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Multiple Darboux–Bäcklund transformations via truncated Painlevé expansion and Lie point symmetry approach*

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For a given truncated Painlevé expansion of an arbitrary nonlinear Painlevé integrable system, the residue with respect to the singularity manifold is known as a nonlocal symmetry, called the residual symmetry, which is proved to be localized to Lie point symmetries for suitable prolonged systems. Taking the Korteweg–de Vries equation as an example, the n-th binary Darboux–Bäcklund transformation is re-obtained by the Lie point symmetry approach accompanied by the localization of the n-fold residual symmetries.

Keywords: residue symmetry, multiple Darboux–Bäcklund transformation, Lie point symmetry approach

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1. Introduction

To understand an essentially nonlinear world, scientists have established various integrable systems, such as the Korteweg–de Vries (KdV) equation,[1,2] the nonlinear Schrödinger equation (NLS),[3,4] the sine-Gordon (SG) equation,[5–8] the Boussinesq equation,[9] the Kadomtzev–Pedviashvili (KP) equation,[10] and so on.

The symmetry approach is one of the most powerful methods to study exact solutions of a nonlinear system no mater whether the model is integrable or not. However, some types of physically important solutions, for instance, the n-soliton ($n \geq 2$) solutions of integrable systems, cannot be obtained by means of the standard Lie point symmetry group method. In order to conquer this difficulty, we will develop a new method to change the situation for Painlevé integrable models by introducing proper higher dimensional solution spaces, so that the Darboux–Bäcklund transformations (and then the multiple soliton solutions) can be obtained from the Lie point symmetry approach.

The Painlevé analysis is one of the best approaches to study integrability of nonlinear systems. Its essential idea is to study the analytic property, and thus, the residue with respect to the singular manifold must be one of the most important subjects for Painlevé integrable models. In fact, Newell, Tabor, and Zeng have pointed out that the coefficient of ϕ^{-1} in the expansions for the dependent variables is related to new functions, which are the "square" of the functions satisfying Lax equations, and it is also a symmetry (usually it is named as the square eigenfunction symmetry for the KdV-type systems and the AKNS systems). In Ref. [11], we re-named these types of symmetries as residual symmetries because it is valid not only for integrable systems but also for non-integrable models. There are no Lax pairs and then no eigenfunctions for nonintegrable systems. Even for integrable systems, the nonlocal symmetries may not be proportional to or derivatives of the square eigenfunctions.

Recently, the nonlocal symmetries related to the infinitesimal forms of the Darboux transformation (DT) and/or Bäcklund transformations (BT) have been widely applied to obtain quite general solutions of some important nonlinear systems such as the KdV equation,[12] modified KdV equation,[13] Harry–Dym equation,[14,15] Kawamoto equation,[16] Sawada–Kortera equation,[17] Kadomtsev–Petviashivili equation,[10,18] NLS equation,[19,20] water wave equation,[21] Burgers equation,[22,23] and Broer–Kaup system.[24] To find integrable properties and exact solutions of Painlevé integrable systems, we need to localize the nonlocal residue symmetries by introducing suitable prolonged systems,[25–28] so that the n-th (Darboux–)Bäcklund transformations can be obtained via the Lie point symmetry approach.

This paper is organized in the following way. In Section 2, we first summarize the known facts related to the residual symmetries to several theorems for some Painlevé integrable systems. The theorems are illustrated by some concrete examples. In Section 3, taking the KdV equation as an example, we demonstrate that arbitrary numbers of residual symmetries can be localized to establish finite transformations which are equivalent to the second type of multiple Darboux–Bäcklund transformations. The last section is a summary and discussion.

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2. Residual symmetries of Painlevé integrable systems

It is known[29–32] that for almost all integrable systems, there are possible variants possessing the Painlevé property, that is to say, they are Painlevé integrable. For the Painlevé integrable systems, for instance, a single component derivative polynomial system,[33]

$$P(u)=0, \tag{1}$$

there exists a truncated Painlevé expansion

$$u=\sum_{i=0}^{\alpha} u_i \phi^{i-\alpha}, \tag{2}$$

for a positive integer α.

