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Quantum parameter estimation in a spin-boson dephasing quantum system by periodical projective measurements*

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In this paper, we explore how to estimate the phase damping parameter γ and the tunneling amplitude parameter Δ from a spin-boson dephasing quantum model by periodical projective measurements. The preparation of initial states is accomplished by performing the period measurements in our scheme. The parameter γ can be always estimated when projective measurement bases are chosen as $\theta = \pi/2$ and $\phi = 0$. Based on the estimated value of γ and the interval information of Δ, we can select another measurement bases ($\theta = \pi/4$ and $\phi = \pi/2$) to obtain the estimated value of Δ. A coherent control is indispensable to estimate Δ if γ is in the interval of Δ; whereas the control is not necessary if γ is out of the known interval of Δ. We establish the relation between the optimal period time and the parameter γ or Δ in terms of Fisher information. Although the optimal measurement period cannot be selected beforehand, the aforementioned relation can be utilized to adjust the measurement period to approach the optimal one.

Keywords: parameter estimation, periodical projective measurement, spin-boson model, dephasing quantum system

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1. Introduction

Quantum metrology uses quantum techniques such as entanglement to yield higher statistical precision than purely classical approaches.[1] Quantum metrology is playing an important role in precision measurement such as gravitational wave detection,[2] phase shifts in optical interferometry,[3,4] transition frequencies in atomic spectroscopy,[5,6] magnetic field detection,[7,8] and so on. Quantum metrology includes not only measuring the physical parameters with a precision beating the classical physics limit, but also measuring the quantum parameters precisely.[9]

The process of quantum parameter estimation is usually composed of several stages:[10] initial state preparation, dynamic process evolution, measurement, and data processing. Parameters to be estimated are loaded into the system during the dynamic evolution process. Then they are inferred from the measurement results. The parameters in quantum metrology are maybe classical ones like phase and frequency or maybe specific ones in quantum system such as the physical quantities describing decoherence phenomena.

The Cramér–Rao inequality[11] states that

$$V_\alpha \geqslant \frac{1}{F_\alpha}, \tag{1}$$

where α is the parameter to be estimated, V_α and F_α are the corresponding variance and Fisher information of α, respectively. The Fisher information is an essential quantity for extracting information about a parameter from a measurement-induced probability distribution and can be calculated as follows:

$$F_\alpha = \sum_i (\partial_\alpha P_i)^2 / P_i, \tag{2}$$

where P_i is the possibility corresponding to the result i obtained by a POVM measurement operator. Equation (1) means that the Fisher information saturates the bound of the estimation precision and it applies to any parameter estimation problem, including quantum parameter estimation.[11]

A spin-boson model describes the interaction between a qubit and a dissipative bosonic bath. This model effectively characterizes the dissipation-induced decoherence in quantum microscopic system, thus it is a frequently used paradigm for numerous physical and chemical processes.[12] The model has been applied to noisy quantum dots,[13] the decoherence of qubits in quantum computation,[14,15] superconducting circuits,[16,17] cavity quantum electrodynamics,[18,19] and so on.

Dephasing process, also known as phase damping process, is uniquely quantum mechanical, which describes the loss of quantum information without loss of energy.[20] The dephasing process is quite ubiquitous in quantum metrology so that it has been studied a lot,[21–35] such as the frequency estimation in atomic system,[21–26] BEC system,[26–29] nuclear magnetic resonance system,[30] magnetic-field gradient

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estimation,[31] and so on. As dephasing is a noise process, it is usually to study the destructive effect of dephasing and how to suppress the effect. However in particular systems such as BEC system,[29] it can enhance the parameter estimation precision with the help of decoherence-induced nonlinearity interaction.

The phase damping coefficient is often regarded as a known constant or can be calculated analytically in the aforementioned researches. In comparision with these researches, we display a process to estimate two parameters, including the phase damping parameter and tunneling amplitude parameter, with periodical projective measurements from a two-state dephasing model. The state prepared by periodical projective measurement in a period is reused as the initial state in the next period, not wasted. We conclude that the phase damping parameter γ can be always estimated by the parameters of projective measurement bases chosen as $\theta = \pi/2$ and $\phi = 0$. The tunneling amplitude parameter Δ can be estimated based on several factors: the measurement parameters $\theta = \pi/4$ and $\phi = \pi/2$, the estimated value of γ and the interval of Δ as prior information. If γ is in the interval of Δ, control is also needful. Control is not necessary to estimate Δ when γ is not in the interval of Δ. We also investigate how to select the optimal period time for each parameter estimation. Although the optimal measurement period cannot be selected beforehand, measurement period can be adjusted to approach the optimal one utilizing the relation between the optimal period time and the parameter γ or Δ in terms of Fisher information.

The rest of the paper is organized as follows. In Section 2, we first describe the whole process. Then we show the evolution process model, the preparation of the initial state and measurement results after the evolution with parameterization. Section 3 investigates the estimation of γ and Δ with different projective measurement bases. In Section 4, there is a brief conclusion.

