ISSN 1674-1056
http://cpb.iphy.ac.cn
http://iopscience.iop.org/cpb
中国物理 B
Chinese Physics
中国物理
Chinese Physics B

Constructing (2+1)-dimensional *N*=1 supersymmetric integrable systems from the Hirota formalism in the superspace

Jian-Yong Wang(王建勇), Xiao-Yan Tang(唐晓艳), Zu-Feng Liang(梁祖峰)
Citation: Chin. Phys. B . 2018, 27(4): 040203. **doi:** 10.1088/1674-1056/27/4/040203

Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

An extension of integrable equations related to AKNS and WKI spectral problems and their reductions

Xian-Guo Geng(耿献国), Yun-Yun Zhai(翟云云)
Chin. Phys. B . 2018, 27(4): 040201. **doi:** 10.1088/1674-1056/27/4/040201

Integrability classification and exact solutions to generalized variable-coefficient nonlinear evolution equation

Han-Ze Liu(刘汉泽), Li-Xiang Zhang(张丽香)
Chin. Phys. B . 2018, 27(4): 040202. **doi:** 10.1088/1674-1056/27/4/040202

Improved reproducing kernel particle method for piezoelectric materials

Ji-Chao Ma(马吉超), Gao-Feng Wei(魏高峰), Dan-Dan Liu(刘丹丹)
Chin. Phys. B . 2018, 27(1): 010201. **doi:** 10.1088/1674-1056/27/1/010201

Improved control for distributed parameter systems with time-dependent spatial domains utilizing mobile sensor—actuator networks

Jian-Zhong Zhang(张建中), Bao-Tong Cui(崔宝同), Bo Zhuang(庄波)
Chin. Phys. B . 2017, 26(9): 090201. **doi:** 10.1088/1674-1056/26/9/090201

Generalized symmetries of an *N*=1 supersymmetric Boiti–Leon–Manna–Pempinelli system

Wang Jian-Yong, Tang Xiao-Yan, Liang Zu-Feng, Lou Sen-Yue
Chin. Phys. B . 2015, 24(5): 050202. **doi:** 10.1088/1674-1056/24/5/050202

Constructing (2+1)-dimensional $\mathcal{N}=1$ supersymmetric integrable systems from the Hirota formalism in the superspace*

Jian-Yong Wang(王建勇)[1,†], Xiao-Yan Tang(唐晓艳)[2], and Zu-Feng Liang(梁祖峰)[3]

[1] *Department of Mathematics and Physics, Quzhou University, Quzhou* 324000, *China*

[2] *Shanghai Key Laboratory of Trustworthy Computing, East China Normal University, Shanghai* 200062, *China*

[3] *Department of Physics, Hangzhou Normal University, Hangzhou* 310036, *China*

(Received 23 December 2017; revised manuscript received 15 January 2018; published online 29 March 2018)

The $\mathcal{N}=1$ supersymmetric extensions of two integrable systems, a special negative Kadomtsev–Petviashvili (NKP) system and a (2+1)-dimensional modified Korteweg–de Vries (MKdV) system, are constructed from the Hirota formalism in the superspace. The integrability of both systems in the sense of possessing infinitely many generalized symmetries are confirmed by extending the formal series symmetry approach to the supersymmetric framework. It is found that both systems admit a generalization of W_∞ type algebra and a Kac-Moody–Virasoro type subalgebra. Interestingly, the first one of the positive flow of the supersymmetric NKP system is another $\mathcal{N}=1$ supersymmetric extension of the (2+1)-dimensional MKdV system. Based on our work, a hypothesis is put forward on a series of (2+1)-dimensional supersymmetric integrable systems. It is hoped that our work may develop a straightforward way to obtain supersymmetric integrable systems in high dimensions.

Keywords: formal series symmetry approach, generalized symmetry, supersymmetric NKP system, supersymmetric MKdV system

PACS: 02.30.Jr, 11.30.Pb, 02.20.Sv **DOI:** 10.1088/1674-1056/27/4/040203

1. Introduction

Supersymmetry is one of the most significant ideas in theoretical physics, combining bosonic and fermionic fields in a unified way. The mathematical formulation of this idea is based on the introduction of the anticommuting fermionic variables along with the bosonic ones. During the past decades, studies of supersymmetric integrable systems and their possible applications in physics have been a subject of considerable interest, and a number of well-known integrable systems, such as the sine-Gordon equation,[1] the Korteweg–de Vries (KdV) equation,[2] and Kadomtsev–Petviashvili (KP) hierarchies,[3] have been embedded into their supersymmetric counterparts. It turns out that these supersymmetric integrable systems possess similar remarkable properties. However, most studies on the supersymmetric integrable systems are in (1+1) dimensions. To our knowledge, the KP hierarchy and (2+1)-dimensional KdV equation (also named as the Boiti–Leon–Manna–Pempinelli (BLMP) equation)[4,5] are the very few exceptions that have been extended in the supersymmetric framework. In the viewpoint of physics, the (2+1)-dimensional field theory is particularly interesting since the theory is simple yet possesses a rich asymptotic symmetry and provides us with a deeper insight in general.[6] Consequently, it is within the scope of the integrable systems higher than (1+1) dimensions, such as the KP[7] and supersymmetric KP hierarchies,[8] that the W_∞ algebras were identified. Therefore, it is significant if one can construct supersymmetric extensions of the (2+1)-dimensional integrable systems, especially those related to the W_∞-type algebra.

In addition, the realizations of the W_∞ algebra in the (2+1)-dimensional supersymmetric integrable system are mostly based on the extensions of the bosonic Lax pair to super one. In the classical integrable system context, Lou has established the formal series symmetry approach for the realization of W_∞, which has been utilized to demonstrate that some famous (2+1)-dimensional models, such as the KP hierarchy,[7] the 3D Toda field theory,[9] and the integrable dispersive long wave equations,[10] admit one or more sets of infinitely many generalized symmetries and the corresponding symmetry algebras are the generalizations of the W_∞ algebra.

In this paper, we consider the $\mathcal{N}=1$ supersymmetric extensions of two bilinear integrable systems, the bilinear negative KP (NKP) system[11,12]

$$(D_t+D_x^2)F\cdot G=0, \tag{1a}$$

$$D_y(D_t+D_x^2)F\cdot G=0, \tag{1b}$$

and the (2+1)-dimensional modified KdV (MKdV) system

$$(D_t+D_x^3)F\cdot G=0, \tag{2a}$$

$$D_y(D_t+D_x^3)F\cdot G=0. \tag{2b}$$

The bilinear NKP system (1) can also be named as the (2+1)-dimensional (1+1)-component Ablowitz–Kaup–

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11605102, 11475052, 11675055, and 11626140).

†Corresponding author. E-mail: jywangqz@126.com

© 2018 Chinese Physical Society and IOP Publishing Ltd

Newell–Segur (AKNS) system, the (2+1)-dimensional Broer–Kaup (BK) system[13] and the (2+1)-dimensional sinh-Gordon equation.[14] The bilinear MKdV system (2) has been rediscovered several times by Ito,[15] Hietarinta,[16] Lou and Li,[17] respectively. Many remarkable properties of the system (2) have been established. For instance, its N-soliton solutions, Bäklund transformation, and Lax representation were obtained in Ref. [18], and its integrability in the sense of possessing infinitely many generalized symmetries was confirmed in Ref. [17].

This paper is organized as follows. In the next section, the supersymmetric extension of the bilinear NKP system (1) and its nonlinear version are proposed. Then, by extending the formal series symmetry approach to the supersymmetric framework, it is confirmed that the supersymmetric NKP system admits a set of infinitely many generalized symmetries with an arbitrary function $f(t)$. Finally, symmetry algebras of the supersymmetric NKP system are presented, including a limit case $f(t)=1$ which implies the commutativity of the positive flow. In Section 3, similar study is conducted on the MKdV system (2). The last section gives a short summary and discussion.

