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Comparative investigation of freezing phenomena for quantum coherence and correlations*

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We show that the freezing phenomenon, exhibited by a specific class of two-qubit state under local nondissipative decoherent evolutions, is a common feature of the relative entropy measure of quantum coherence and correlation. All those measurement outcomes, preserve a constant value in the considered noisy channels, but the condition, property and mechanism of the freezing phenomenon for quantum coherence are different from those of the quantum correlation.

Keywords: freezing phenomenon, quantum coherence, quantum discord

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1. Introduction

Quantum coherence arising from quantum superposition is a fundamental feature of quantum mechanics. It has been widely used as a resource and root concept in quantum information processing,[1] quantum metrology,[2–5] entanglement creation,[6,7] thermodynamics,[8–12] and quantum biology.[13–16] Recently, a rigorous theory of coherence as a physical resource has been developed.[17–19] Within such a physical resource frame of coherence, some coherence measures have been proposed and investigated, such as the l_1 norm of coherence and the relative entropy of coherence.[17]

Quantum coherence in a multipartite system involves the essence of quantum correlations. One of the potential quantum correlations is quantum discord, which may even exist in a separable state with vanished entanglement.[20–24] Quantum discord also is a crucial resource for the development of quantum technologies, such as quantum communication,[25,26] quantum computation,[27,28] etc.

Quantum coherence and discord are both useful physical resources, but the quantum coherence and discord of a quantum state are often destroyed by the noise of the environment. A challenge in exploiting these resources is to protect them from decoherence caused by noise. One of the most fascinating phenomena observed in the dynamics of quantum coherence and discord is the possibility for its freezing, that is, complete time invariance without any external control, in the presence of particular initial states and noisy evolutions. In fact numerous studies have focused on the freezing phenomenon for quantum discord and coherence. Under the local nondissipative decoherence evolution, it has been observed that a number of known discord-type measures all remain constant for a finite time interval in Markovian conditions[29–33] and for multiple intervals,[34–36] or forever[37] in non-Markovian conditions, when considering two non-interacting qubits initially in a specific class of Bell-diagonal (BD) states. Recently, the quantum coherence can also remain frozen under local nondissipative decoherence channels, for the particular initial BD states have been studied.[38–41]

These freezing phenomena are quite appealing since it implies that every protocol relying on discord or coherence as a resource will run with a performance unaffected by noise in the specific dynamical conditions. Currently, these investigations only prove the occurrence of freezing under some particular condition by considering the specific decoherence channel. However, it is natural to ask what properties these freezing phenomena have and why these freezing phenomena can occur. This work addresses such issues. Furthermore, we know that the quantum coherence and correlations have intimate relations between them. Many efforts have been devoted to the investigation of the connections between quantum coherence and quantum correlations in multipartite systems,[42–46] Through comparative investigation of the freezing phenomena for quantum coherence and discord we can better understand the relations between them.

As is well known, the different measures of quantum resource are not identical and conceptually different. Different quantum resources do not coincide with each other for different measures, and a direct comparison of two notions is rather meaningless. So we only focus on the measures based on the relative entropy, which enjoys the properties of physical interpretation and being easily computed.

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The rest of the paper is organized as follows. In Section 2, we review the definitions of the relative entropy of coherence and discord, and some paradigmatic instances of incoherent channels. In Section 3, we derive the conditions for freezing the quantum coherence and analyze the properties and reasons for the frozen quantum coherence. In Section 4, we derive the conditions for freezing the quantum discord and analyze the properties and reasons for the frozen quantum discord. In Section 5 we compare these freezing phenomena and draw some conclusions from the study in this paper.

2. Preliminaries

The relative entropy between two quantum states ρ and σ is defined as

$$S(\rho||\sigma) \equiv -\mathrm{tr}(\rho\log\sigma) - S(\rho), \tag{1}$$

where $S(\rho) = -\mathrm{tr}(\rho\log\rho)$ is the von Neumann entropy of ρ.[1] The relative entropy is a non-negative quantity, and due to this property it often appears in the context of distance measure though technically it is not symmetric. The relative entropy of discord was first introduced by Modi *et al.*[23] and the relative entropy of coherence was defined in Ref. [17].

