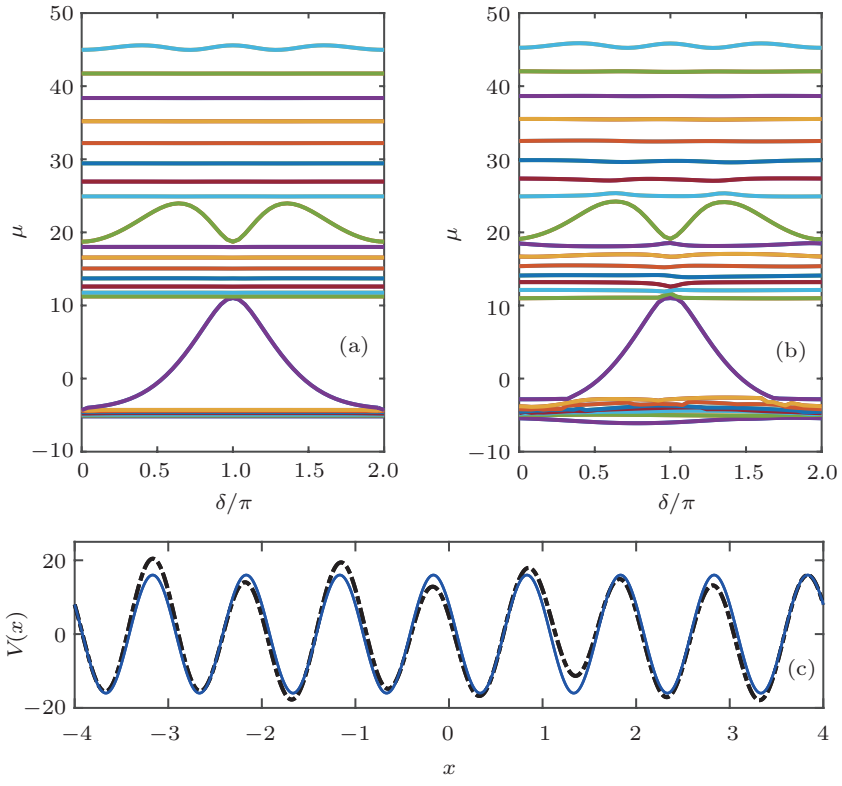


Fig. 4. (color online) Nonlinear energies as a function of the phase δ from Eq. (1). The interaction parameter has been taken as $g=300$. (a) Results for the periodic potential $V(x)$ without disorder. (b) Results for a disordered potential $V(x)+V_r(x)$. (c) Comparison of the periodic potential $V(x)$ (blue solid line) and the disorder potential $V(x)+V_r(x)$ (black dashed line).

On the other hand, the above model is natural to relate the adiabatic variations of the potential in time $V(x,t)$, i.e., the quantized adiabatic particle transport problem. For simplicity of the discussion, the particle in a band is viewed as a set of charge localized in the center of the Wannier functions. With the increase of the time, the potential will translate its position and the particle will move with the potential. Although the Wannier function recovers its original shape after a cycle of the potential, its position or a particle has slid by a lattice-constant. The average particle current over a time cycle is the Chern number.[25,26] We follow the method in Ref. [27] to perform the computation of the Chern number directly. We find that the Chern numbers for the Boson in the three sub-bands are 1, -2, and 1, respectively, for both the linear ($g = 0$) and nonlinear ($g = 300$) cases.

3.4. Composition relation

As discussed in Refs. [10], [22], [28], and [29], gap solitons develop in the linear band gaps and originate from the bound states of a single parabola. So they can be divided into different families according to the locations of the band gaps. On the other hand, the nonlinear Bloch band can be viewed as a lifted linear Bloch band by increasing the nonlinear interaction. However, the linear Bloch band can be viewed as an evolution from the discrete energy levels of an individual parabola. In particular, the gap solitons match the Wannier function well when the periodic potential is strong. Therefore, the gap solitons and nonlinear Bloch waves should share certain common features. This is called the 'composition relation'. By numerical analysis, an almost perfect match is found between the nonlinear Bloch waves and the gap solitons within a single unit cell of $V(x)$. The good matches give strong evidences to verify that gap solitons are the building blocks of the nonlinear Bloch waves. The issue is whether the composition relation is still valid under the open boundary condition.