2. Symmetries and symmetry algebras of the supersymmetric NKP system

2.1. Supersymmetric extension of the bilinear NKP system and its nonlinear version

Our candidate for the supersymmetric extension of the bilinear NKP system (1) is

$$(D_t+D_x^2)F\cdot G=0, \tag{3a}$$
$$\mathcal{S}_y(D_t+D_x^2)F\cdot G=0, \tag{3b}$$

where F and$G$ are bosonic functions of the super spacetime (x,y,t,θ), and θ is a Grassmann odd variable satisfying $\theta^2=0$. The superversion of the Hirota derivative is defined as

$$\mathcal{S}_yD_t^mD_x^nF\cdot G=(\mathcal{D}_{\theta_1}-\mathcal{D}_{\theta_2})\left(\frac{\partial}{\partial t_1}-\frac{\partial}{\partial t_2}\right)^m\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)^nF(x_1,y_1,t_1,\theta_1)\cdot G(x_2,y_2,t_2,\theta_2)\Big|_{\substack{\theta_1=\theta_2\\ y_1=y_2\\ x_1=x_2\\ t_1=t_2}},$$

and $\mathcal{D}_\theta=\partial_\theta+\theta\partial_y$ is the usual super derivative.

It is obvious that we extend the bilinear NKP system to its supersymmetric counterpart only by changing one of the Hirota operators. However, this seemingly simple extension is not trivial and worth further considerations, as the Hirota bilinear method is powerful and effective not only for finding soliton solutions but also for searching new integrable systems in the classic[16,19–21] and supersymmetric[22] contexts. The coupled system (3) can also be viewed as a (2+1)-dimensional generalization of the bilinear supersymmetric two-boson system,[13,23–26] since it reduces to the supersymmetric two-boson system when $y=x$.

Before proceeding further, let us first introduce the transformations

$$F=\exp\left\{\frac{1}{2}[(v-u)+\theta(\phi-\psi)]\right\},$$
$$G=\exp\left\{\frac{1}{2}[(v+u)+\theta(\phi+\psi)]\right\} \tag{4}$$

with super even (bosonic) functions u and v, and super odd (fermionic) functions ψ and ϕ being functions of (x,y,t). Under the transformations (4), equation (3) can be expressed in the nonlinear version through the direct calculations (a more convenient way to derive the nonlinear version is given in Appendix A),

$$u_t-u_x^2-v_{xx}=0, \tag{5a}$$
$$\psi_t-2u_x\psi_x-\phi_{xx}=0, \tag{5b}$$
$$\phi_t-2u_x\phi_x-\psi_{xx}=0, \tag{5c}$$
$$v_{yt}-2u_xv_{xy}-u_{xxy}-2\psi_x\phi_x=0. \tag{5d}$$

In the fermionic limit $\psi=\phi=0$, equation (5) reduces to a (2+1)-dimensional coupled Burgers system[27]

$$u_t-u_x^2-v_{xx}=0, \tag{6a}$$
$$v_{yt}-2u_xv_{xy}-u_{xxy}=0, \tag{6b}$$

which can be reformed to the bilinear NKP system (1) under the bi-logarithmic transformations

$$u=\ln\left(\frac{F}{G}\right),\qquad v=\ln(FG). \tag{7}$$

Thus, the coupled system (5) deserves the name of the supersymmetric NKP (SNKP) system.

2.2. Infinitely many generalized symmetries of the SNKP system

A symmetry $\sigma=(U,\xi,\eta,V)^{\mathrm{T}}$ of the SNKP system is defined as a solution of its linearized equations by assuming that Eq. (5) is form-invariant under the infinitesimal transformations

$$u\to u+\varepsilon U,\quad \psi\to\psi+\varepsilon\xi,$$
$$\phi\to\phi+\varepsilon\eta,\quad v\to v+\varepsilon V, \tag{8}$$

where ε is an infinitesimal parameter, ξ and η are fermionic functions of (t,x,y,ψ,ϕ,u,v), while U and V are bosonic func-

tions of (t,x,y,ψ,ϕ,u,v). Inserting the infinitesimal transformations (8) into Eq. (5) and picking up the coefficients of ε, we directly obtain the linearized system that the symmetry should satisfy

$$U_t - 2u_xU_x - V_{xx} = 0, \tag{9a}$$
$$\xi_t - 2u_x\xi_x - 2\psi_xU_x - \eta_{xx} = 0, \tag{9b}$$
$$\eta_t - 2u_x\eta_x - 2\phi_xU_x - \xi_{xx} = 0, \tag{9c}$$
$$V_{yt} - 2v_{xy}U_x - 2u_xV_{xy} - U_{xxy} - 2\xi_x\phi_x - 2\psi_x\eta_x = 0. \tag{9d}$$

Extending the formal series symmetry approach[7,9,10] to the supersymmetric framework, the components of the symmetry should be reformed as

$$U = \sum_{k=0}^{\infty} f^{(n-k+1)}U_n[k], \quad V = \sum_{k=0}^{\infty} f^{(n-k+2)}V_n[k], \tag{10a}$$
$$\eta = \sum_{k=0}^{\infty} f^{(n-k+1)}\eta_n[k], \quad \xi = \sum_{k=0}^{\infty} f^{(n-k+1)}\xi_n[k], \tag{10b}$$

where f is an arbitrary function of t, $f^{(n-k)}$ denotes the $(n-k)$-th derivative of f with respect to t, $U_n[k]$, $\xi_n[k]$, $\eta_n[k]$, and $V_n[k]$ are functions of (x,y,u,ψ,ϕ,v) and their arbitrary derivatives, but not t-dependent explicitly, $U_n[k]$ and $V_n[k]$ are bosonic functions, while $\xi_n[k]$ and $\eta_n[k]$ are fermionic functions.

Substituting Eq. (10) into the symmetry definition equation (9) and collecting the terms of the same derivative order of f yield

$$\sum_{k=0}^{\infty} f^{(n-k+2)}(U_n[k] - V_{nxx}[k]) + f^{(n-k+1)}(U_{nt}[k] - 2u_xU_{nx}[k]) = 0, \tag{11a}$$
$$\sum_{k=0}^{\infty} f^{(n-k+2)}\xi_n[k] + f^{(n-k+1)}(\xi_{nt}[k] - 2u_x\xi_{nx}[k] - 2\psi_xU_n[k] - \eta_{nxx}[k]) = 0, \tag{11b}$$
$$\sum_{k=0}^{\infty} f^{(n-k+2)}\eta_n[k] + f^{(n-k+1)}(\eta_{nt}[k] - 2u_x\eta_{nx}[k] - 2\phi_xU_n[k] - \xi_{nxx}[k]) = 0, \tag{11c}$$
$$\sum_{k=0}^{\infty} f^{(n-k+3)}V_{ny}[k] + f^{(n-k+2)}(V_{nyt}[k] - 2u_xV_{nxy}[k]) - f^{(n-k+1)}(2v_{xy}U_{nx}[k] + U_{nxxy}[k] + 2\xi_{nx}[k]\phi_x + 2\psi_x\eta_{nx}[k]) = 0. \tag{11d}$$

Since f is an arbitrary function of t, equation (11) must hold at any derivative order of f. Therefore, by setting to zero all of the coefficients of the different derivative orders of f, we obtain the following overdetermined system

$$V_{ny}[0] = 0, \quad V_n[0] = g(x) = g, \tag{12a}$$
$$U_n[0] = V_{nxx}[0], \quad V_n[1] = \xi_n[0] = \eta_n[0] = 0, \tag{12b}$$
$$U_n[k-1] = -U_{nt}[k-2] + 2u_xU_{nx}[k-2] + V_{nxx}[k-1], \tag{12c}$$
$$\xi_n[k-1] = -\xi_{nt}[k-2] + 2u_x\xi_{nx}[k-2] + 2\psi_xU_{nx}[k-2] + \eta_{nxx}[k-2], \tag{12d}$$
$$\eta_n[k-1] = -\eta_{nt}[k-2] + 2u_x\eta_{nx}[k-2] + 2\phi_xU_{nx}[k-2] + \xi_{nxx}[k-2], \tag{12e}$$
$$V_{ny}[k] = -V_{nyt}[k-1] + 2u_xV_{nxy}[k-1] + 2v_{xy}U_{nx}[k-2] + U_{nxxy}[k-2] + 2\xi_{nx}[k-2]\phi_x + 2\psi_x\eta_{nx}[k-2] \tag{12f}$$

for $k = 2,3,\ldots,n+1$, where we have defined $\partial_y\partial_y^{-1} = \partial_y^{-1}\partial_y = 1$.