Given any quantum state ρ, one can list the relative entropy of discord D and the relative entropy of coherence C as follows:

$$D(\rho) = \min_{\sigma\in J} S(\rho||\sigma), \tag{2}$$

$$C(\rho) = \min_{\sigma\in\tau} S(\rho||\sigma), \tag{3}$$

where J and τ denote a set of classical states and a set of incoherent states, respectively. Let χ and ρ^d be the closest classical state and the closest incoherent state to the quantum state ρ, respectively. The relative entropy of discord and coherence can be rewritten as follows:

$$D(\rho) = \min_{\sigma\in J} S(\rho||\sigma) = S(\rho||\chi) = S(\chi) - S(\rho), \tag{4}$$

$$C(\rho) = \min_{\sigma\in\tau} S(\rho||\sigma) = S\left(\rho||\rho^d\right) = S\left(\rho^d\right) - S(\rho). \tag{5}$$

The BD states are structurally simple states which nonetheless remain of high relevance to theoretical and experimental research in quantum information, as they include the well-known Bell and Werner states.[47] Usually, the BD states have two forms. The first form of the BD states is described by diagonal elements in the basis of the four maximally entangled Bell states, that is,

$$\rho = \sum_{i=1}^{4} \lambda_i |\psi_i\rangle\langle\psi_i|, \tag{6}$$

where the $|\psi_i\rangle$'s are the four Bell states $|\Phi^{\pm}\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\Psi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, with conditions $\lambda_i \geq 0$ and $\sum_{i=1}^{4} \lambda_i = 1$. The second form of the BD states is described by the Bloch representation in the computational basis, which has density operators in the form of

$$\rho = \frac{1}{4}\left(I\otimes I + \sum_{i=1}^{3} c_i\sigma_i\otimes\sigma_i\right), \tag{7}$$

where the σ_i's are Pauli operators and $0 \leq |c_i| \leq 1$, I is the identity operator, which are also described by the triple $\{c_1, c_2, c_3\}$.

As is well known, for any quantum state ρ, the closest incoherent state ρ^d is the diagonal version of ρ, which only retains the diagonal elements of ρ.[17] Therefore, for any BD states $\rho = \sum_{i=1}^{4} \lambda_i |\psi_i\rangle\langle\psi_i|$, the closest incoherent state is

$$\rho^d = \sum_{i=1}^{4} \lambda_i (|\psi_i\rangle\langle\psi_i|)^d, \tag{8}$$

where $(|\psi_i\rangle\langle\psi_i|)^d$ is the matrix containing only the leading diagonal elements of $|\psi_i\rangle\langle\psi_i|$. For any particular BD states $\rho = \sum_{i=1}^{4} \lambda_i |\psi_i\rangle\langle\psi_i|$, where λ_i are ordered in nonincreasing size, i.e., $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. The closest classical state is given by

$$\begin{aligned}\chi \;=\; & \frac{q}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|) \\ & + \frac{1-q}{2}(|\psi_3\rangle\langle\psi_3| + |\psi_4\rangle\langle\psi_4|)\end{aligned} \tag{9}$$

with $q = \lambda_1 + \lambda_2$.[23]