2. Spin-boson model, preparation of initial states, and parameterization process

First we state the whole process briefly. The whole progress of parameter estimation is simply depicted in Fig. 1. The spin-boson model has two parameters γ and Δ to be estimated. γ is loaded into the system during the evolution and Δ is an original parameter of the model. The whole evolution is a recycling process with the period time in addition to the quantum parameter estimation process described in Section 1. The initial state in each period is prepared by projective measurement in last period. The projective measurement operators are denoted as M_+ and M_-. Two possible initial states marked as $\rho_+(0)$ and $\rho_-(0)$ corresponding to M_+ and M_-, respectively, can be prepared by the two different measurement operators. Then the initial state evolves for the time T and the corresponding density matrix becomes $\rho_+(T)$ (or $\rho_-(T)$) corresponding to $\rho_+(0)$ (or $\rho_-(0)$) after the evolution. Making measurement on $\rho_+(T)$ to get two different states corresponding to $\{M_+, M_-\}$ and these two states are regarded as initial states participating in the next evolution period. The situation of $\rho_-(T)$ is analogous to the one of $\rho_+(T)$. It should be noted that the projective measurement is instantaneous so that the period time is the same as the evolution time T and we call T the period time later. In summary, four measurement results are generated during the whole evolution process in terms of two possible initial states and two measurement results corresponding to each initial state. We denote the probabilities as $P(+|+)$, $P(-|+)$, $P(+|-)$, and $P(-|-)$ corresponding to these four results, respectively. The parameters are inferred with these probabilities. The details of each stage for the estimation process are described as follows.

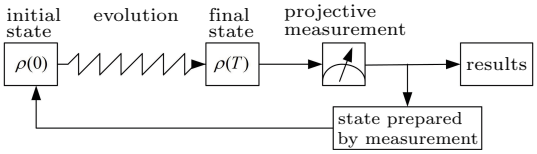


Fig. 1. The whole process of parameter estimation by periodical projective measurements. The states prepared by projective measurements in previous period are regarded as the initial states in the next period.

2.1. Evolution process model for the parameters to be estimated

We consider a model of a two-state system coupling to a bosonic heat bath.[36] The dynamics of this model can be described by spin-boson master equation[12,37] as follows:

$$\frac{\partial \rho(t)}{\partial t} = -\frac{\mathrm{i}}{\hbar}[H_R, \rho] - \frac{1}{\hbar}\tilde{D}(t)[\sigma_z, [\sigma_z, \rho]] - \frac{\mathrm{i}}{\hbar}z^*(t)\sigma_x\rho(t) + \frac{\mathrm{i}}{\hbar}z(t)\rho(t)\sigma_x - \frac{1}{\hbar}z(t)\sigma_z\rho\sigma_y - \frac{1}{\hbar}z^*(t)\sigma_y\rho\sigma_z, \quad (3)$$

where $\sigma_\alpha(\alpha \in \{x, y, z\})$ is the Pauli matrix, and $H_R = \hbar\Delta\sigma_x/2$ is the Hamiltonian of the spin qubit. In Eq. (3)

$$\tilde{D}(t) = \int_0^t \mathrm{d}s\nu(s)\cos\Delta s,$$
$$z(t) = \int_0^t \mathrm{d}s\,(\nu(s) + \mathrm{i}\eta(s))\sin\Delta s \quad (4)$$

are coefficients of dephasing and of system decay, respectively. In Eq. (4), the bare frequency Δ denotes the tunneling amplitude between two states of the spin system, the noise kernel $\nu(t)$ and the dissipation kernel $\eta(t)$ can be respectively expressed as

$$\nu(t) = \int_0^\infty \mathrm{d}\omega J(\omega)\cos\omega t(1 + 2N(\omega)),$$
$$\eta(t) = \int_0^\infty \mathrm{d}\omega J(\omega)\sin\omega t, \quad (5)$$

where $J(\omega)$ is spectral density of the environment and $N(\omega)$ is the mean occupation number of the environment oscillators.

We focus on the dephasing effect neglecting the decay influenced by the environment in the system, and assumed $\tilde{D}(t)=\hbar\gamma/4>0$ and $z(t)=0$. The spin-boson master equation is reduced as follows:

$$\frac{\partial\rho(t)}{\partial t}=-\frac{\mathrm{i}}{\hbar}[\hat{H},\rho]+\frac{\gamma}{2}(\sigma_z\rho\sigma_z-\rho), \tag{6}$$

where $\gamma>0$ to be estimated is the dephasing rate characterizing the daphasing effect influenced by the environment, and σ_z is the Pauli-Z matrix. The control Hamiltonian $\hat{H}$ in Eq. (6) is the original Hamiltonian $\hat{H}_1=\hbar\Delta\sigma_x/4$ (for calculating conveniently later in the paper, we modify the H_R in Eq. (3) by a coefficient 1/2 to get H_1) modulated by control fields $\hat{H}_2=\hbar[\omega_0(t)\sigma_x+\omega_1(t)\sigma_y+\omega_2(t)\sigma_z]/2$ and can be generally described as a composition of Pauli matrices

$$\begin{aligned}\hat{H}&=\hat{H}_1+\hat{H}_2\\&=\frac{\hbar}{2}\left[\left(\omega_0(t)+\frac{\Delta}{2}\right)\sigma_x+\omega_1(t)\sigma_y+\omega_2(t)\sigma_z\right].\end{aligned} \tag{7}$$

The description of Eq. (6) may occur in some quantum systems. For example, a controlled superconducting qubit stays in a superposition without decay between the two energy levels, while the relative phase of the two energy levels may shift due to the effect of the environment. It means that the information of the qubit is lost and the dephasing may lead to phase flip.