It is obvious from Eq. (12) that $U_n[k-1]$, $\xi_n[k-1]$, $\eta_n[k-1]$, and $V_n[k]$ can be solved recursively for any given g. We can write the result in a more compact manner

$$\begin{pmatrix} U_n[k-1] \\ \xi_n[k-1] \\ \eta_n[k-1] \\ V_n[k] \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & \partial_x^2 \\ 2\psi_x\partial_x & a_{22} & \partial_x^2 & 0 \\ 2\phi_x\partial_x & \partial_x^2 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} U_n[k-2] \\ \xi_n[k-2] \\ \eta_n[k-2] \\ V_n[k-1] \end{pmatrix}$$
$$= \begin{pmatrix} a_{11} & 0 & 0 & \partial_x^2 \\ 2\psi_x\partial_x & a_{22} & \partial_x^2 & 0 \\ 2\phi_x\partial_x & \partial_x^2 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}^{k-1} \begin{pmatrix} g_{xx} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

with $k = 2,3,\ldots,n+2$, and the operators in the matrix defined as

$$a_{11} = a_{22} = a_{33} = -\partial_t + 2u_x\partial_x, \quad a_{41} = \partial_x^2 + 2\partial_y^{-1}v_{xy}\partial_x,$$
$$a_{42} = -2\partial_y^{-1}\phi_x\partial_x, \quad a_{43} = 2\partial_y^{-1}\psi_x\partial_x,$$
$$a_{44} = -\partial_t + 2\partial_y^{-1}u_x\partial_x\partial_y.$$

Essentially, the recursion formula (13) should be truncated to guarantee the existence of the formal series symmetries. For the SNKP system, we find $(U_n[n+1], \xi_n[n+1], \xi_n[n+1], V_n[n+2])^{\mathrm{T}}$ satisfy the truncated condition

$$\begin{pmatrix} a_{11} & 0 & 0 & \partial_x^2 \\ 2\psi_x\partial_x & a_{22} & \partial_x^2 & 0 \\ 2\phi_x\partial_x & \partial_x^2 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} U_n[n+1] \\ \xi_n[n+1] \\ \eta_n[n+1] \\ V_n[n+2] \end{pmatrix} = 0.$$

In other words, $(U_n[n+1], \xi_n[n+1], \xi_n[n+1], V_n[n+2])^{\mathrm{T}}$ also satisfies the symmetry definition equation (9).

It is noted from Eq. (13) that one can deduce the explicit generalized symmetries by inserting the exact expression of g. Here, we present the first four generalized symmetries in detail, obtained by choosing $g = x^3/12$, $x^4/96$, $x^5/960$, and

$x^6/11520$, respectively,

$$\sigma_0(f)=\begin{pmatrix}\frac{1}{2}f_t x+fu_x\\ f\psi_x\\ f\phi_x\\ \frac{1}{12}f_{ttt}x^3+fv_x\end{pmatrix}, \tag{14}$$

$$\sigma_1(f)=\begin{pmatrix}\frac{1}{8}f_{tt}x^2+\frac{1}{2}f_t xu_x+fu_t\\ \frac{1}{2}f_t x\psi_x+f\psi_t\\ \frac{1}{2}f_t x\phi_x+f\phi_t\\ \frac{1}{96}f_{tttt}x^4+\frac{1}{2}f_t xv_x+fv_t\end{pmatrix}, \tag{15}$$

$$\sigma_2(f)=\begin{pmatrix}U_2\\ \xi_2\\ \eta_2\\ V_2\end{pmatrix} \tag{16}$$

with

$$\begin{aligned}U_2=&\frac{1}{48}f_{ttt}x^3+\frac{1}{8}f_{tt}x^2u_x+\frac{1}{4}[2x(u_x^2+v_{xx})+v_x]\\&+f\Big[u_{xxx}+\frac{3}{2}u_xv_{xx}+u_x^3+\frac{3}{2}\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)_x\Big],\\ \xi_2=&\frac{1}{8}f_{tt}x^2\psi_x+\frac{1}{4}[2x(2u_x\psi_x+\phi_{xx})+\phi_x]\\&+f\Big[\psi_{xxx}+\frac{3}{2}(2u_x^2+v_{xx})\psi_x+3u_x\phi_{xx}+\frac{3}{2}u_{xx}\phi_x\Big],\\ \eta_2=&\frac{1}{8}f_{tt}x^2\phi_x+\frac{1}{4}[2x(2u_x\phi_x+\psi_{xx})+\psi_x]\\&+f\Big[\phi_{xxx}+\frac{3}{2}(2u_x^2+v_{xx})\phi_x+3u_x\psi_{xx}+\frac{3}{2}u_{xx}\psi_x\Big],\\ V_2=&\frac{1}{960}f_{ttttt}x^5+\frac{1}{8}f_{tt}x^2v_x\\&+\frac{1}{4}\{2x[2\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)+u_{xx}]+u_x\}\\&+f\Big[v_{xxx}+\frac{3}{2}\partial_y^{-1}(2u_x^2v_{xy}+2u_xu_{xxy}+v_{xx}v_{xy}\\&+u_{xx}u_{xy}+4u_x\psi_x\phi_x+\psi_x\psi_{xx}-\phi_x\phi_{xx})\Big],\end{aligned}$$