A symmetry of Eq. (1) is defined as a solution of its linearized equation

$$P'(u)\sigma=\lim_{\epsilon=0}\frac{\mathrm{d}P(u+\epsilon\sigma)}{\mathrm{d}\epsilon}=0. \tag{3}$$

Substituting the truncated Painlevé expansion (2) into the original nonlinear system (1), and then vanishing the coefficients of different powers of ϕ, we have the following theorem.

Theorem 1 (Residual symmetry theorem)

The residue of the truncated Painlevé expansion (2) with respect to the singular manifold ϕ, i.e., $u_{\alpha-1}$, is a symmetry of Eq. (1) with the solution u_α.

Poof Though it is a known fact as mentioned in Ref. [33], we still write a proof procedure here for completeness. Substituting Eq. (2) into Eq. (1), we obtain

$$\begin{aligned} P(u)&=P\left(\sum_{i=0}^{\alpha} u_i\phi^{i-\alpha}\right)\\ &=P(u_\alpha)+\left[P'(u_\alpha)u_{\alpha-1}\right]\frac{1}{\phi}+O\left(\frac{1}{\phi^2}\right)=0. \end{aligned} \tag{4}$$

Vanishing the coefficients of ϕ^0 and ϕ^{-1}, we have

$$P(u_\alpha)=0, \tag{5}$$

and

$$P'(u_\alpha)u_{\alpha-1}=0, \tag{6}$$

respectively. Comparing Eqs. (5) and (6) with Eqs. (1) and (3), we find that u_α is a solution of the original model, and the residue

$$\sigma^{u_\alpha}=u_{\alpha-1} \tag{7}$$

is a symmetry of Eq. (1) with respect to the solution u_α. Theorem 1 is thus proved. ■

It is also known that for Painlevé integrable system (1),

$$u_\alpha=u_\alpha(\phi) \tag{8}$$

transforms the original nonlinear system (1) to its Schwarzian form

$$P_s(\phi)=0. \tag{9}$$

The Schwarzian form (9) is invariant under the Möbious transformation

$$\phi\longrightarrow\frac{a\phi+b}{c\phi+d},\quad ac\neq bd, \tag{10}$$

which hints Eq. (9) possesses three symmetries, $\sigma^\phi=a_1$, $\sigma^\phi=b_1\phi$, and

$$\sigma^\phi=c_1\phi^2 \tag{11}$$

with arbitrary constants a_1, b_1, and c_1. To the best of our knowledge, the residual symmetry (7) is just related to the Möbious transformation symmetry (11) by the linearized equation of Eq. (8), i.e.,

$$\sigma^{u_\alpha}=(u_\alpha)'_\phi\sigma^\phi, \tag{12}$$

with Eqs. (7) and (11), while the constant c_1 is different for different models.

In many cases, the detailed forms of the residual symmetries can be simply fixed by the dimensional analysis. For instance, if the nonlinear system (1) possesses the quasilinear evolution form

$$u_t=au_{x^b}+P_1(u,u_x,\ldots,u_{x^c}), \tag{13}$$

for constants a, b, and $c<b$, and the dimensional degree, δ, of $1/u$ is smaller than or equal to b (assuming the dimensional degree of x is 1), then the residual symmetry of Eq. (13) reads

$$\sigma^u=c_0\phi_{x^\delta}, \tag{14}$$

where c_0 is a constant dependent on different models, hereafter we drop out the indices α without loss of generality.