We consider paradigmatic instances of incoherent channels which embody typical noise sources in quantum information processing. [1] The bit flip, bit-phase flip, and phase flip channels which are represented in Kraus form by $K_0^{F_i} = \sqrt{1-q/2}I$, $K_{j,k\neq i}^{F_i} = 0$, $K_i^{F_i} = \sqrt{q/2}\sigma_i$, with $i = 1$, $i = 2$, and $i = 3$, respectively, with I being the identity operator, σ_i the ith Pauli matrix, and $q = 1 - \mathrm{e}^{-\gamma t}$ encoding the strength of the noise. Here t represents time and γ the decoherence rate. The actions of two independent and identical local noisy channels on each qubit of a two-qubit system map the system state ρ into the evolved state $\rho(t)$. Local bit flip channels on each qubit map initial BD states with $\{c_1, c_2, c_3\}$ to BD states with $\{c_1, c_2\,\mathrm{e}^{-2\gamma t}, c_3\,\mathrm{e}^{-2\gamma t}\}$. Local bit-phase flip channels on each qubit map initial BD states with $\{c_1, c_2, c_3\}$ to BD states with $\{c_1\,\mathrm{e}^{-2\gamma t}, c_2, c_3\,\mathrm{e}^{-2\gamma t}\}$. Local phase flip channels on each qubit map initial BD states with $\{c_1, c_2, c_3\}$ to BD states with $\{c_1\,\mathrm{e}^{-2\gamma t}, c_2\,\mathrm{e}^{-2\gamma t}, c_3\}$.[32]

3. Freezing phenomenon for quantum coherence

Bromley *et al.* studied the conditions for frozen quantum coherence in Ref. [38]. They first showed the properties of the evolution of a quantum system under particular noisy channels, and then proved that all bona fide distance-based coherence monotones are left invariant during the entire evolution under certain conditions of the initial states. Conversely, in

this section, we first show that the relative entropy of coherence is determined only by one element of the BD states with $\{c_1, c_2, c_3\}$ under certain conditions, then we identify the conditions for the freezing of quantum coherence in terms of the properties of the evolution of a quantum system under particular noisy channels. Furthermore, we study the properties and reasons for the freezing phenomenon for quantum coherence.

For convenience, in the following, we define the density operators $\delta_{1,2} = |\Phi^{\pm}\rangle\langle\Phi^{\pm}|$ and $\delta_{3,4} = |\Psi^{\pm}\rangle\langle\Psi^{\pm}|$, where $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$ are the usual Bell basis. Then a BD state has the form $\rho = \sum_i \lambda_i \delta_i$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{4} \lambda_i = 1$. Comparing it with the second form of BD states,

$$\rho = \frac{1}{4}\left(\mathrm{I}\otimes\mathrm{I} + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i\right),$$

we obtain the following formulas:

$$\lambda_1 = \frac{1}{4}(1 + c_1 + c_3 - c_2), \quad \lambda_2 = \frac{1}{4}(1 + c_2 + c_3 - c_1),$$
$$\lambda_3 = \frac{1}{4}(1 + c_1 + c_2 - c_3), \quad \lambda_4 = \frac{1}{4}(1 - c_1 - c_3 - c_2). \quad (10)$$

According to formulas (5) and (8), we obtain the relative entropy of coherence as follows:

$$C(\rho) = 1 + \lambda_1 \log\frac{\lambda_1}{\lambda_1+\lambda_2} + \lambda_2 \log\frac{\lambda_2}{\lambda_1+\lambda_2} + \lambda_3 \log\frac{\lambda_3}{\lambda_3+\lambda_4} + \lambda_4 \log\frac{\lambda_4}{\lambda_3+\lambda_4}. \quad (11)$$

Analyzing formula (11) by considering formula (10) we can obtain the following results:

If $c_2 = -c_1 c_3$, then

$$C(\rho) = \frac{1}{2}[(1+c_1)\log(1+c_1) + (1-c_1)\log(1-c_1)]; \quad (12)$$

If $c_1 = -c_2 c_3$, then

$$C(\rho) = \frac{1}{2}[(1+c_2)\log(1+c_2) + (1-c_2)\log(1-c_2)]. \quad (13)$$

As we know, local bit flip channels on each qubit map initial BD states with $\{c_1(0), c_2(0), c_3(0)\}$ to BD states with $\{c_1(0), c_2(0)\,\mathrm{e}^{-2\gamma t}, c_3(0)\,\mathrm{e}^{-2\gamma t}\}$. Note that $c_1(0)$ is unaffected by the environment noise. If we choose the condition $c_2(0) = -c_1(0)c_3(0)$ for the initial BD states, we can see that the evolved BD states also satisfy the condition $c_2(t) = -c_1(t)c_3(t)$. According to formula (12), we know that the quantum coherence is frozen under local bit flip channels for the quantum system in BD states with initial condition $c_2(0) = -c_1(0)c_3(0)$, that is,