and

$$\sigma_3(f)=\begin{pmatrix}U_3\\ \xi_3\\ \eta_3\\ V_3\end{pmatrix} \tag{17}$$

with

$$\begin{aligned}U_3=&\frac{1}{384}f_{tttt}x^4+\frac{1}{48}x^3u_x+\frac{1}{8}f_{tt}[x^2(u_x^2+v_{xx})+xv_x]+\frac{1}{4}\{x[2u_{xxx}+3\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)_x+3u_xv_{xx}+2u_x^3]+2u_{xx}+u_xv_x\\&+2\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)\}+f\{u_x[4u_{xxx}+2\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)_x]+\partial_y^{-1}[4(u_x\psi_x\phi_x)_x+\psi_x\psi_{xxx}-\phi_x\phi_{xxx}+v_{xxx}v_{xy}-u_{xxx}u_{xy}\\&-4u_x(u_{xx}v_{xy}-v_{xx}u_{xy})]+u_x^4+4u_x^2v_{xx}+2u_{xx}^2+v_{xx}^2+v_{xxxx}\},\\ \xi_3=&\frac{1}{48}f_{ttt}x^3\psi_x+\frac{1}{8}f_{tt}[x^2(2u_x\psi_x+\phi_{xx})+x\phi_x]+\frac{1}{4}f_t\{x[3u_{xx}\phi_x+2\psi_{xxx}+6u_x^2\psi_x+6u_x\phi_{xx}+3v_{xx}\psi_x]+3u_x\phi_x+v_x\psi_x+2\psi_{xx}\}\\&+f\{\psi_x[2\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)_x+4u_x(u_x^2+v_{xx})+3u_{xxx}]+4(u_x\psi_{xx})_x+2v_{xx}\phi_{xx}+v_{xxx}\phi_x+6u_x(u_x\phi_x)_x+\phi_{xxxx}\},\\ \eta_3=&\frac{1}{48}f_{ttt}x^3\phi_x+\frac{1}{8}f_{tt}[x^2(2u_x\phi_x+\psi_{xx})+x\psi_x]+\frac{1}{4}f_t\{x[3u_{xx}\psi_x+2\phi_{xxx}+6u_x^2\phi_x+6u_x\psi_{xx}+3v_{xx}\phi_x]+3u_x\psi_x+v_x\phi_x+2\phi_{xx}\}\\&+f\{\phi_x[2\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)_x+4u_x(u_x^2+v_{xx})+3u_{xxx}]+4(u_x\phi_{xx})_x+2v_{xx}\psi_{xx}+v_{xxx}\psi_x+6u_x(u_x\psi_x)_x+\psi_{xxxx}\},\\ V_3=&\frac{1}{11520}f_{ttttt}x^6+\frac{1}{48}f_{ttt}x^3v_x+\frac{1}{8}f_{tt}\{x^2[u_{xx}+2\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)]+xu_x\}\\&+\frac{1}{4}f_t\Big\{\Big[3x(u_{xx}u_{xy}+v_{xx}v_{xy}+2u_xu_xxy+2u_x^2v_{xy}+\psi_x\psi_{xx}\\&-\psi_x\psi_{xx}+4u_x\psi_x\phi_x)\Big]+2xv_{xxx}+\frac{1}{2}v_x^2+\frac{3}{2}u_x^2+2v_{xx}\Big\}+f\{2v_x\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)_x+\partial_y^{-1}[4(u_xv_{xxy})_x\\&+4(3u_x^2+v_{xx})\psi_x\phi_x+4u_xv_{xy}(u_x^2+v_{xx})+6u_x(\psi_x\psi_{xx}-\phi_x\phi_{xx}-u_xu_{xx}u_{xy})+3(\psi_x\phi_{xxx}+\psi_{xxx}\phi_x+u_{xxx}v_{xy})\\&-2v_x(u_xv_{xy}+\psi_x\phi_x)_x+2\psi_{xx}\phi_{xx}+2u_{xxy}v_{xx}+u_{xy}v_{xxx}]+6u_x^2u_{xx}+u_{xxxx}\}.\end{aligned}$$

In the derivation of the symmetries (14)–(17), we have used two tips to avoid the confusion caused by the anti-commuting fermionic functions. First, the x-derivatives of ψ should be placed in front of the x-derivatives of ϕ, for instances, the term $\phi_x\psi_{xxx}$ should be reformed as $-\psi_{xxx}\phi_x$. Second, the lower order of the x-derivatives of ψ or ϕ should be placed in front of their higher orders, for instances, the term $\phi_{xxx}\phi_x$ should be rewritten as $-\phi_x\phi_{xxx}$.

Actually, the arbitrary function g in the general symmetry formula can be further determined with the help of the dimensional analysis. Due to the linearity of the symmetry definition equation, the arbitrary function g can be expanded as a Lau-

rent series. Assume the variable x has the dimension $[L]^1$, then from Eq. (5), we find that y, t, u, v, ψ, and ϕ have the dimensions $[L]^0$, $[L]^2$, $[L]^0$, $[L]^0$, $[L]^0$, and $[L]^0$, respectively. Thereafter, we know from Eqs. (14)–(17) that $V_0(f)/f$, $V_1(f)/f$, $V_2(f)/f$, …, $V_n(f)/f$ have the dimensions $[L]^{-1}$, $[L]^{-2}$, $[L]^{-3}$, …, $[L]^{-(n+1)}$, correspondingly. That is to say, $f^{(n+2)}g/f$, one term of $V_n(f)/f$, must have the dimension $[L]^{-(n+1)}$, which means g must have the dimension $[L]^{n+3}$. Consequently, g can be determined as

$$g=\frac{x^{n+3}}{2^{n+1}(n+3)!}. \tag{18}$$

Substituting Eq. (18) into Eq. (10), the symmetries of the SNKP system can be summarized as

$$\sigma_n(f)=\begin{pmatrix} \dfrac{f^{(n+1)}x^{n+1}}{2^{n+1}(n+1)!}+\displaystyle\sum_{k=1}^{n+1}f^{n-k+1}U_n[k] \\ \displaystyle\sum_{k=1}^{n+1}f^{n-k+1}\xi_n[k] \\ \displaystyle\sum_{k=1}^{n+1}f^{n-k+1}\eta_n[k] \\ \dfrac{f^{(n+2)}x^{n+3}}{2^{n+1}(n+3)!}+\displaystyle\sum_{k=2}^{n+2}f^{n-k+2}V_n[k] \end{pmatrix} \tag{19}$$

for $n=0,1,2,\ldots$, with

$$\begin{pmatrix} U_n[k-1] \\ \xi_n[k-1] \\ \eta_n[k-1] \\ V_n[k] \end{pmatrix}=\frac{1}{2^{n+1}(n+1)!}\begin{pmatrix} a_{11} & 0 & 0 & \partial_x^2 \\ 2\psi_x\partial_x & a_{22} & \partial_x^2 & 0 \\ 2\phi_x\partial_x & \partial_x^2 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}^{k-1} \cdot\begin{pmatrix} x^{n+1} \\ 0 \\ 0 \\ 0 \end{pmatrix},\quad k=1,2,\ldots,n+2.$$

2.3. Symmetry algebras of the SNKP system

The detailed calculations show that the generalized symmetries $\sigma_n(f)$ constitute a closed infinite dimensional Lie algebra due to the presence of the arbitrary function $f(t)$

$$[\sigma_n(f_1),\sigma_m(f_2)]=-\frac{1}{2}\sigma_{n+m-1}\cdot\left((m+1)\dot{f}_1f_2-(n+1)f_1\dot{f}_2\right), \tag{20}$$

where the dot over a function denotes the derivative of the function with respect to its variable, and the Lie product $[\ ,\ ]$ is defined as

$$[A(u),B(u)]=A'B-BA'=\frac{\partial}{\partial\varepsilon}\left[A(u+\varepsilon B(u))-B(u+\varepsilon A(u))\right]\Big|_{\varepsilon=0}.$$

In the following, we would like to list some interesting subalgebras of Eq. (20), while skipping the tedious concrete verifications.

(i) Commutative algebra In the limit case of $f_1=f_2=1$, the generalized symmetries given by Eq. (19) degenerate to the t-independent symmetries

$$\sigma_n=\frac{1}{2^{n+1}(n+1)!}\begin{pmatrix} a_{11} & 0 & 0 & \partial_x^2 \\ 2\psi_x\partial_x & a_{22} & \partial_x^2 & 0 \\ 2\phi_x\partial_x & \partial_x^2 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}^n\begin{pmatrix} x^{n+1} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which constitute a commutative algebra

$$[\sigma_n,\sigma_m]=0.$$

Such an algebra implies the commutativity of the positive flow. Thus, it is confirmed that the SNKP system is integrable in the sense of possessing infinitely many generalized symmetries.

(ii) Virasoro algebra Taking $f=t$, the generalized symmetries degenerate to the so-called τ symmetries, $\sigma_n(t)=\tau_n$, which constitute a centerless Virasoro algebra

$$[\tau_n,\tau_m]=\frac{(n-m)}{2}\tau_{n+m-1}.$$

(iii) W_∞ algebra It is remarkable that by restricting the arbitrary function f to be a Laurent polynomial ($f=t^r$ and $\sigma_n(t^r)=\sigma_n^r$, $r\neq1$), the general algebra (20) reduces to the W_∞ algebra

$$[\sigma_n^r,\sigma_m^s]=-\frac{1}{2}[r(m+1)-s(n+1)]\sigma_{n+m-1}^{r+s-1}. \tag{21}$$

Although σ_n^r represents a generator of the conformal spin n, it is still difficult to obtain higher spin generators in other types of the representations of the W_∞ algebra. Fortunately, in our case, generators for any higher conformal spins can be easily obtained from Eq. (21).