Some well-known concrete examples of Eq. (13) include the KdV equation

$$u_t+u_{xxx}-6uu_x=0, \tag{15}$$

the SK equation

$$u_t=u_{xxxxx}+45u^2u_x+15(uu_{xx})_x=0, \tag{16}$$

the KK equation

$$u_t=u_{xxxxx}+\frac{15}{2}uu_{xxx}+\frac{75}{4}u_xu_{xx}+\frac{45}{4}u^2u_x=0, \tag{17}$$

and the fifth order KdV equation

$$u_t=u_{xxxxx}+30u^3u_x+20u_xu_{xx}+10uu_{xxx}. \tag{18}$$

It is clear that for all these models, the dimensional degree of $1/u$ is 2, and thus, their residual symmetries are all given by

$$\sigma^u=-2\phi_{xx}, \tag{19}$$

by dropping out possible trivial constant multiplications for different models. In fact, many other types of important nonlinear systems, such as the Boussinesq equation

$$u_{tt}+u_{xxxx}+6(uu_x)_x=0, \tag{20}$$

and the KP equation

$$(u_t+u_{xxx}+6uu_x)_x+su_{yy}=0, \tag{21}$$

also possess the residual symmetry (19). In addition, it is easy to determine that the constant c_1 in Eq. (11) is

$$c_1=1, \quad (\delta=2) \tag{22}$$

for all the models listed in Eqs. (15)–(21).

It is known[25–28] that for some kinds of nonlocal symmetries, the localization procedure can be used such that the nonlocal symmetries of the original nonlinear system become local ones for a suitably prolonged system. It is interesting that if the residual symmetry given by Eq. (14) is related to the Möbious transformation symmetry (11), then we have the following Bäcklund transformation theorem:

Theorem 2 Bäcklund transformation theorem

If the nonlinear system (1) possesses a Schwarzian form and the residual symmetry (14) is related to the Möbious transformation symmetry (11), then the model possesses the Bäcklund transformation

$$U(\varepsilon)=u-\frac{c_0}{c_1}\left(\ln(c_1\varepsilon f-1)\right)_{x^\delta}, \tag{23}$$

$$F(\varepsilon)=\frac{f}{1-\varepsilon c_1 f}, \tag{24}$$

where $\{u,\ f\}$ and $\{U(\varepsilon),\ F(\varepsilon)\}$ are all solutions of the nonlinear system (1) and its Schwarzian form.

Proof To prove the Bäcklund transformation Theorem 2 is simply to solve the initial value problem

$$\frac{\mathrm{d}U(\epsilon)}{\mathrm{d}\epsilon}=c_0\left(F(\epsilon)\right)_{x^\delta}, \quad U(0)=u, \tag{25}$$

$$\frac{\mathrm{d}F(\epsilon)}{\mathrm{d}\epsilon}=c_1F(\epsilon)^2, \quad F(0)=f. \tag{26}$$

It is quite trivial to see that the solution of Eq.(26) is just Eq. (24), and substituting Eq. (24) into Eq. (25) yields Eq. (23) immediately. Theorem 2 is thus proved. ■

Theorem 2 is known for various integrable systems. For instance, according to Eqs. (19) and (22), the KdV, SK, KK, fifth order KdV, Boussinesq, and KP equations possess a common Bäcklund transformation Theorem 2 with $c_0=-2$, $c_1=1$, and $\delta=2$.

3. Multiple Darboux–Bäcklund transformations of KdV equation related to *n*-fold residual symmetries

The transformation

$$u=-\frac{1}{2}\frac{\phi_{xxx}}{\phi_x}+\frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}-\lambda \tag{27}$$

changes the KdV equation (15) to its Schwarzian form

$$S+C+\lambda=0, \ \ C\equiv\frac{\phi_t}{\phi_x}, \ \ S\equiv\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\frac{\phi_{xx}^2}{\phi_x^2}. \tag{28}$$

The residual symmetry of the KdV equation becomes a local Lie point symmetry

$$\sigma^u=-2\phi_{xx}, \tag{29a}$$

$$\sigma^\phi=\phi^2, \tag{29b}$$

$$\sigma^g=2g\phi, \tag{29c}$$

$$\sigma^h=2(h\phi+g^2), \tag{29d}$$

for the prolonged system

$$u_t+u_{xxx}+6uu_x=0, \tag{30a}$$

$$u=-\frac{1}{2}\frac{\phi_{xxx}}{\phi_x}+\frac{1}{4}\frac{\phi_{xx}^2}{\phi_x^2}-\lambda, \tag{30b}$$

$$g=\phi_x, \tag{30c}$$

$$h=g_x. \tag{30d}$$

The finite transformation of the Lie point symmetry [Eqs. (29a)–(29d)] is just the Theorem 2 of the last section.