$$C(\rho(t)) = C(\rho(0)) = \frac{1}{2}H(c_1(0)), \quad (14)$$

where $H(x) = [(1+x)\log(1+x) + (1-x)\log(1-x)]$. It is easy to check whether $H(x)$ is a symmetric convex function when $|x| \leq 1$ and the symmetric point is zero. Due to the fact that the function $C(\rho(t))$ is a symmetric convex function and $|c_1(0)| \leq 1$, we can find when $|c_1(0)| \to 1$, the frozen quantum coherence approaches to a maximum value. Thus, the maximum value of the frozen quantum coherence is reached when $c_1(0) = \pm 1$ and $c_2(0) = \mp c_3(0)$, that is, $C(\rho(t)) = C(\rho(0)) = 1$. The corresponding initial BD states are

$$\rho = \frac{1}{2}[(1 + c_3(0))\delta_1 + (1 - c_3(0))\delta_3], \quad (15)$$

and

$$\rho = \frac{1}{2}[(1 + c_3(0))\delta_2 + (1 - c_3(0))\delta_4]. \quad (16)$$

Now, we analyze why the freezing phenomenon can occur. When the initial BD states with the condition $c_2(0) = -c_1(0)c_3(0)$, the evolved BD states can be written as

$$\rho(t) = \frac{1}{4}\begin{bmatrix} 1+c_3(t) & 0 & 0 & c_1(0)(1+c_3(t)) \\ 0 & 1-c_3(t) & c_1(0)(1-c_3(t)) & 0 \\ 0 & c_1(0)(1-c_3(t)) & 1-c_3(t) & 0 \\ c_1(0)(1+c_3(t)) & 0 & 0 & 1+c_3(t) \end{bmatrix}. \quad (17)$$

Considering the definition of the l_1 norm of quantum coherence $C(\rho) = \sum_{i\neq j}|\rho_{ij}|$,[17] we can directly obtain the reasons for the freezing of quantum coherence. When the quantum systems undergo local identical bit flip channels, as time goes on, the value of the element $|\rho_{14}(t)| = |c_1(0)|(1+c_3(t))$ becomes smaller, and the value of the other element $|\rho_{23}(t)| = |c_1(0)|(1-c_3(t))$ becomes larger, but the summation of these two elements is a constant, $|\rho_{14}(t)| + |\rho_{23}(t)| = 2|c_1(0)|$. That is to say, the l_1 norm of quantum coherence is frozen, $C(\rho(t)) = |c_1(0)|$. So we believe that the freezing phenomenon for quantum coherence is the coherence interchange in the quantum system by local identical bit flip channels.

We know that local bit-phase flip channels on each qubit map initial BD states with $\{c_1, c_2, c_3\}$ to BD states with $\{c_1\,\mathrm{e}^{-2\gamma t}, c_2, c_3\,\mathrm{e}^{-2\gamma t}\}$. Similarly, according to formula (13), we know that the quantum coherence is frozen under local bit-phase flip channels for the quantum system in BD states with initial condition $c_1(0) = -c_2(0)c_3(0)$. The formula

of the frozen quantum coherence is $C(\rho(t)) = C(\rho(0)) = \frac{1}{2}H(c_2(0))$. The properties and the reasons for the frozen quantum coherence are analogous to the above.

However, from formula (11) of the relative entropy of coherence, we cannot find a function of c_3 which is similar to function (12) of c_1 or the function (13) of c_2. That is to say, the freezing phenomenon for quantum coherence cannot occur when the quantum system undergoes local phase flip channels.