It is also noted that in addition to the symmetries $\sigma_n(f)$ in Eq. (19), there may be other types of symmetries for the SNKP system. In a similar way, one can obtain a special simple symmetry with an arbitrary function $p=p(y)$

$$Y_0(p)=\begin{pmatrix} pu_y \\ p\psi_y+\dfrac{1}{2}p_y\psi \\ p\phi_y+\dfrac{1}{2}p_y\phi \\ pv_y \end{pmatrix}. \tag{22}$$

The commutation relation between $\sigma_n(f)$ and $Y_0(p)$ is

$$[\sigma_n(f),\ Y_0(p)]=0,\quad n=0,1,2,\ldots.$$

From Eqs. (14), (15), and (22), it is obvious that the SNKP admits the following Lie point symmetries

$$X(h)=\sigma_0(h),\quad T(f)=\sigma_1(f),\quad Y(p)=Y_0(-p),$$

with $f\equiv f(t)$, $h\equiv h(t)$, and $p\equiv p(y)$. The nonzero commutation relations between these symmetries read

$$[T(f_1),\ T(f_2)]=T(f_1\dot{f}_2-\dot{f}_1f_2), \tag{23a}$$

$$[T(f),\ X(h)] = X\left(f\dot{h} - \frac{1}{2}h\dot{f}\right), \tag{23b}$$

$$[Y(p_1),\ Y(p_2)] = Y(p_1\dot{p}_2 - \dot{p}_1 p_2), \tag{23c}$$

which constitute an infinite dimensional Kac–Moody–Virasoro-type algebra typical for the integrable systems.

3. Symmetries and symmetry algebras of the (2+1)-dimensional supersymmetric MKdV system

3.1. Supersymmetric extension of the bilinear MKdV system and its nonlinear version

The supersymmetric extension of the MKdV system can be taken as

$$(D_t + D_x^3)F\cdot G = 0, \tag{24a}$$

$$\mathcal{S}_y(D_t + D_x^3)F\cdot G = 0. \tag{24b}$$

By using the transformations (4), we can write Eq. (24) in its nonlinear version

$$u_t + u_x^3 + u_{xxx} + 3u_x v_{xx} = 0, \tag{25a}$$

$$\psi_t + \psi_{xxx} + 3u_x^2\psi_x + 3v_{xx}\psi_x + 3u_x\phi_{xx} = 0, \tag{25b}$$

$$\phi_t + \phi_{xxx} + 3u_x^2\phi_x + 3v_{xx}\phi_x + 3u_x\psi_{xx} = 0, \tag{25c}$$

$$v_{yt} + v_{xxxy} + 3v_{xx}v_{xy} + 3u_x^2 v_{xy} + 3u_x u_{xxy} + 6u_x\psi_x\phi_x + 3\psi_x\psi_{xx} - 3\phi_x\phi_{xx} = 0. \tag{25d}$$

In the fermionic limit $\psi = \phi = 0$, equation (25) becomes

$$u_t + u_x^3 + u_{xxx} + 3u_x v_{xx} = 0, \tag{26a}$$

$$v_{yt} + 3v_{xx}v_{xy} + v_{xxxy} + 3u_x^2 v_{xy} + 3u_x u_{xxy} = 0, \tag{26b}$$

which is an interesting couple of the MKdV equation and the potential BLMP equation.[28] By using the bi-logarithmic transformations (7), one can convert the coupled system (26) into the bilinear MKdV system (2). Thus, this system (25) deserves the name of the supersymmetric MKdV (SMKdV) system.

3.2. Infinitely many generalized symmetries of the SMKdV system

A symmetry $\sigma = (U,\xi,\eta,V)^{\mathrm{T}}$ of the SMKdV system is the solution of its linearized system of equations

$$U_t + 3u_x^2 U_x + U_{xxx} + 3v_{xx}U_x + 3u_x V_{xx} = 0, \tag{27a}$$

$$\xi_t + \xi_{xxx} + 3u_x^2\xi_x + 6u_x\psi_x U_x + 3v_{xx}\xi_x + 3\psi_x V_{xx} + 3\phi_{xx}U_x + 3u_x\eta_{xx} = 0, \tag{27b}$$

$$\eta_t + \xi_{xxx} + 3u_x^2\eta_x + 6u_x\phi_x U_x + 3v_{xx}\eta_x + 3\phi_x V_{xx} + 3\psi_{xx}U_x + 3u_x\xi_{xx} = 0, \tag{27c}$$

$$V_{yt} + 3v_{xy}V_{xx} + 3v_{xx}V_{xy} + V_{xxxy} + 3u_x^2 V_{xy} + 6u_x v_{xy}U_x + 3u_{xxy}U_x + 3u_x U_{xxy} + 6\psi_x\phi_x U_x + 6u_x\phi_x\eta_x - 6u_x\phi_x\xi_x + 3\psi_x\xi_{xx} - 3\psi_{xx}\xi_x + 3\phi_{xx}\eta_x - 3\phi_x\eta_{xx} = 0. \tag{27d}$$

where U and V are bosonic functions, while ξ and η are fermionic functions.

Analogous to the SNKP case, U, ξ, η, and V should be expanded in the form of

$$U = \sum_{k=0}^{\infty} f^{(n-k)}U_n[k], \qquad \xi = \sum_{k=0}^{\infty} f^{(n-k)}\xi_n[k], \tag{28a}$$

$$\eta = \sum_{k=0}^{\infty} f^{(n-k)}\eta_n[k], \qquad V = \sum_{k=0}^{\infty} f^{(n-k+1)}V_n[k]. \tag{28b}$$

Substituting Eq. (28) into the symmetry definition equation (27) and collecting the coefficients of the different derivative orders of f, we obtain the overdetermined equations

$$V_{ny}[0] = 0, \quad V_n[0] = g(x) = g, \tag{29a}$$

$$U_n[0] = -3u_x V_{nxx}[0], \quad \xi_n[0] = -3\psi_x V_{nxx}[0], \tag{29b}$$

$$\eta_n[0] = -3\phi_x V_{nxx}[0], \quad V_n[1] = -3v_x V_{nxx}[0], \tag{29c}$$

$$U_n[k-1] = -\{U_{nt}[k-2] + 3u_x^2 U_{nx}[k-2] + U_{nxxx}[k-2] + 3v_{xx}U_{nx}[k-2] + 3u_x V_{nxx}[k-1]\}, \tag{29d}$$

$$\xi_n[k-1] = -\{\xi_{nt}[k-2] + \xi_{nxxx} + 3u_x^2\xi_{nx}[k-2] + 6u_x\psi_x U_{nx}[k-2] + 3v_{xx}\xi_{nx}[k-2] + 3\psi_x V_{nxx}[k-1] + 3\phi_{xx}U_{nx}[k-2] + 3u_x\eta_{nxx}[k-2]\}, \tag{29e}$$

$$\eta_n[k-1] = -\{\eta_{nt}[k-2] + \eta_{nxxx} + 3u_x^2\eta_{nx}[k-2] + 6u_x\phi_x U_{nx}[k-2] + 3v_{xx}\eta_{nx}[k-2] + 3\phi_x V_{nxx}[k-1] + 3\psi_{xx}U_{nx}[k-2] + 3u_x\xi_{nxx}[k-2]\}, \tag{29f}$$