Because the symmetry equation of a nonlinear system is linear and the Schwarzian form of the original nonlinear system possesses infinite solutions, we get infinite residual symmetries ϕ_{i,x^δ}, $i=1,2,\ldots$. For the KdV equation (15), the infinite residual symmetries can be written as

$$\sigma_n^u=-2\sum_{i=1}^{n}c_i\phi_{i,xx} \tag{31}$$

for arbitrary n, where ϕ_i, $i=1,\ldots,n$ are all different solutions of the Schwarzian KdV equation (28), namely, for different $\lambda=\lambda_i$, $\lambda_i\neq\lambda_j$, $\forall i\neq j$.

Similar to the $n=1$ case, to find the finite transformations of Eq. (31), one has to introduce a suitably enlarged system such that the nonlocal symmetry can be localized to a Lie point symmetry. Fortunately, it is easy to deduce the following localization theorem:

Theorem 3 Localization theorem

If $\{u,\ \phi_i,\ g_i,\ h_i,\ i=1,\ 2,\ \ldots,\ n\}$ is a solution of the prolonged system

$$u_t+u_{xxx}+6uu_x=0, \tag{32a}$$

$$u=-\frac{1}{2}\frac{\phi_{i,xxx}}{\phi_{i,x}}+\frac{1}{4}\frac{\phi_{i,xx}^2}{\phi_{i,x}^2}-\lambda_i, \tag{32b}$$

$$g_i=\phi_{i,x}, \tag{32c}$$

$$h_i=g_{i,x},\ i=1,\ \ldots,\ n, \tag{32d}$$

then the symmetry (31) is localized to a Lie point symmetry

$$\sigma^u=-2\sum_{j=1}^{n}c_jh_j, \tag{33a}$$

$$\sigma^{\phi_i} = c_i\phi_i^2 + \sum_{j\neq i}^{n} \frac{c_j}{4}\frac{(g_ih_j - g_jh_i)^2}{g_ig_j(\lambda_i-\lambda_j)^2}, \tag{33b}$$

$$\sigma^{g_i} = 2c_ig_i\phi_i + \sum_{j\neq i}^{n} c_j\frac{g_ih_j - g_jh_i}{\lambda_i-\lambda_j}, \tag{33c}$$

$$\sigma^{h_i} = 2c_i(h_i\phi_i + g_i^2) + \sum_{j\neq i}^{n}\frac{c_j}{2}\left[4g_ig_j + \frac{(g_i^2h_j^2 - g_j^2h_i^2)}{g_ig_j(\lambda_i-\lambda_j)}\right]. \tag{33d}$$

Proof To prove Theorem 3, the first step is to write down the linearized system of the prolonged system as

$$\sigma_t^u + \sigma_{xxx}^u + 6(u\sigma^u)_x = 0, \tag{34a}$$

$$\sigma^u = -\frac{1}{2}\frac{\sigma_{xxx}^{\phi_i}}{\phi_{i,x}} + \frac{1}{2}\frac{\sigma_x^{\phi_i}\phi_{i,xxx}}{\phi_{i,x}^2} + \frac{1}{2}\frac{\phi_{i,xx}\sigma_{xx}^{\phi_i}}{\phi_{i,x}^2} - \frac{1}{2}\frac{\phi_{i,xx}^2\sigma_x^{\phi_i}}{\phi_{i,x}^3}, \tag{34b}$$

$$\sigma^{g_i} = \sigma_x^{\phi_i}, \tag{34c}$$

$$\sigma^{h_i} = \sigma_x^{g_i},\ i = 1,\ \ldots,\ n. \tag{34d}$$

In Theorem 3, there are n independent but similar symmetries related to $c_k \neq 0$, $c_j = 0$, $\forall j \neq k$, and $k = 1,\ 2,\ \ldots,\ n$.