4. Freezing phenomenon for quantum discord

Cianciaruso *et al.* have studied the conditions for frozen quantum discord in Ref. [32]. They first showed the properties of the evolution of a quantum system interacting with a non-dissipative decohering environment, and then proved that the quantum discord measured by geometric approach remains constant during the evolution for a paradigmatic class of two-qubit state. In this section, we first show that the relative entropy of discord is determined only by one element of the BD states with $\{c_1, c_2, c_3\}$ under certain conditions, then we identify the conditions for freezing the quantum discord in terms of the properties of the evolution of a quantum system under particular noisy channels. Particularly, we obtain two conditions for freezing the quantum discord under certain noisy channels.

For calculating the relative entropy of discord, we list the ordering for all $\{\lambda_i\}$ in nonincreasing size for the BD states $\rho = \sum_i \lambda_i \delta_i$, i.e., $\lambda_{\max} \geq \lambda_{\sec} \geq \lambda_{thi} \geq \lambda_{\min}$. According to formulas (4) and (9), we obtain the formula of the relative entropy of discord as follows:

$$\begin{aligned} D(\rho) = {} & 1 + \sum_{i=1}^{4} \lambda_i \log \lambda_i - (\lambda_{\max} + \lambda_{\sec}) \log (\lambda_{\max} + \lambda_{\sec}) \\ & - (\lambda_{thi} + \lambda_{\min}) \log (\lambda_{thi} + \lambda_{\min}). \end{aligned} \tag{18}$$

Considering that the condition $\{\lambda_1, \lambda_2\} \geq \{\lambda_3, \lambda_4\}$ turns into $\{(1+c_1+c_3-c_2), (1+c_2+c_3-c_1)\} \geq \{(1+c_1+c_2-c_3), (1-c_1-c_3-c_2)\}$, after calculation, we obtain

$$-c_3 \leq \{c_1, c_2\} \leq c_3. \tag{19}$$

Considering that the condition $\{\lambda_1, \lambda_2\} \leq \{\lambda_3, \lambda_4\}$ turns into $\{(1+c_1+c_3-c_2), (1+c_2+c_3-c_1)\} \leq \{(1+c_1+c_2-c_3), (1-c_1-c_3-c_2)\}$, after calculation, we obtain

$$c_3 \leq \{c_1, c_2\} \leq -c_3. \tag{20}$$

Unifying these two conditions (19) and (20), we can see that the condition $\{\lambda_1, \lambda_2\} \geq \{\lambda_3, \lambda_4\}$ or $\{\lambda_1, \lambda_2\} \leq \{\lambda_3, \lambda_4\}$ becomes $\{|c_1|, |c_2|\} \leq |c_3|$ in the BD states

$$\rho = \frac{1}{4}\left(\mathrm{I} \otimes \mathrm{I} + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i\right).$$

Similarly, we can prove that the condition $\{\lambda_1, \lambda_3\} \geq \{\lambda_2, \lambda_4\}$ or $\{\lambda_1, \lambda_3\} \leq \{\lambda_2, \lambda_4\}$ becomes $\{|c_2|, |c_3|\} \leq |c_1|$, and the condition $\{\lambda_1, \lambda_4\} \geq \{\lambda_2, \lambda_3\}$ or $\{\lambda_1, \lambda_4\} \leq \{\lambda_2, \lambda_3\}$ turns into $\{|c_1|, |c_3|\} \leq |c_2|$.

According to formula (18) by considering formulas (10), we can obtain the formulas of the quantum discord in various conditions.