$$\eta_n[k-1] = -\{\eta_{nt}[k-2] + \eta_{nxxx} + 3u_x^2\eta_{nx}[k-2] + 6u_x\phi_x U_{nx}[k-2] + 3v_{xx}\eta_{nx}[k-2] + 3\phi_x V_{nxx}[k-1] + 3\psi_{xx}U_{nx}[k-2] + 3u_x\xi_{nxx}[k-2]\}, \tag{29g}$$

$$\begin{aligned}V_{ny}[k] = -\{&V_{nyt}[k-1] + 3v_{xy}V_{nxx}[k-1] + 3v_{xx}V_{nxy}[k-1] + V_{nxxxy}[k-1]\\ &+ 3u_x^2V_{nxy}[k-1] + 6u_xv_{xy}U_{nx}[k-2] + 3u_{xxy}U_{nx}[k-2] + 3u_xU_{nxxy}[k-2]\\ &+ 6\psi_x\phi_xU_{nx}[k-2] - 6u_x\phi_x\xi_{nx}[k-2] + 6u_x\psi_x\eta_{nx}[k-2] - 3\psi_{xx}\xi_{nx}[k-2]\\ &+ 3\psi_x\xi_{nxx}[k-2] + 3\phi_{xx}\eta_{nx}[k-2] - 3\phi_x\eta_{nxx}[k-2]\}\end{aligned} \tag{29h}$$

for $k = 2,3,\ldots,n+1$.

From Eq. (29), it is clear that $U_n[k-2]$, $\xi_n[k-2]$, $\eta_n[k-2]$, and $V_n[k-1]$ can be solved recursively for any given g, namely,

$$\begin{aligned}\begin{pmatrix} U_n[k-1]\\ \xi_n[k-1]\\ \eta_n[k-1]\\ V_n[k]\end{pmatrix} &= -\begin{pmatrix} A_{11} & 0 & 0 & 3u_x\partial_x^2\\ A_{21} & A_{11} & 3u_x\partial_x^2 & 3\psi_x\partial_x^2\\ A_{31} & 3u_x\partial_x^2 & A_{11} & 3\phi_x\partial_x^2\\ A_{41} & A_{42} & A_{43} & A_{44}\end{pmatrix}\begin{pmatrix} U_n[k-2]\\ \xi_n[k-2]\\ \eta_n[k-2]\\ V_n[k-1]\end{pmatrix}\\ &= 3\begin{pmatrix} A_{11} & 0 & 0 & 3u_x\partial_x^2\\ A_{21} & A_{11} & 3u_x\partial_x^2 & 3\psi_x\partial_x^2\\ A_{31} & 3u_x\partial_x^2 & A_{11} & 3\phi_x\partial_x^2\\ A_{41} & A_{42} & A_{43} & A_{44}\end{pmatrix}^{k-1} g_{xx}\begin{pmatrix} u_x\\ \psi_x\\ \phi_x\\ v_x\end{pmatrix},\end{aligned} \tag{30}$$

for $k = 1,2,\ldots,n+2$, and the operators A_{11}, A_{21}, A_{31}, A_{41}, A_{42}, A_{43}, and A_{44} are defined as

$$\begin{aligned}&A_{11} = \partial_t + \partial_x^3 + 3u_x^2\partial_x + 3v_{xx}\partial_x, \quad A_{21} = 6u_x\psi_x\partial_x + 3\phi_{xx}\partial_x, \quad A_{31} = 6u_x\phi_x\partial_x + 3\psi_{xx}\partial_x,\\ &A_{41} = 3\partial_y^{-1}(2u_xv_{xy}\partial_x + u_{xxy}\partial_x + u_x\partial_x^2\partial_y + 2\psi_x\phi_x\partial_x), \quad A_{42} = -3\partial_y^{-1}(2u_x\phi_x\partial_x + \psi_{xx}\partial_x - 3\psi_x\partial_x^2),\\ &A_{43} = 3\partial_y^{-1}(2u_x\psi_x\partial_x + \phi_{xx}\partial_x - 3\phi_x\partial_x^2), \quad A_{44} = \partial_t + \partial_x^3 + 3\partial_y^{-1}(v_{xy}\partial_x^2 + v_{xx}\partial_x\partial_y + 3u_x^2\partial_x\partial_y).\end{aligned}$$

Essentially, to guarantee the existence of the formal series symmetries, we need the truncated condition

$$\begin{pmatrix} A_{11} & 0 & 0 & 3u_x\partial_x^2\\ A_{21} & A_{11} & 3u_x\partial_x^2 & 3\psi_x\partial_x^2\\ A_{31} & 3u_x\partial_x^2 & A_{11} & 3\phi_x\partial_x^2\\ A_{41} & A_{42} & A_{43} & A_{44}\end{pmatrix}\begin{pmatrix} U_n[n]\\ \xi_n[n]\\ \eta_n[n]\\ V_n[n+1]\end{pmatrix} = 0, \tag{31}$$

which means $(U_n[n],\xi_n[n],\xi_n[n],V_n[n+1])^{\mathrm{T}}$ also satisfies the symmetry definition equation (27).

With the recursion relation (30), we can deduce the first three generalized symmetries by choosing $g = x^2/6$, $x^3/54$, and $x^4/648$, respectively,

$$K_0(f) = -\begin{pmatrix} fu_x\\ f\psi_x\\ f\phi_x\\ -\frac{1}{6}f_tx^2 + fv_x\end{pmatrix}, \tag{32}$$

$$K_1(f) = -\frac{1}{3}\begin{pmatrix} f_txu_x + 3fu_t\\ f_tx\psi_x + 3f\psi_t\\ f_tx\phi_x + 3f\phi_t\\ -\frac{1}{18}f_{tt}x^3 + f_txv_x + 3fv_t\end{pmatrix}, \tag{33}$$

$$K_2(f) = \begin{pmatrix} U_2\\ \xi_2\\ \eta_2\\ V_2\end{pmatrix} \tag{34}$$

with

$$\begin{aligned}U_2 = &-\frac{1}{18}f_{tt}x^2u_x + \frac{1}{3}f_t[x(u_x^3 + u_{xxx} + 3u_xv_{xx}) + u_{xx} + u_xv_x]\\ &- f\Big\{5u_x\partial_y^{-1}[2(u_x\psi_x\phi_x)_x + (u_x^2v_{xy})_x + \psi_x\psi_{xxx} - \phi_x\phi_{xxx} + u_xu_{xxxy}\\ &+ v_{xxx}v_{xy}] + 5u_x\left[\frac{3}{2}(u_{xx}^2 + v_{xx}^2) + u_x(u_xv_{xx} + u_{xxx}) + v_{xxxx}\right] + 5(u_{xx}v_{xx})_x + u_x^5 + u_{xxxxx}\Big\},\\ \xi_2 = &-\frac{1}{18}f_{tt}x^2\psi_x + \frac{1}{3}f_t[x(3u_x^2\psi_x + \psi_{xxx} + 3v_{xx}\psi_x + 3u_x\phi_{xx}) + \psi_{xx} + v_x\psi_x + u_x\phi_x]\\ &- f\Big\{5\psi_x\partial_y^{-1}[2(u_x\psi_x\phi_x)_x + (u_x^2v_{xy})_x + \psi_x\psi_{xxx} - \phi_x\phi_{xxx} + u_xu_{xxxy} + v_{xxx}v_{xy}]\\ &+ 5\psi_x\left[\frac{3}{2}(u_{xx}^2 + v_{xx}^2) + 3u_x^2v_{xx} + u_x^4 + 2u_xu_{xxx} + v_{xxxx}\right]\end{aligned}$$

$$+5[(u_{xx}\phi_{xx}+v_{xx}\psi_{xx})_x+3u_x(v_{xx}\phi_{xx}+u_{xx}\psi_{xx})+2u_x^2(\psi_{xx}+u_x\phi_x)_x+u_x(\phi_{xxxx}+v_{xxx}\phi_x)]+\psi_{xxxxx}\Big\},$$