2.2. Preparation of initial states by projective measurements

We consider projective measurements on two-dimensional quantum systems as follows:

$$M_+=|\psi\rangle\langle\psi|,\ \ M_-=|\psi^\perp\rangle\langle\psi^\perp|, \tag{8}$$

where $|\psi\rangle=\cos(\theta/2)|0\rangle+\mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2)|1\rangle$ and $|\psi^\perp\rangle=-\mathrm{e}^{-\mathrm{i}\phi}\sin(\theta/2)|0\rangle+\cos(\theta/2)|1\rangle$. The indexes $+$ and $-$ in Eq. (8) refer to measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ prior to the measurement, then the probabilities that results $+$ and $-$ occur are given by

$$p_+=\mathrm{tr}(M_+\rho M_+^\dagger),\ \ p_-=\mathrm{tr}(M_-\rho M_-^\dagger), \tag{9}$$

and the corresponding states of system after measurement will become

$$\frac{M_+\rho M_+^\dagger}{\mathrm{tr}(M_+\rho M_+^\dagger)},\ \ \frac{M_-\rho M_-^\dagger}{\mathrm{tr}(M_-\rho M_-^\dagger)}. \tag{10}$$

The measurement operators satisfy the completeness equation

$$M_+^\dagger M_++M_-^\dagger M_-=\boldsymbol{I}, \tag{11}$$

where $\boldsymbol{I}$ is the identity operator.

We denote

$$\rho_+(0)=M_+,\ \ \rho_-(0)=M_-, \tag{12}$$

where $\rho_\pm(0)$ are the density matrix corresponding to the different initial states prepared by $M_\pm$, respectively.

2.3. Parameterization

For the sake of simplicity, we first parameterize ρ by using the Bloch vector $\boldsymbol{v}$ with the real components

$$v_\alpha=\mathrm{tr}(\rho\sigma_\alpha). \tag{13}$$

We then have

$$\rho=\frac{1}{2}(I+\boldsymbol{v}\cdot\boldsymbol{\sigma}). \tag{14}$$

Substituting Eq. (14) and Eq. (7) into Eq. (6), equation (6) can be reduced to the following Bloch equation (For simplicity, we further set $\hbar=1$).

$$\begin{pmatrix}\dot{v}_x\\ \dot{v}_y\\ \dot{v}_z\end{pmatrix}=\begin{pmatrix}-\gamma & -\omega_2(t) & \omega_1(t)\\ \omega_2(t) & -\gamma & -(\omega_0(t)+\Delta/2)\\ -\omega_1(t) & \omega_0(t)+\Delta/2 & 0\end{pmatrix}\begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix}=B\boldsymbol{v}(0), \tag{15}$$

where B is the transition matrix, $\boldsymbol{v}(0)=\boldsymbol{v}_+(0)$ or $\boldsymbol{v}(0)=\boldsymbol{v}_-(0)$, and $\boldsymbol{v}_\pm(0)$ will be shown in Eq. (18).

Our goal is to explore how to estimate the two parameters γ and Δ with control. For simplicity, we adopt the simple constant control below:

$$\omega_0(t)=\omega_0,\ \ \omega_1(t)\equiv 0,\ \ \omega_2(t)\equiv 0. \tag{16}$$

At this time,

$$B=\begin{pmatrix}-\gamma & 0 & 0\\ 0 & -\gamma & -W/2\\ 0 & W/2 & 0\end{pmatrix}, \tag{17}$$

where we denote $W=2\omega_0+\Delta$.

After projective measurement, the initial state in each period can be presented with Bloch vector corresponding $\rho_\pm(0)=M_\pm$ with Eq. (13) as follows:

$$\boldsymbol{v}_+(0)=\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix},\ \ \boldsymbol{v}_-(0)=-\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}, \tag{18}$$

where the index "+" and "−" represent the state prepared by M_+ or M_-, respectively.

The $\rho_\pm(0)$ evolves for a period time T, it means that the

Bloch vector $\boldsymbol{v}(0)$ evolves for T, thus we have

$$\boldsymbol{v}(T) = \mathrm{e}^{BT}\boldsymbol{v}(0) = \begin{pmatrix} v_x(T) \\ v_y(T) \\ v_z(T) \end{pmatrix}, \tag{19}$$

where $\boldsymbol{v}(0) = \boldsymbol{v}_{\pm}(0)$. With Eq. (14), we have

$$\rho(T) = \frac{1}{2}\begin{pmatrix} 1+v_z(T) & v_x(T)-\mathrm{i}v_y(T) \\ v_x(T)+\mathrm{i}v_y(T) & 1-v_z(T) \end{pmatrix}. \tag{20}$$