If $|c_1| \leq |c_3|$ and $c_2 = -c_1 c_3$, then

$$D(\rho) = \frac{1}{2}[(1+c_1)\log(1+c_1) + (1-c_1)\log(1-c_1)]. \tag{21}$$

If $|c_2| \leq |c_3|$ and $c_1 = -c_2 c_3$, then

$$D(\rho) = \frac{1}{2}[(1+c_2)\log(1+c_2) + (1-c_2)\log(1-c_2)]. \tag{22}$$

If $|c_3| \leq |c_1|$, and $c_2 = -c_1 c_3$, then

$$D(\rho) = \frac{1}{2}[(1+c_3)\log(1+c_3) + (1-c_3)\log(1-c_3)]. \tag{23}$$

If $|c_2| \leq |c_1|$, and $c_3 = -c_1 c_2$, then

$$D(\rho) = \frac{1}{2}[(1+c_2)\log(1+c_2) + (1-c_2)\log(1-c_2)]. \tag{24}$$

If $|c_3| \leq |c_2|$ and $c_1 = -c_2 c_3$, then

$$D(\rho) = \frac{1}{2}[(1+c_3)\log(1+c_3) + (1-c_3)\log(1-c_3)]. \tag{25}$$

If $|c_1| \leq |c_2|$ and $c_3 = -c_1 c_2$, then

$$D(\rho) = \frac{1}{2}[(1+c_1)\log(1+c_1) + (1-c_1)\log(1-c_1)]. \tag{26}$$

As the local bit flip channels on each qubit map initial BD states with $\{c_1(0), c_2(0), c_3(0)\}$ to BD states with $\{c_1(0), c_2(0)\,\mathrm{e}^{-2\gamma t}, c_3(0)\,\mathrm{e}^{-2\gamma t}\}$, note that $c_1(0)$ is unaffected by the environment noise. According to formula (21), one can see that when the initial BD states satisfy the conditions $|c_1(0)| \leq |c_3(0)|$ and $c_2(0) = -c_1(0)c_3(0)$, the quantum discord is determined only by $c_1(0)$. Obviously, the corresponding evolved BD states satisfy the condition $c_2(t) = -c_1(t)c_3(t)$, so when the evolved BD states also satisfy the condition $|c_1(t)| \leq |c_3(t)|$, i.e., $|c_1(0)| \leq |c_3(0)|\,\mathrm{e}^{-2\gamma t}$, the quantum discord is also determined only by $c_1(0)$ and the freezing phenomenon is present. We define a threshold time t' by

$$t' = -\frac{1}{2\gamma}\ln\frac{|c_1(0)|}{|c_3(0)|}, \tag{27}$$

the quantum discord stays constant for $0 \leq t \leq t'$. Similarly, according to formula (26), we can see when the initial BD states satisfy the conditions $|c_1(0)| \leq |c_2(0)|$ and $c_3(0) = -c_1(0)c_2(0)$, the quantum discord is frozen within the threshold time t'',

$$t'' = -\frac{1}{2\gamma}\ln\frac{|c_1(0)|}{|c_2(0)|}. \tag{28}$$

Therefore, we can obtain two conditions for freezing the quantum discord when the quantum systems undergo local identical bit flip channels. The first condition is the initial BD states with $|c_1(0)| \leq |c_3(0)|$ and $c_2(0) = -c_1(0)c_3(0)$, and the threshold time is t'. The second condition is the initial BD states with $|c_1(0)| \leq |c_2(0)|$ and $c_3(0) = -c_1(0)c_2(0)$, and the threshold time is t''. The frozen quantum discord is

$$D(\rho(t)) = D(\rho(0)) = \frac{1}{2}H(c_1(0)). \tag{29}$$

We know that $H(x)$ is a symmetric convex function when $|x| \leq 1$. When $|c_1(0)| \to 1$, the frozen quantum discord approaches to a maximum value, but with the values $|c_1(0)|$ increasing, the values of the threshold time decrease. So for prolonging the freezing time, we should make $|c_1(0)|$ smaller where some quantum task in employing the quantum discord is guaranteed. We now analyze how to use these two conditions of freezing quantum discord when the quantum systems undergo local identical bit flip channels. If given $|c_1(0)|$ and $|c_3(0)|$ in the BD states, when $|c_1(0)| \leq |c_3(0)|$, we can adjust $|c_2(0)|$ such that it satisfies the condition $c_2(0) = -c_1(0)c_3(0)$. When $|c_1(0)| \geq |c_3(0)|$, we can adjust $|c_2(0)|$ such that it satisfies the conditions $c_2(0) = -c_3(0)/c_1(0)$ and $|c_1(0)| \leq |c_2(0)|$, thus we have $|c_1(0)| \geq |c_3(0)| \geq |c_1(0)|^2$. That is to say, for the BD states with given $|c_1(0)|$ and $|c_3(0)|$, when $|c_1(0)| \leq |c_3(0)|$, we can freeze the quantum discord via the first condition. When $|c_1(0)| \geq |c_3(0)| \geq |c_1(0)|^2$, we can freeze the quantum discord via the second condition, and the threshold time t'' can be rewritten as