$$\eta_2=-\frac{1}{18}f_{tt}x^2\phi_x+\frac{1}{3}f_t[x(3u_x^2\phi_x+\phi_{xxx}+3v_{xx}\phi_x+3u_x\psi_{xx})+\phi_{xx}+v_x\phi_x$$
$$+u_x\psi_x]-f\Big\{5\phi_x\partial_y^{-1}[2(u_x\phi_x\psi_x)_x+(u_x^2v_{xy})_x+\psi_x\psi_{xxx}-\phi_x\phi_{xxx}$$
$$+u_xu_{xxxy}+v_{xxx}v_{xy}]+5\phi_x\left[\frac{3}{2}(u_{xx}^2+v_{xx}^2)+3u_x^2v_{xx}+u_x^4+2u_xu_{xxx}+v_{xxxx}\right]$$
$$+5[(u_{xx}\psi_{xx}+v_{xx}\phi_{xx})_x+3u_x(u_{xx}\phi_{xx}+v_{xx}\psi_{xx})+2u_x^2(\phi_{xx}+u_x\psi_x)_x+u_x(\psi_{xxxx}+v_{xxx}\psi_x)]+\phi_{xxxxx}\Big\},$$

$$V_2=\frac{1}{648}f_{ttt}x^4-\frac{1}{18}f_{tt}x^2v_x+\frac{1}{3}f_t\Big[xv_{xxx}+3x\partial_y^{-1}(v_{xx}v_{xy}+u_x^2v_{xy}+u_xu_{xxy}$$
$$+2u_x\psi_x\phi_x+\psi_x\psi_{xx}-\phi_x\phi_{xx})+\frac{1}{2}(u_x^2+v_x^2)+v_{xx}\Big]$$
$$-f\Big\{5v_x\partial_y^{-1}[2(u_x\phi_x\psi_x)_x+(u_x^2v_{xy})_x+\psi_x\psi_{xxx}-\phi_x\phi_{xxx}+u_xu_{xxxy}+v_{xxx}v_{xy}]$$
$$+5\partial_y^{-1}\Big[2u_x^2(u_xu_{xy})_x-v_x(u_xu_{xxxy}+v_{xxx}v_{xy})+\frac{3}{2}v_{xy}(u_{xx}^2+v_{xx}^2)$$
$$+3u_{xx}(\psi_x\phi_x)_x+3u_x(\psi_{xxx}\phi_x+\psi_x\phi_{xxx})-v_x(\psi_x\psi_{xxx}-\phi_x\phi_{xxx}),$$
$$+(4u_x^2+3v_{xx})(\psi_x\psi_{xx}-\phi_x\phi_{xx})+u_xu_{xxxxy}+v_{xxxx}v_{xy}+3u_x(u_{xx}v_{xx})_y$$
$$+(u_x^4+2u_xu_{xxx})v_{xy}+2(2u_x^3+u_{xxx}+3u_xv_{xx})\psi_x\phi_x+u_x^2(3v_{xx}v_{xy}-v_xv_{xxy})$$
$$-u_x(3u_{xy}v_{xx}+2u_{xx}v_xv_{xxy})-2u_x\psi_x(u_x\phi_x)_x-2u_xv_x\psi_{xx}\phi_x\Big]$$
$$+5(u_{xx}u_{xxx}+v_{xx}v_{xxx}+2u_x^2v_{xxx})+v_{xxxxx}\Big\}.$$

Similarly, with the help of the dimensional analysis, the only possibility of g is identified as

$$g=\frac{x^{n+2}}{3^{n+1}(n+2)!}. \tag{35}$$

Then the substitution of Eq. (35) into Eq. (28) leads to the symmetries of the SMKdV system

$$K_n(f)=\begin{pmatrix}\sum\limits_{k=0}^{n}f^{(n-k)}U_n[k]\\ \sum\limits_{k=0}^{n}f^{(n-k)}\xi_n[k]\\ \sum\limits_{k=0}^{n}f^{(n-k)}\eta_n[k]\\ \dfrac{f^{n+1}x^{n+2}}{3^{n+1}(n+2)!}+\sum\limits_{k=1}^{n+1}f^{(n-k+1)}V_n[k]\end{pmatrix} \tag{36}$$

for $n=0,1,2,\ldots$, with

$$\begin{pmatrix}U_n[k-1]\\ \xi_n[k-1]\\ \eta_n[k-1]\\ V_n[k]\end{pmatrix}=\frac{1}{3^nn!}\begin{pmatrix}A_{11}&0&0&3u_x\partial_x^2\\ A_{21}&A_{11}&3u_x\partial_x^2&3\psi_x\partial_x^2\\ A_{31}&3u_x\partial_x^2&A_{11}&3\phi_x\partial_x^2\\ A_{41}&A_{42}&A_{43}&A_{44}\end{pmatrix}^{k-1}$$
$$\cdot x^n\begin{pmatrix}u_x\\ \psi_x\\ \phi_x\\ v_x\end{pmatrix},\quad k=1,2,\ldots,n+2.$$

3.3. Symmetry algebras of the SMKdV system

Detailed calculations show that the generalized symmetries $K_n(f)$ constitute a closed infinite dimensional Lie algebra

$$[K_n(f_1),\ K_m(f_2)]=\frac{1}{3}K_{n+m-1}$$
$$\cdot\big[(2m+1)\dot{f}_1f_2-(2n+1)f_1\dot{f}_2\big]. \tag{37}$$

Now, we list some interesting subalgebras of Eq. (37) to see the commutation relation clearly.

(i) Commutative algebra In the limit case $f_1=f_2=1$, we have the t-independent symmetries

$$K_n=\frac{1}{3^nn!}\begin{pmatrix}A_{11}&0&0&3u_x\partial_x^2\\ A_{21}&A_{11}&3u_x\partial_x^2&3\psi_x\partial_x^2\\ A_{31}&3u_x\partial_x^2&A_{11}&3\phi_x\partial_x^2\\ A_{41}&A_{42}&A_{43}&A_{44}\end{pmatrix}^nx^n\begin{pmatrix}u_x\\ \psi_x\\ \phi_x\\ v_x\end{pmatrix}, \tag{38}$$

which constitute a commutative algebra

$$[K_n,K_m]=0. \tag{39}$$

Therefore, it is confirmed that the SMKdV system is integrable in the sense of possessing infinitely many generalized symmetries.

(ii) Virasoro algebra In this case, we take $f=t$ and thus $K(t)=\tau_n$, which constitute a centerless Virasoro algebra

$$[K_n(t),K_m(t)]=\frac{2(m-n)}{3}K_{n+m-1}(t). \tag{40}$$

(iii) W_∞ algebra In the case of $f=t^r$ $(r\neq 1)$, $K_n(t^r)\equiv K_n^r$ constitutes the W_∞ algebra

$$\begin{aligned}&[K_n^r,\ K_m^s]=\frac{1}{3}[r(2m+1)-s(2n+1)]K_{n+m-1}^{r+s-1},\\&K_n^r\equiv K_n(t^r).\end{aligned} \tag{41}$$

Remarkably, the SMKdV system share the same special simple symmetry with an arbitrary function $p\equiv p(y)$ as the SNKP system

$$Y_0(p)=\begin{pmatrix}pu_y\\ p\psi_y+\frac{1}{2}p_y\psi\\ p\phi_y+\frac{1}{2}p_y\phi\\ pv_y\end{pmatrix}. \tag{42}$$

Direct calculations show that the commutation relation between $K_n(f)$ and $Y_0(g)$ is

$$[K_n(f),\ Y_0(p)]=0,\quad n=0,1,2,\ldots. \tag{43}$$

Interestingly, there also exists an infinite dimensional Kac-Moody-Virasora type subalgebra obtained from Eqs. (32), (33), and (34) as

$$X(h)=K_0(-f),\quad T(f)=K_1(-f),\quad Y(p)=Y_0(-p), \tag{44}$$

where $f\equiv f(t)$, $h\equiv h(t)$, $p\equiv p(y)$, and the nonzero commutation relations read

$$\begin{aligned}&[T(f_1),\ T(f_2)]=T(f_1\dot{f}_2-\dot{f}_1f_2),\\&[T(f),\ X(h)]=X\left(f\dot{h}-\frac{1}{3}h\dot{f}\right),\\&[Y(p_1),\ Y(p_2)]=Y(p_1\dot{p}_2-\dot{p}_1p_2).\end{aligned}$$

4. Summary and discussion

In this paper, we have presented the supersymmetric extensions of two integrable systems, the NKP system and the (2+1)-dimensional MKdV system, from the Hirota formalism in the superspace. By extending the formal series symmetry theory to the supersymmetric framework, it is confirmed that both models admit a set of infinitely many generalized symmetries with an arbitrary function $f(t)$. Due to the presence of the arbitrary function, the generalized symmetries of both models constitute a generalization of the W_∞ algebra.