For any fixed $c_k \neq 0$ and $c_j = 0$, $j \neq k$, from Eq. (29), we have

$$\sigma^u = -2c_kh_k, \tag{35a}$$

$$\sigma^{\phi_k} = c_k\phi_k^2, \tag{35b}$$

$$\sigma^{g_k} = 2c_kg_k\phi_k, \tag{35c}$$

$$\sigma^{h_k} = 2c_k(h_k\phi_k + g_k^2). \tag{35d}$$

For $j \neq k$, eliminating u from Eq. (32b) with $i = k$ and Eq. (32b) with $i = j$, we obtain

$$\phi_{j,xxx} = 2(\lambda_k-\lambda_j)\phi_{j,x} + \frac{\phi_{j,xx}^2}{2\phi_{j,x}} + \frac{\phi_{k,xxx}\phi_{j,x}}{\phi_{k,x}} - \frac{\phi_{k,xx}^2\phi_{j,x}}{2\phi_{k,x}^2}. \tag{36}$$

Substituting Eq. (35a) into Eq. (34b) with $i = j$ and eliminating $\phi_{j,xxx}$ via Eq. (36) yield

$$\sigma^{\phi_j} = \frac{c_k(g_kh_j - h_kg_j)^2}{4g_jg_k(\lambda_k-\lambda_j)^2}. \tag{37}$$

Differentiating Eq. (37) with respect to x once and twice, respectively, immediately results in

$$\sigma^{g_j} = c_k\frac{g_kh_j - g_jh_k}{\lambda_k-\lambda_j}, \tag{38}$$

and

$$\sigma^{h_j} = \frac{c_k}{2}\left[4g_kg_j + \frac{g_k^2h_j^2 - g_j^2h_k^2}{g_kg_j(\lambda_k-\lambda_j)}\right]. \tag{39}$$

After taking the linear combination of the above results for all $k = 1,\ 2,\ \ldots,\ n$, Theorem 3 is proved. ■

Whence a nonlocal symmetry is localized to a Lie point symmetry, to find its finite transformation becomes a standard trick to solve its initial value problem according to Lie's first principle.[34] For the Lie point symmetry (33), the corresponding initial value problem reads

$$\frac{\mathrm{d}U(\epsilon)}{\mathrm{d}\epsilon} = -2\sum_{j=1}^{n} c_jH_j(\epsilon)_j, \tag{40a}$$

$$\frac{\mathrm{d}\Phi_i(\epsilon)}{\mathrm{d}\epsilon} = c_i\Phi_i(\epsilon)^2 + \sum_{j\neq i}^{n}\frac{c_j}{4}\frac{(G_i(\epsilon)H_j(\epsilon) - G_j(\epsilon)H_i(\epsilon))^2}{G_i(\epsilon)G_j(\epsilon)(\lambda_i-\lambda_j)^2}, \tag{40b}$$

$$\frac{\mathrm{d}G_i(\epsilon)}{\mathrm{d}\epsilon} = 2c_iG_i(\epsilon)\Phi_i(\epsilon) + \sum_{j\neq i}^{n} c_j\frac{G_i(\epsilon)H_j(\epsilon) - G_j(\epsilon)H_i(\epsilon)}{\lambda_i-\lambda_j}, \tag{40c}$$

$$\frac{\mathrm{d}H_i(\epsilon)}{\mathrm{d}\epsilon} = 2c_i[H_i(\epsilon)\Phi_i(\epsilon) + G_i^2(\epsilon)] + \sum_{j\neq i}^{n}\frac{c_j}{2}\left[4G_i(\epsilon)G_j(\epsilon) + \frac{G_i^2(\epsilon)H_j^2(\epsilon) - G_j^2(\epsilon)H_i^2(\epsilon)}{G_i(\epsilon)G_j(\epsilon)(\lambda_i-\lambda_j)}\right], \tag{40d}$$

$$U(0) = u,\ \Phi_i(0) = \phi_i,\ G_i(0) = g_i,\ H_i(0) = h_i,\ i = 1,\ \ldots,\ n. \tag{40e}$$

Solving the above initial valued problem (40), we obtain the following n-th Bäclund transformation theorem.