$$t'' = -\frac{1}{2\gamma}\ln\frac{|c_1(0)|^2}{|c_3(0)|}.$$

But when $|c_3(0)| \leq |c_1(0)|^2$, we cannot freeze the quantum discord. Similarly, if given $|c_1(0)|$ and $|c_2(0)|$ in the BD states, when $|c_1(0)| \leq |c_2(0)|$, we can freeze the quantum discord via the second condition, when $|c_1(0)| \geq |c_2(0)| \geq |c_1(0)|^2$, we can freeze the quantum discord via the first condition, but when $|c_2(0)| \leq |c_1(0)|^2$, we cannot freeze the quantum discord.

We now show why the frozen quantum discord can occur and why the freezing phenomenon is present in a finite time. For the initial BD states with $\{c_1(0), -c_1(0)c_3(0), c_3(0)\}$, the closest classical states are $\{0,0,c_3(0)\}$ when $|c_1(0)| \leq |c_3(0)|$. For the evolved BD states $\{c_1(0), -c_1(0)c_3(0)\,e^{-2\gamma t}, c_3(0)\,e^{-2\gamma t}\}$, when $|c_1(0)| \leq |c_3(0)|\,e^{-2\gamma t}$, the closest classical states are $\{0,0,c_3(0)\,e^{-2\gamma t}\}$; when $|c_1(0)| \geq |c_3(0)|\,e^{-2\gamma t}$, the closest classical states are $\{c_1(0),0,0\}$. Cianciaruso *et al.*[32] have proved that any contractive distance D satisfies the following translational invariance properties for BD states:

$$D(\{c_1, -c_1c_3, c_3\}, \{c_1,0,0\}) = D(\{0,0,c_3\}, \{0,0,0\}), \tag{30}$$

and

$$D(\{c_1, -c_1c_3, c_3\}, \{0,0,c_3\}) = D(\{c_1,0,0\}, \{0,0,0\}). \tag{31}$$

So within the threshold time, we have

$$\begin{aligned} D\big(\{c_1(0), -c_1(0)c_3(0)\,e^{-2\gamma t}, c_3(0)\,e^{-2\gamma t}\}, \{0,0,c_3(0)\,e^{-2\gamma t}\}\big) &= D(\{c_1(0), -c_1(0)c_3(0), c_3(0)\}, \{0,0,c_3(0)\}) \\ &= D(\{c_1(0),0,0\}, \{0,0,0\}), \end{aligned} \tag{32}$$

the freezing phenomenon for quantum discord is present. But we can see that the classical correlations vary with time. When the evolution time is beyond the threshold time, the closest classical states change, and we have

$$D\big(\{c_1(0), -c_1(0)c_3(0)\,e^{-2\gamma t}, c_3(0)\,e^{-2\gamma t}\}, \{c_1(0),0,0\}\big) = D\big(\{0,0,c_3(0)\,e^{-2\gamma t}\}, \{0,0,0\}\big), \tag{33}$$

the quantum discord of the evolved states varies with time. But the value of the classical correlation is a constant.