After the evolution, projective measurements are made to gain the measurement results and to prepare the initial state for the next evolution period. Considering different forms of the initial states described by $\boldsymbol{v}_+(0)$ and $\boldsymbol{v}_-(0)$, there are four combination for the probabilities of the corresponding results obtained by the operators M_+ and M_- for the density matrix ρ. They are expressed as

$$\begin{aligned} P(+|+) &= \mathrm{tr}(M_+\rho_+(T)M_+^{\dagger}), \\ P(-|+) &= \mathrm{tr}(M_-\rho_+(T)M_-^{\dagger}), \end{aligned} \tag{21}$$

$$\begin{aligned} P(+|-) &= \mathrm{tr}(M_+\rho_-(T)M_+^{\dagger}), \\ P(-|-) &= \mathrm{tr}(M_-\rho_-(T)M_-^{\dagger}), \end{aligned} \tag{22}$$

where $\rho_{\pm}(T)$ represent $\rho_{\pm}(0) = M_{\pm}$ evolving for time T.

3. Estimation of unknown parameters

We present how to estimate γ and Δ from the model built in Section 2. We give the conclusion of this part firstly:

(i) The parameter γ can be always estimated with periodical projective measurements by choosing measurement bases where $\theta = \pi/2$ and $\phi = 0$.

(ii) To estimate Δ, $\theta = \pi/4$ and $\phi = \pi/2$, the estimated value γ and the *a priori* information about Δ are necessary.

Supposing the *a priori* information of Δ is given as $\Delta \in [\bar{\Delta} - \delta\Delta, \bar{\Delta} + \delta\Delta]$. There are two cases under this condition, namely: i) γ satisfies $\bar{\Delta} - \delta\Delta \leqslant \gamma \leqslant \bar{\Delta} + \delta\Delta$; ii) γ is out of $[\bar{\Delta} - \delta\Delta, \bar{\Delta} + \delta\Delta]$. In case i), the control is necessary to estimate Δ. In case ii), Δ can be estimated without control. In this paper, we mainly focus on the case i).

3.1. Estimation of γ and Δ with different measurement bases

In case i) where $\bar{\Delta} - \delta\Delta \leqslant \gamma \leqslant \bar{\Delta} + \delta\Delta$, we can adjust the ratio $\eta = \gamma/W = \gamma/(2\omega_0 + \Delta)$ by changing the control value ω_0 to drive the system to $\eta < 1$. Thus, the system becomes underdamped with the help of control. The estimation process is presented as follows.

With $\eta < 1$ and Eq. (15),

$$\mathrm{e}^{BT} = \begin{pmatrix} \mathrm{e}^{-\gamma T} & 0 & 0 \\ 0 & \mathrm{e}^{-\gamma T/2}[\cos(sT) - (\gamma/2s)\sin(sT)] & \mathrm{e}^{-\gamma T/2}(\Delta/2s)\sin(sT) \\ 0 & -\mathrm{e}^{-\gamma T/2}(\Delta/2s)\sin(sT) & \mathrm{e}^{-\gamma T/2}[\cos(sT) + (\gamma/2\omega)\sin(sT)] \end{pmatrix}, \tag{23}$$

where

$$s = \frac{1}{2}\sqrt{(W^2 - \gamma^2)} = \frac{1}{2}\sqrt{(2\omega_0 + \Delta)^2 - \gamma^2}, \tag{24}$$

and $\boldsymbol{v}(0) = \boldsymbol{v}_+(0)$ or $\boldsymbol{v}(0) = \boldsymbol{v}_-(0)$.

$$\boldsymbol{v}_+(T) = \mathrm{e}^{BT}\boldsymbol{v}_+(0) = \begin{pmatrix} \mathrm{e}^{-\gamma T}\sin\theta\cos\phi \\ \mathrm{e}^{-\gamma T/2}[\cos(sT) - (\gamma/2s)\sin(sT)]\sin\theta\sin\phi + \mathrm{e}^{-\gamma T/2}(\Delta/2s)\sin(sT)\cos\theta \\ \mathrm{e}^{-\gamma T/2}[\cos(sT) + (\gamma/2s)\sin(sT)]\cos\theta - \mathrm{e}^{-\gamma T/2}(\Delta/2s)\sin(sT)\sin\theta\sin\phi \end{pmatrix}, \tag{25}$$

with Eq. (18), we have

$$\boldsymbol{v}_-(T) = \mathrm{e}^{BT}\boldsymbol{v}_-(0) = -\mathrm{e}^{BT}\boldsymbol{v}_+(0) = -\boldsymbol{v}_+(T). \tag{26}$$