Theorem 4 The n-th Bäclund transformation theorem

If $\{u,\ \phi_i,\ g_i,\ h_i,\ i = 1,\ 2,\ \ldots,\ n\}$ is a solution of the prolonged KdV system (32), so is $\{U(\epsilon),\ \Phi_i(\epsilon),\ G_i(\epsilon),\ H_i(\epsilon),\ i = 1,\ 2,\ \ldots,\ n\}$, where

$$U(\epsilon) = u + 2(\ln\Delta)_{xx}, \tag{41a}$$

$$\Phi_i(\epsilon) = -\frac{\Delta_i}{\Delta}, \tag{41b}$$

$$G_i(\epsilon) = \Phi_{i,x}(\epsilon), \tag{41c}$$

$$H_i(\epsilon) = \Phi_{i,xx}(\epsilon) \tag{41d}$$

with Δ and Δ_i being the determinants of

$$M=\begin{pmatrix} c_1\epsilon\phi_1-1 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1j} & \cdots & c_1\epsilon w_{1n}\\ c_2\epsilon w_{12} & c_2\epsilon\phi_2-1 & \cdots & c_2\epsilon w_{2j} & \cdots & c_2\epsilon w_{2n}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ c_j\epsilon w_{1j} & c_j\epsilon w_{2j} & \cdots & c_j\epsilon\phi_j-1 & \cdots & c_j\epsilon w_{jn}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{jn} & \cdots & c_n\epsilon\phi_n-1 \end{pmatrix}, \quad w_{ij}\equiv\frac{g_ih_j-g_jh_i}{2\sqrt{g_ig_j}(\lambda_i-\lambda_j)}, \tag{42}$$

and

$$M_i=\begin{pmatrix} c_1\epsilon\phi_1-1 & c_1\epsilon w_{12} & \cdots & c_1\epsilon w_{1,i-1} & c_1\epsilon w_{1i} & c_1\epsilon w_{1,i+1} & \cdots & c_1\epsilon w_{1n}\\ c_2\epsilon w_{12} & c_2\epsilon\phi_2-1 & \cdots & c_2\epsilon w_{2,i-1} & c_2\epsilon w_{2i} & c_2\epsilon w_{2,i+1} & \cdots & c_2\epsilon w_{2n}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ c_{i-1}\epsilon w_{1,i-1} & c_{i-1}\epsilon w_{2,i-1} & \cdots & c_{i-1}\epsilon\phi_{i-1}-1 & c_{i-1}\epsilon w_{i-1,i} & c_{i-1}\epsilon w_{i-1,i+1} & \cdots & c_{i-1}\epsilon w_{i-1,n}\\ w_{1i} & w_{2i} & \cdots & w_{i,i-1} & \phi_i & w_{i,i+1} & \cdots & w_{in}\\ c_{i+1}\epsilon w_{1,i+1} & c_{i+1}\epsilon w_{2,i+1} & \cdots & c_{i+1}\epsilon w_{i-1,i+1} & c_{i+1}\epsilon w_{i,i+1} & c_{i+1}\epsilon\phi_{i+1}-1 & \cdots & c_{i+1}\epsilon w_{i+1,n}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ c_n\epsilon w_{1n} & c_n\epsilon w_{2n} & \cdots & c_n\epsilon w_{i-1,n} & c_n\epsilon w_{in} & c_n\epsilon w_{i+1,n} & \cdots & c_n\epsilon\phi_n-1 \end{pmatrix}, \tag{43}$$

respectively.

Because the Schwarzian KdV system (28) possesses the Möbious transformation invariance (10), it is evident that the group parameters ϵ and c_i, $\forall i=1,\ \ldots,\ n$ in Theorem 4 can be simply taken as 1 and $c_j\epsilon f_j-1$ can also be simply replaced by f_j without loss of generality.