Under the open boundary condition, the system still exists in extended states. As shown in Fig. 5 by blue solid lines, their wave functions can extend the whole region. The extended states can also be viewed as an evolution from the discrete states of an individual parabola. Due to the overlap of the adjacent wave functions of the discrete states, the degenerate energy levels will split. The splitting of energy levels forms a series of energy bands. The local wave pockets will then become extended states. The good correspondences between the gap solitons and the eigen states of a parabola supports that the gap solitons can be used as the Wannier function to build the extended states.

We solve the nonlinear Schrödinger equation [Eq. (1)] with a parabolic potential [Eq. (3)] to obtain the gap solitons and then calculate the nearest-neighbor exchange coupling parameter J. Neglecting the second and third neighbor exchange interaction and solving the tight-binding Hamiltonian under the open boundary condition, we obtain the tight-binding wave functions as shown in Fig. 5 (black circled lines). Comparing the blue solid lines and the black circled lines in Fig. 5, we find that the states given by the two methods coincide well. Since the wave function overlap of the gap solitons in the low energy states is less than that in the high energy states, the splitting in the low energy levels is less than that in the high energy levels. As is shown in Fig. 2, the high energy band has a large band width.

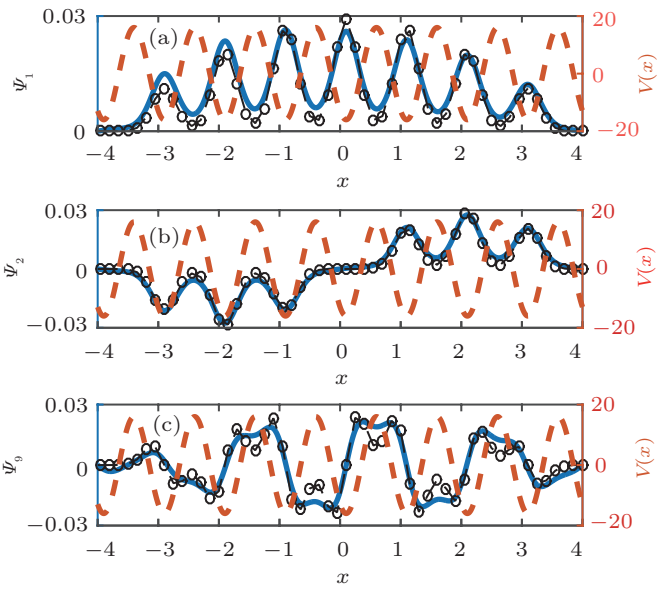


Fig. 5. (color online) The figure shows three extended states and how the three states can be obtained by the tight binding approximate method under the open boundary condition. The wave functions shown by blue solid lines are obtained by solving the nonlinear Schrödinger equation (1) with the difference method directly. The black circled lines show the wave functions by the tight binding approximate method.

The extended state $\Psi_1(x)$ in Fig. 5(a) is the ground state of the system and the envelope of the tight-binding wave function is Gauss type. The gap soliton used as the Wannier function is from the ground state without a node. The extended state $\Psi_2(x)$ as shown in Fig. 5(b) is the first excited state of the system and the tight-binding wave function has a node. The gap soliton is from the ground state. So the sign of the wave function $\Psi_2(x)$ is positive in one half and negative in the other half. The extend state $\Psi_9(x)$ shown in Fig. 5(c) is in the second band. The gap soliton used as the Wannier function is from the excited state with a node. The tight-binding wave function is also Gaussian. However, the nearest-neighbor exchange interaction J is positive, which is in contrast to the case in Figs. 5(a) and 5(b) where the nearest-neighbor exchange interaction J is negative. The sign of the tight-binding wave function in one site is different from that of the neighbor site. The good matches verify that gap solitons are the building blocks

of the extend states. The so-called "composition relation" is still valid under the open boundary condition.

4. Summary

In summary, we have explored the nontrivial states of the interacting Bosons confined in a 1D cosine potential under the open boundary condition. Our study reveals that a kind of wave exists in this system: edge gap solitons. The localized waves originate from the edge parabolas and exhibit nontrivial topological natures. We find that the problem connects to the quantized adiabatic particle transport. With the linear stability analysis, it is found that the stable edge gap solitons can exist near the bottom of the linear band gap. The numerical results verify that the composition relation remains correct under the open boundary condition. Edge gap solitons are thus the fundamental building blocks to extend states. It is expected that the results will be useful for observing topological phases by using a 1D Bose system.

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