With Eq. (21), we obtain

$$\begin{aligned} P(+|+) &= \frac{1}{2}\Big\{1 + \mathrm{e}^{-\gamma T}\sin^2\theta\cos^2\phi + \mathrm{e}^{-\gamma T/2}\Big[\cos(sT) - \frac{\gamma}{2s}\sin(sT)\Big]\sin^2\theta\sin^2\phi + \mathrm{e}^{-\gamma T/2}\frac{\Delta}{2s}\sin(sT)\cos\theta\sin\theta\sin\phi \\ &\quad + \mathrm{e}^{-\gamma T/2}\Big[\cos(sT) + \frac{\gamma}{2s}\sin(sT)\Big]\cos^2\theta - \mathrm{e}^{-\gamma T/2}\frac{\Delta}{2s}\sin(sT)\sin\theta\sin\phi\cos\theta\Big\} \\ &= \frac{1}{2}\Big[1 + \mathrm{e}^{-\gamma T}\sin^2\theta\cos^2\phi + \mathrm{e}^{-\gamma T/2}\cos(sT)(\sin^2\theta\sin^2\phi + \cos^2\theta) \\ &\quad + \mathrm{e}^{-\gamma T/2}\frac{\gamma}{2s}\sin(sT)(-\sin^2\theta\sin^2\phi + \cos^2\theta)\Big], \end{aligned} \tag{27}$$

$$P(-|+) = \frac{1}{2}\Big\{1 - \mathrm{e}^{-\gamma T}\sin^2\theta\cos^2\phi - \mathrm{e}^{-\gamma T/2}\Big[\cos(sT) - \frac{\gamma}{2s}\sin(sT)\Big]\sin^2\theta\sin^2\phi - \mathrm{e}^{-\gamma T/2}\frac{\Delta}{2s}\sin(sT)\cos\theta\sin\theta\sin\phi$$

$$
\begin{aligned}
&- \mathrm{e}^{-\gamma T/2}\left[\cos(sT)+\frac{\gamma}{2s}\sin(sT)\right]\cos^2\theta + \mathrm{e}^{-\gamma T/2}\frac{\Delta}{2s}\sin(sT)\sin\theta\sin\phi\cos\theta\Big\}\\
&=\frac{1}{2}\Big[1-\mathrm{e}^{-\gamma T}\sin^2\theta\cos^2\phi-\mathrm{e}^{-\gamma T/2}\cos(sT)(\sin^2\theta\sin^2\phi+\cos^2\theta)\\
&\quad+\mathrm{e}^{-\gamma T/2}\frac{\gamma}{2s}\sin(sT)(\sin^2\theta\sin^2\phi-\cos^2\theta)\Big]. \qquad (28)
\end{aligned}
$$

We can testify

$$P(-|-)=P(+|+),\ \ P(+|-)=P(-|+). \qquad (29)$$

We can estimate γ and Δ in succession. In other words, Δ cannot be estimated without the estimation value of γ underlying the observation of Eqs. (27) and (28). To estimate the parameters γ and Δ in succession, we design the following two steps: firstly, to estimate γ, setting $\theta=\pi/2$ and $\phi=0$; secondly, to estimate Δ, setting $\theta=\pi/4$ and $\phi=\pi/2$ so that $\sin^2\theta\sin^2\phi=\cos^2\theta$. Here we set the period time T to be fixed, but T may be different for estimation of the two parameters as Remark 2 shows.

In the first step, equations (27) and (28) will become

$$P(+|+)=\frac{1}{2}(1+\mathrm{e}^{-\gamma T}),$$

$$P(-|+)=\frac{1}{2}(1-\mathrm{e}^{-\gamma T}), \qquad (30)$$

thus,

$$\gamma=-\frac{1}{T}\ln(P(+|+)-P(-|+)). \qquad (31)$$

In the second step, equations (27) and (28) will become

$$
\begin{aligned}
P'(+|+)&=\frac{1}{2}[1+\mathrm{e}^{-\gamma T/2}\cos(sT)],\\
P'(-|+)&=\frac{1}{2}[1-\mathrm{e}^{-\gamma T/2}\cos(sT)], \qquad (32)
\end{aligned}
$$

where

$$s=\frac{1}{T}\arccos[\mathrm{e}^{\gamma T/2}(P'(+|+)-P'(-|+))+2k_0^\star\pi]. \qquad (33)$$

Here $k_0^\star$ may be expressed as

$$k_0^\star=\mathrm{flr}\left(\frac{1}{2}\frac{\sqrt{(2\omega_0+\bar{\Delta}-\delta\Delta)^2-\gamma^2}}{\sqrt{(2\omega_0+\bar{\Delta}+\delta\Delta)^2-\gamma^2}-\sqrt{(2\omega_0+\bar{\Delta}-\delta\Delta)^2-\gamma^2}}\right), \qquad (34)$$

where the flr function is a function that takes as input a real number x and gives an output the greatest integer $\mathrm{flr}(x)$ that is less than or equal to x. The process to determine $k_0^\star$ is shown in Appendix A.

With Eq. (24) and $\Delta>0$, we may obtain

$$\Delta=\sqrt{4s^2+\gamma^2}-2\omega_0, \qquad (35)$$

where γ is shown in Eq. (31), ω_0 is the control value we can choose and s is determined by Eq. (33).