According to Theorem 4, infinitely many new solutions can be generated from any seed solutions of the KdV equation and its Schwarzian form. For instance, starting from the trivial vacuum solution

$$u=0, \tag{44}$$

we obtain a special solution of Eq. (32b) and the Schwarzian KdV Eq. (28) with $\lambda=\lambda_i$ in the form of

$$\phi_i=\exp(k_ix-k_i^3t),\quad \lambda_i=-\frac{1}{4}k_i^2, \tag{45}$$

and then

$$w_{ij}=\frac{2\sqrt{k_ik_j}}{k_i+k_j}\exp\left\{\frac{k_i+k_j}{2}[x+(k_i^2-k_ik_j+k_j^2)t]\right\}. \tag{46}$$

Substituting the special solution (45) and (46) with Eq. (44) into Theorem 4, we reobtain the multiple soliton solutions of the KdV equation

$$\begin{aligned}\Delta=&1-c_1a\exp(k_1x-k_1^3t)-c_2a\exp(k_2x-k_2^3t)\\&+c_1c_2a^2\frac{(k_1-k_2)^2}{(k_1+k_2)^2}\exp[(k_1+k_2)x-(k_1^3+k_2^3)t]\end{aligned} \tag{47}$$

for the two-soliton solution.

Furthermore, due to Eq. (32b), it is easy to prove that w_{ij} in Eq. (42) satisfy

$$w_{ij,x}=\sqrt{g_ig_j}=\psi_i\psi_j,\ \forall i,j=1,\ 2,\ \ldots,\ n, \tag{48}$$

where ψ_i, $i=1,\ 2,\ \ldots,\ n$ are just the usual spectral functions of the Lax pair

$$\psi_{i,xx}+(u+\lambda_i)\psi_i=0,\ i=1,\ 2,\ \ldots,\ n \tag{49}$$

of the KdV equation. Therefore, Theorem 4 is equivalent to the known second type of Darboux–Bäcklund transformation.[35,36] In other words, Theorem 4 can be restated as:

Theorem 5 Equivalent DT theorem

If u is a solution of the KdV equation and ψ_i, $i=1,\ \ldots,\ n$ are spectral functions related to u with spectral parameters λ_i, so is u_n with

$$u_n=u+2[\ln(\det M)]_{xx},\ M_{ij,x}=\psi_i\psi_j. \tag{50}$$

4. Summary and discussion

In summary, it is shown that for Painlevé integrable systems, infinitely many nonlocal symmetries defined as residual symmetries can be readily read out from the residual of the truncated Painlevé expansions. The residual symmetries are nonlocal for the original nonlinear system. However, the residual symmetries can be localized to Lie point symmetries by prolonging the original system to an enlarged system. Consequently, starting from the Lie point symmetries of the prolonged system, the n-th Dabourx–Bäcklund transformation (and then the n-soliton solution) can be straightforwardly obtained by using Lie's first principle.

If the forms of the residual symmetries are the same for different nonlinear systems, a common first Bäcklund transformation can be obtained. For instance, for the KdV equation (15), the fifth order KdV equation (18), the SK system (16), the KK equation (17), the Boussinesq equation (20), and the KP system (21), the once Bäcklund transformation possesses the same Theorem 3 with $c_0=-2$, $c_1=1$, and

$\delta = 2$. However, the multiple BTs will be quite different for different models.

The explicit forms of the finite transforms for n-fold residual symmetries are obtained for the KdV equation. The result is equivalent to the second type of the n-th Darboux–Bäcklund (or named Levi) transformation.[35,36] Though Theorem 4 is equivalent to the known DT Theorem 5, three points need to be further emphasized. (i) The second type of DT and then the multiple soliton solutions can be obtained via the Lie point symmetry method accompanied by the localization approach. (ii) The group parameter ϵ is necessary to find the second type of DT via the Lie point symmetry approach, however, it is only auxiliary because it can be absorbed by the Möbious transformation invariance of the Schwarzian systems. Other group parameters c_i can also be absorbed. (iii) The commutable theorems for the Bäcklund transformations and Darboux transformations become trivial in the Lie point symmetry theory due to the commutativity in the addition algorithm. It is noted that the proposed method is applicable to any nonlinear system as long as it can pass the Painléve test. Residual symmetries, especially the n-th DTs related to the other models listed in this paper, will be further studied in our future researches.

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