Interestingly, it is found that the first one of the positive flows of the SNKP system (taking $f=1$ in Eq. (16)) is another (2+1)-dimensional supersymmetric extension of the MKdV equation in the potential form

$$\begin{aligned}u_{t_1}&=u_{xxx}+u_x^3+\frac{3}{2}u_xv_{xx}+\frac{3}{2}\partial_y^{-1}(u_xv_{xy}+\psi_x\phi_x)_x,\\ \psi_{t_1}&=\psi_{xxx}+\frac{3}{2}(2u_x^2+v_{xx})\psi_x+3u_x\phi_{xx}+\frac{3}{2}u_{xx}\phi_x,\\ \phi_{t_1}&=\phi_{xxx}+\frac{3}{2}(2u_x^2+v_{xx})\phi_x+3u_x\psi_{xx}+\frac{3}{2}u_{xx}\psi_x,\\ v_{t_1}&=v_{xxx}+\frac{3}{2}\partial_y^{-1}(v_{xx}v_{xy}+2u_x^2v_{xy}+2u_xu_{xxy}\\&\quad+u_{xx}u_{xy}+4u_x\psi_x\phi_x+\psi_x\psi_{xx}-\phi_x\phi_{xx}).\end{aligned}$$

Together with our previous work,[29,30] we are quite sure that the symmetry algebras of this new (2+1)-dimensional supersymmetric MKdV system is isomorphic to the SMKdV system (25).

Based on our work, we would like to put forward a novel hypothesis on a series of (2+1)-dimensional supersymmetric integrable systems

$$\begin{aligned}&(D_t+D_x^n)F\cdot G=0,\\&\mathcal{S}_y(D_t+D_x^n)F\cdot G=0\end{aligned}$$

with an integer $n\geq 2$. It is noted that $n=2$ and $n=3$ correspond to the SNKP system (3) and the SMKdV system (24), respectively. Nonetheless, it is difficult to prove the integrable property in the symmetry sense for $n>3$, due to the difficulty in determining the function of integration. Therefore, the integrability should be proved via other means, such as the criteria that the existence of three super-soliton solutions. It is hoped that this hypothesis may establish a straightforward way for the supersymmetric extensions of the integrable systems in high dimensions.

Appendix A: Derivation of Eq. (5) and Eq. (25)

Let us expand Eq. (3) as

$$\begin{aligned}&F_{xx}G-2F_xG_x+FG_{xx}+F_tG-FG_t=0,\\&F_{xx\theta}G-F_{xx}G_\theta-2F_{x\theta}G_x+2F_xG_{x\theta}+F_\theta G_{xx}-FG_\theta+F_{t\theta}G-F_tG_\theta-F_\theta G_t+FG_{t\theta}=-\theta(F_{xxy}G-F_{xx}G_y-2F_{xy}G_x\\&\quad+2F_xG_{xy}+F_yG_{xx}-FG_y+F_{ty}G-F_tG_y-F_yG_t+FG_{ty}).\end{aligned} \tag{A1}$$

It is noted that as F and G are both bosonic fields, there are no anti-commutation relations in Eq. (A1). Introducing F and G in the following exponential forms

$$F = \mathrm{e}^{(p-q)/2}, \quad G = \mathrm{e}^{(p+q)/2}, \tag{A2}$$

with p and q being bosonic functions of (x,y,t,θ), we can readily obtain

$$\begin{aligned} &q_t - q_x^2 - p_{xx} = 0, \\ &p_{t\theta} - 2q_x p_{x\theta} - q_{xx\theta} + \theta(p_{yt} - 2q_x p_{xy} - q_{xxy}) = 0. \end{aligned} \tag{A3}$$

By substituting $p = v + \theta\phi$ and $q = u + \theta\psi$ into Eq. (A3) and then collecting the coefficients of θ, equation (5) is derived. Following the same procedure, one can also obtain the (2+1)-dimensional supersymmetric MKdV system (25).

References

[1] Di Vecchia P and Ferrara S 1977 *Nucl. Phys. B* **130** 93
[2] Mathieu P 1988 *J. Math. Phys.* **29** 2499
[3] Manin Yu I and Radul A O 1985 *Commun. Math. Phys.* **98** 65
[4] Delisle L and Mosaddeghi M 2013 *J. Phys. A: Math. Theor.* **46** 115203
[5] Wang J Y, Tang X Y, Liang Z F and Lou S Y 2015 *Chin. Phys. B* **24** 050202
[6] Henneaux M and Rey S J 2010 *J. High Energy Phys.* **12** 007
[7] Lou S Y 1993 *J. Phys. A: Math. Gen.* **26** 4387
[8] Ghosh S and Paul S K 1995 *Phys. Lett. B* **341** 293
[9] Lou S Y 1993 *Phys. Rev. Lett.* **71** 4099
[10] Lou S Y 1994 *J. Phys. A: Math. Gen.* **27** 3235
[11] Lou S Y 1998 *Phys. Scr.* **57** 481
[12] Gegenhasi, Hu X B and Wang H Y 2007 *Phys. Lett. A* **360** 439
[13] Fan E G 2011 *Stud. Appl. Math.* **127** 284
[14] Jia M, Li J H and Lou S Y 2008 *J. Phys. A: Math. Theor.* **41** 275204
[15] Ito M 1980 *J. Phys. Soc. Jpn.* **49** 771
[16] Hietarinta J 1987 *J. Math. Phys.* **28** 2094
[17] Li J H and Lou S Y 2008 *Chin. Phys. B* **17** 747
[18] Peng L Z, Zhang M X, Gegenhasi and Liu Q P 2008 *Phys. Lett. A* **372** 4197
[19] Hietarinta J 1987 *J. Math. Phys.* **28** 1732
[20] Hietarinta J 1987 *J. Math. Phys.* **28** 2586
[21] Hietarinta J 1988 *J. Math. Phys.* **29** 628
[22] Sarma D 2004 *Nucl. Phys. B* **681** 351
[23] Liu Q P and Yang X X 2006 *Phys. Lett. A* **351** 131
[24] Fan E G and Hon Y C 2012 *J. Math. Phys.* **53** 013503
[25] Xue L L and Liu Q P 2013 *Phys. Lett. A* **377** 828
[26] Brunelli J C and Das A 1994 *Phys. Lett. B* **337** 303
[27] Wang J Y, Liang Z F and Tang X Y 2014 *Phys. Scr.* **89** 025201
[28] Tang X Y, Li J, Liang Z F and Wang J Y 2014 *Phys. Lett. A* **378** 1439
[29] Wang J Y, Yu J and Lou S Y 2010 *Commun. Theor. Phys.* **53** 999
[30] Wang J Y, Hu H W and Yu J 2011 *J. Math. Phys.* **52** 103704