3.2. Estimation γ and Δ from measurement results

Firstly we set $\theta=\pi/2$ and $\phi=0$ to estimate γ. Measurements are made to get a sequence of results. We denote N_1 as the number of the same results appearing adjacently and N_2 as the number of the different results appearing adjacently. That's to say, we suppose the sequence of the results is $\{R_i|R_i\in\{+,-\},\ i=1,\ldots,N\}$, where N is the times of experiment, then we observe and count the results. If $R_{j-1}R_j\in\{++,--\}$ $(j=2,\ldots,N)$ appears, N_1 increases by 1; if $R_{j-1}R_j\in\{+-,-+\}$ appears, N_2 increases by 1. $N_1+N_2=N-1$. When N is a large number,

$$
\begin{aligned}
\frac{N_1-N_2}{N-1}&=P(+|+)+P(-|-)-P(-|+)-P(+|-)\\
&=\mathrm{e}^{-\hat{\gamma}T}. \qquad (36)
\end{aligned}
$$

Thus,

$$\hat{\gamma}=-\frac{1}{T}\ln\frac{N_1-N_2}{N-1}. \qquad (37)$$

To estimate Δ, we suppose that we know the *a priori* information as $\Delta\in[\bar{\Delta}-\delta\Delta,\bar{\Delta}+\delta\Delta]$. We set $\theta=\pi/4$ and $\phi=\pi/2$. Again, we denote N_1' as the number of the same results appearing adjacently and N_2' as the number of the different results appearing adjacently. We suppose the sequence of the results is $\{R_i\in\{+,-\},i=1,\ldots,N'\}$, then we observe and count the results. If $R_{j-1}R_j\in\{++,--\}$ $(j=2,\ldots,N')$ appears, N_1' increases by 1; if $R_{j-1}R_j\in\{+-,-+\}$ appears, N_2' increases by 1. $N_1'+N_2'=N'-1$. When N' is a large number, $\hat{\Delta}=\sqrt{4s^2+\hat{\gamma}^2}-2\omega_0$ with Eq. (35), where ω_0 is the control value,

$$s=\frac{1}{T}\arccos\left[\mathrm{e}^{\hat{\gamma}T/2}\left(\frac{N_1'-N_2'}{N'-1}\right)+2k_0^\star\pi\right], \qquad (38)$$

and $k_0^\star$ is determined by Eq. (34).

Remark 1 It is known that $\Delta\in[\bar{\Delta}-\delta\Delta,\bar{\Delta}+\delta\Delta]$, we denote $\Delta_{\min}=\bar{\Delta}-\delta\Delta$ and $\Delta_{\max}=\bar{\Delta}+\delta\Delta$. In case ii), γ is out of $[\bar{\Delta}-\delta\Delta,\bar{\Delta}+\delta\Delta]$. It means that $\gamma/\Delta_{\min}<1$ or $\gamma/\Delta_{\max}>1$. If we set the control $\omega_0=0$ in Eq. (24), we have $s^2>0$ or $s^2<0$ with respect to $\gamma/\Delta_{\min}<1$ or $\gamma/\Delta_{\max}>1$. This means that equation (23) will become underdamped or overdamped, and we can testify that Δ can be estimated without control in both cases.

Remark 2 It is an important problem to choose optimal period time to make the parameters achieve the highest estimation precision. For different parameter estimation process, the optimal time may be different. Here we set the period time as T_1 and T_2 estimating γ and Δ, respectively. We explore the problem by means of Fisher information[11] of the parameters with Eq. (2):

$$F_\gamma = \frac{T_1^2 \mathrm{e}^{-2\gamma T_1}}{1-\mathrm{e}^{-2\gamma T_1}}, \tag{39}$$

$$F_s = \frac{T_2^2 \mathrm{e}^{-\gamma T_2}[\sin(sT_2)]^2}{1-\mathrm{e}^{-\gamma T_2}[\cos(sT_2)]^2}. \tag{40}$$

We can obtain the relation between each parameter and the corresponding optimal period time as follows when $\mathrm{d}F_\gamma/\mathrm{d}T_1 = 0$ and $\mathrm{d}F_s/\mathrm{d}T_2 = 0$, respectively:

$$1-\gamma T_1 = \mathrm{e}^{-2\gamma T_1}, \tag{41}$$

$$(2-\gamma T_2)\sin(sT_2) + 2sT_2\cos(sT_2) = 0. \tag{42}$$

The details to obtain Eqs. (39)–(42) and some discussions are shown in Appendix B.

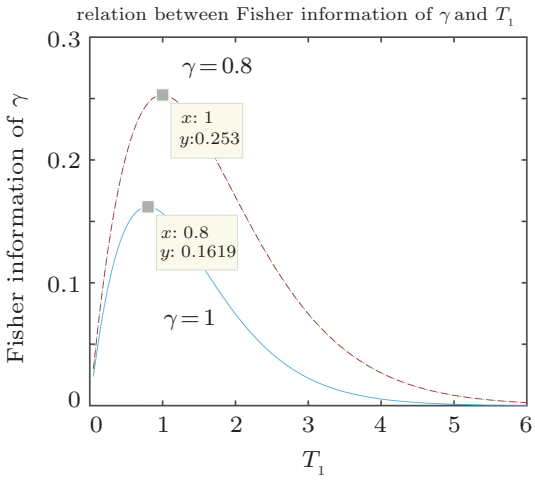


Fig. 2. (color online) Relation between T_1 and Fisher information of γ.