For the initial BD states with $\{c_1(0), c_2(0), -c_1(0)c_2(0)\}$, the closest classical states are $\{0,c_2(0),0\}$ when $|c_1(0)| \leq |c_2(0)|$. For the evolved BD states $\{c_1(0), c_2(0)\,e^{-2\gamma t}, -c_1(0)c_2(0)\,e^{-2\gamma t}\}$, when $|c_1(0)| \leq |c_2(0)|\,e^{-2\gamma t}$, the closest classical states are $\{0,c_2(0)\,e^{-2\gamma t},0\}$; when $|c_1(0)| \geq |c_2(0)|\,e^{-2\gamma t}$, the closest classical states are $\{c_1(0),0,0\}$. Using similar methods to the above, we can explain why the freezing phenomenon for quantum discord can occur and the freezing phenomenon be present in a finite time.

Similarly, as local bit phase flip channels on each qubit map initial BD states with $\{c_1,c_2,c_3\}$ to BD states with $\{c_1\,e^{-2\gamma t}, c_2, c_3\,e^{-2\gamma t}\}$. According to formulas (22) and (24), we can see when the quantum systems undergo local identical bit-phase flip channels, the freezing phenomenon for quantum discord can be present in a finite time when the BD states satisfy the conditions $c_1 = -c_2c_3$ and $|c_2| \leq |c_3|$, or the conditions $c_3 = -c_1c_2$ and $|c_2| \leq |c_1|$. The frozen quantum discord is described as $D(\rho(t)) = D(\rho(0)) = (1/2)H(c_2(0))$, as local phase flip channels on each qubit map initial BD states with $\{c_1,c_2,c_3\}$ to BD states with $\{c_1\,e^{-2\gamma t}, c_2\,e^{-2\gamma t}, c_3\}$. According to formulas (23) and (25), we can see when the quan-

tum systems undergo local identical phase flip channels, the freezing phenomenon for quantum discord can occur in a finite time when the BD states satisfy the conditions $c_2 = -c_1 c_3$ and $|c_3| \leq |c_1|$, or the conditions $c_1 = -c_2 c_3$ and $|c_3| \leq |c_2|$. The frozen quantum discord is described as $D(\rho(t)) = D(\rho(0)) = (1/2)H(c_3(0))$.

5. Conclusions

In this paper, in terms of the definitions of the relative entropy of coherence and discord, we have derived the conditions for freezing phenomenon when the quantum systems undergo bit flip, bit-phase flip, and phase flip channels, respectively. We find that the freezing phenomenon for quantum coherence can occur only in bit flip and bit-phase flip channels, but the freezing phenomenon for quantum discord can be present in bit flip, bit phase flip, and phase flip channels. The conditions for freezing the discord are stricter than for freezing the coherence, but the discord can be frozen via two conditions. We have investigated the properties of the frozen quantum coherence and frozen quantum discord. We find that the quantum coherence is frozen forever under certain conditions, but the freezing phenomenon for quantum discord is present within a finite time under some conditions. Furthermore, the frozen quantum coherence can reach its maximal value, but the frozen quantum discord cannot reach its maximal value, and the threshold time decreases with the increase of the value of frozen quantum discord. We show that the reasons for freezing the quantum coherence are the coherence interchange in the quantum systems, and the freezing of quantum discord is caused by the properties of the measurement which are used to measure quantum discord.

From a fundamental perspective it is important to understand in depth the physical origin of frozen quantum coherence and the relation between quantum coherence and quantum discord. Here, we identify all the conditions for the freezing phenomenon in the mathematically rigorous form, and a comparison among these conditions can help us better understand the freezing phenomenon and the relations between quantum coherence and quantum discord. The reasons for the freezing phenomenon, we provide here, may describe the essence of the freezing phenomenon for quantum coherence and discord, so this work may provide some clues to its further investigation. Recently, the connection between quantum coherence and discord has been established in Ref. [48], through our work we can see the difference between the quantum coherence and discord.

Our result also has an influence from an applicative point of view: the property immune from the noise makes quantum coherence and quantum correlations important for the realisation of quantum technologies. Further research on this question can lead to a more efficient exploitation of coherence and discord for empowering the performance of real-world quantum technologies to be applied to communication, computation, sensing, and metrology.

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