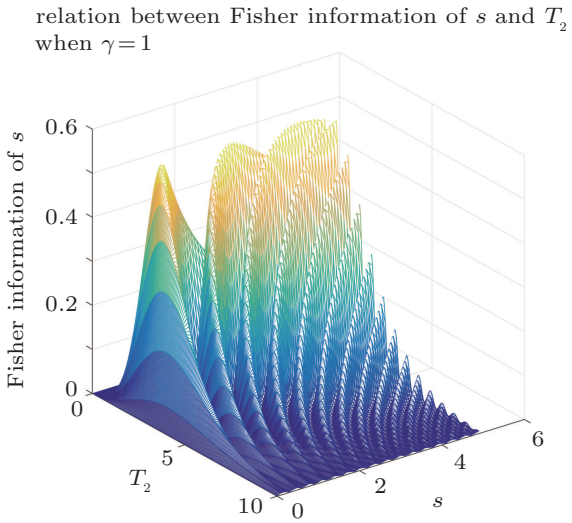


Fig. 3. (color online) Relation between T_2 and Fisher information of s when $\gamma = 1$.

Figures 2 and 3 depict the relation between the Fisher information of each parameter to be estimated and the corresponding optimal period time. Figure 4 shows the solution of Eq. (41) for $\mathrm{d}F_\gamma/\mathrm{d}T_1 = 0$. From these relations, we conclude that we cannot select the optimal time of either γ or Δ before each estimation, as the period time is coupled to the parameter to be estimated. However, we can adjust the period time to help approach the optimal precision of each parameter to be estimated based on the relations.

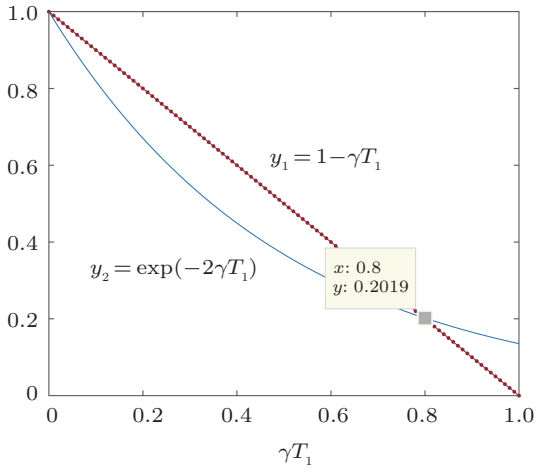


Fig. 4. (color online) Solution of Eq. (41) for $\mathrm{d}F_\gamma/\mathrm{d}T_1 = 0$.

4. Conclusion

In summary, we study how to estimate two parameters in a spin-boson dephasing system by periodical projective measurements. The essential to solve the problem is the possibility of the same or different results appearing adjacently during the whole parameter estimation process. The periodical projective measurements in a period prepare the initial state used in the next period. The period time is coupled to the parameter to be estimated and affects the estimation precision as the Remark 2 shows. Different measurement bases help to simplify the expression of the possibilities, from which the parameters are estimated. The phase damping parameter γ can be always estimated as long as the measurement parameters satisfy $\theta = \pi/2$ and $\phi = 0$. The parameter Δ can be estimated by selecting the measurement bases satisfying $\theta = \pi/4$ and $\phi = \pi/2$ when γ and the interval of Δ are known. If γ is in the interval of Δ, the control is essential to estimate Δ. If γ is not in the interval of Δ, the control is not necessary. We also gain the relation between the optimal time and the parameter to be estimated based on Fisher information, respectively for γ and Δ. As each period time is coupled to the corresponding parameter to be estimated, we cannot select the optimal time for either γ or Δ before the estimation. However, the period time can be adjusted to approach the optimal one with the relations in Eqs. (41)–(42) corresponding to γ and Δ.

The projective measurements are widely used in quantum system,[20] such as the polarized light detection in optical system, projective measurement of spin in atomic system, projective measurement on a superconducting qubit in a superconducting system and so on. Based on the projective measurements in a ceratin system, we may choose a proper period time to implement periodical projective measurements, recycling the states prepared by the projective measurements.

Appendix A: Determining $k_0^\star$ from the *a priori* information of Δ

To estimate Δ, the *a priori* information is necessary. Supposing we know the *a priori* information of Δ as

$$\Delta \in \left[\bar{\Delta} - \delta\Delta, \bar{\Delta} + \delta\Delta\right], \tag{A1}$$

In case i), $\gamma \in \left[\bar{\Delta} - \delta\Delta, \bar{\Delta} + \delta\Delta\right]$. The process to determine $k_0^\star$ in Eq. (34) is as follows.

Supposing $s \in [\bar{s} - \delta s, \bar{s} + \delta s]$, and

$$\begin{cases} \bar{s}T = 2k_0\pi + \dfrac{\pi}{2}, \\ \delta s T \leqslant \dfrac{\pi}{2}, \end{cases} \tag{A2}$$

where k_0 is a defined integer.

From Eq. (A2), we have

$$k_0 \leqslant \frac{\bar{s} - \delta s}{4\delta s}.$$

We choose

$$k_0^\star = \mathrm{flr}\left(\frac{\bar{s} - \delta s}{4\delta s}\right), \tag{A3}$$

where the flr function is a function that takes as input a real number x and gives an output the greatest integer $\mathrm{flr}(x)$ that is less than or equal to x. Below we attach the formula of $k_0^\star$ to the *a priori* information of Δ in Eq. (A1). $s = \sqrt{(2\omega_0 + \Delta)^2 - \gamma^2}/2 > 0$, we obtain

$$\begin{aligned} s_{\min} &= \sqrt{(2\omega_0 + \bar{\Delta} - \delta\Delta)^2 - \gamma^2}/2 = \bar{s} - \delta s, \\ s_{\max} &= \sqrt{(2\omega_0 + \bar{\Delta} + \delta\Delta)^2 - \gamma^2}/2 = \bar{s} + \delta s, \\ \delta s &= (s_{\max} - s_{\min})/2. \end{aligned}$$

Thus,

$$k_0^\star = \mathrm{flr}\left(\frac{\bar{s} - \delta s}{4\delta s}\right) = \mathrm{flr}\left(\frac{1}{2}\frac{\sqrt{(2\omega_0 + \bar{\Delta} - \delta\Delta)^2 - \gamma^2}}{\sqrt{(2\omega_0 + \bar{\Delta} + \delta\Delta)^2 - \gamma^2} - \sqrt{(2\omega_0 + \bar{\Delta} - \delta\Delta)^2 - \gamma^2}}\right).$$

Appendix B: Relation of each optimal evolution time and the corresponding parameter to be estimated

We explore the problem based on Fisher information of each parameter to be estimated with Eq. (2). The process is as follows:

$$F_\gamma = \frac{[\partial_\gamma P(+|+)]^2}{P(+|+)} + \frac{[\partial_\gamma P(-|+)]^2}{P(-|+)} = \frac{[\partial_\gamma P(-|-)]^2}{P(-|-)} + \frac{[\partial_\gamma P(+|-)]^2}{P(+|-)} = \frac{T_1^2\,\mathrm{e}^{-2\gamma T_1}}{1 - e^{-2\gamma T_1}}.$$

As γ is regarded as a constant to be estimated, we calculate the derivative of F_γ with respect to T_1 and equate the derivative with 0, then we have

$$\begin{aligned} &\frac{\mathrm{d}F_\gamma}{\mathrm{d}T_1} = \frac{2T_1\,\mathrm{e}^{-2\gamma T_1}(1 - \mathrm{e}^{-2\gamma T_1} - \gamma T_1)}{(1 - \mathrm{e}^{-2\gamma T_1})^2}, \\ &1 - \gamma T_1 = \mathrm{e}^{-2\gamma T_1}. \end{aligned} \tag{B1}$$

We then regard γT_1 as a whole variable, and find the solution of Eq. (41).

Figure 3 shows that $\gamma T_1 \approx 0.8$ is the solution of $\mathrm{d}F_\gamma/\mathrm{d}T_1 = 0$ and $\gamma T_1 \approx 0.8$ matches the maximum points emerging in Fig. 1 where $T_1 = 1$ if $\gamma = 0.8$ and $T_1 = 0.8$ if $\gamma = 1$. It means that there exists an optimal period time T_1 for γ. However, as the γ is to be estimated, we cannot select the optimal time T_1 exactly to make the Fisher information of γ achieve the highest precision before estimation.

For the parameter Δ to be estimated, there also exists the problem for selecting optimal period time T_2 and the calculation process is analogue to the one of estimating γ. The Fisher information about Δ can be calculated directly with Eq. (2). We prefer to calculate the Fisher information about the intermediate variable s rather than the one about Δ because sT_2 appears as a whole in the trigonometric function in Eq. (40). It is convenient for us to observe how to select T_2 in Eq. (42). The Fisher information about Δ is more complex and it is not so easy to gain a right conclusion.

$$F_s = \frac{[\partial_s P(+|+)]^2}{P(+|+)} + \frac{[\partial_s P(-|+)]^2}{P(-|+)} = \frac{[\partial_s P(-|-)]^2}{P(-|-)} + \frac{[\partial_s P(+|-)]^2}{P(+|-)} = \frac{T_2^2\,\mathrm{e}^{-\gamma T_2}[\sin(sT_2)]^2}{1 - \mathrm{e}^{-\gamma T_2}[\cos(sT_2)]^2}.$$

The relation between T_2 and Fisher information of s for $\gamma = 1$ is shown in Fig. 3.

$$\frac{\mathrm{d}F_s}{\mathrm{d}T_2} = \frac{T_2\,\mathrm{e}^{-\gamma T_2}\sin(sT_2)[(2 - \gamma T_2)\sin(sT_2) + 2sT_2\cos(sT_2)]}{\{1 - \mathrm{e}^{-\gamma T_2}[\cos(sT_2)]^2\}^2}. \tag{B2}$$

For a known γ, when the selected T_2 satisfies

$$(2-\gamma T_2)\sin(sT_2)+2sT_2\cos(sT_2)=0,$$

with a certain s to be estimated, the estimation precision of s is optimal so that the estimation precision of Δ is optimal with Eq. (35). The parameter s is still unknown so that the optimal T_2 cannot be selected before estimation. However, equation (42) is helpful to approach the optimal precision of s by adjusting T_2 and thus the optimal precision of Δ is approached with Eq. (35).

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