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Citation: Chin. Phys. B . 2018, 27(4): 040202. **doi:** 10.1088/1674-1056/27/4/040202

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Integrability classification and exact solutions to generalized variable-coefficient nonlinear evolution equation*

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(Received 1 December 2017; revised manuscript received 1 January 2018; published online 29 March 2018)

This paper is concerned with the generalized variable-coefficient nonlinear evolution equation (vc-NLEE). The complete integrability classification is presented, and the integrable conditions for the generalized variable-coefficient equations are obtained by the Painlevé analysis. Then, the exact explicit solutions to these vc-NLEEs are investigated by the truncated expansion method, and the Lax pairs (LP) of the vc-NLEEs are constructed in terms of the integrable conditions.

Keywords: Painlevé test, integrability classification, Lax pair, truncated expansion, exact solution

PACS: 02.30.Jr, 02.30.Ik **DOI:** 10.1088/1674-1056/27/4/040202

1. Introduction

The Painlevé test could to provide useful information for the identification of completely integrable PDEs, which are of importance in nonlinear theory and physical applications. In addition, they also appear to yield other integrable properties such as Lax pairs (LPs), Bäcklund transformations (BTs), rational solutions, and so on (see e.g., Refs. [1]–[3] and the references therein).

On the other hand, to find exact solutions to the nonlinear evolution equations (NLEEs) is always one of the themes in mathematics and physics. In the past few decades, there are noticeable progresses in this field, and various methods have been developed, such as the inverse scattering method,[4,5] Darboux and Bäcklund transformations,[6–8] Lie symmetry analysis,[9–13] dynamical system method,[14,15] trial function method (direct method),[11,16] Painlevé test,[1–3,12,17–19] and so on. In the present paper, we consider the generalized variable-coefficient nonlinear evolution equation (vc-NLEE) as follows:

$$u_t + f(t)u^p u_x + g(t)u_{xx} + h(t)u_{xxx} + l(t)u = 0, \qquad (1)$$

where $u = u(x,t)$ denotes the unknown function of space variable x and time t, the exponent p is a positive integer, the coefficient functions $f = f(t)$, $g = g(t)$, $h = h(t)$, and $l = l(t)$ are all arbitrary analytic functions, $f = f(t) \neq 0$ is assumed throughout this paper (otherwise this equation is linear and excluded from our discussion).

This equation is also called the variable-coefficient generalized Burgers–KdV type equation sometimes. Such vc-NLEE plays a significant role in mathematical physics, integrable system, and physical applications, etc.[17,18] However, to obtain the exact solutions and integrability of the variable-coefficient equation in generalized form are more involved. By the equivalent transformation, the variable-coefficient equation can be transformed into constant-coefficient equation under some conditions, so the exact solutions are obtained accordingly.[20] This is a useful method for dealing with exact solutions to the vc-NLEEs, but the integrability of the vc-NLEE may be different from the constant-coefficient equation. In Ref. [12], the Painlevé test and Lie group analysis are performed for the variable-coefficient nonlinear evolution equations (including the cylindrical KdV type equations), but the equation is not the generalized form and it differs greatly from Eq. (1) in the present paper.

The main purpose of this paper is to investigate the complete integrability classification of the generalized vc-BKdV equation by Painlevé test. The remainder of this paper is organized as follows. In Section 2, the Painlevé test is performed, the complete Painlevé integrability classification of Eq. (1) is given, and the integrable conditions for the vc-NLEEs are obtained accordingly. Then, the exact explicit solutions to the equations are investigated by the truncated expansion method, and the Lax pairs of the vc-NLEEs are constructed in terms of the integrable conditions. Finally, the conclusion and some remarks will be given in Section 3.

2. Painlevé test for vc-NLEEs

In this section, we employ the Kruskal's simplified method[1–3] to carry out the Painlevé test for Eq. (1). Then, the integrable conditions, exact solutions, and other properties of the vc-NLEEs are obtained.

*Project supported by the National Natural Science Foundation of China (Grant No. 11171041) and the High-Level Personnel Foundation of Liaocheng University (Grant No. 31805).

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First of all, we assume that

$$u=\phi^{-\rho}\sum_{j=0}^{\infty}u_j\phi^j, \tag{2}$$

where $\phi=x+\psi(t)$, $u_j=u_j(t)$ $(j=0,1,2,\ldots)$ are analytic functions in a neighborhood of the noncharacteristic singular manifold, $u_0\neq 0$, ρ is a positive integer.

Then, by using the leading order analysis, we obtain the following results:

(i) $h=h(t)=0$. In this case, equation (1) reduces to the variable-coefficient generalized Burgers' equation (vc-GBE)

$$u_t+f(t)u^p u_x+g(t)u_{xx}+l(t)u=0, \tag{3}$$

where $fg\neq 0$. In this case, we have

$$\rho=\frac{1}{p} \tag{4}$$

with p being a positive integer, we choose $p=1$, that is $\rho=1$ and $u_0=2g/f\neq 0$. So the condition $u_0\neq 0$ is satisfied.

(ii) $h=h(t)\neq 0$. In the general case, we have

$$\rho=\frac{2}{p}. \tag{5}$$

In view of p being a positive integer, we choose $p=1$ or $p=2$, respectively.

When $p=1$, we have $\rho=2$ and $u_0=-12h/f\neq 0$. So the condition $u_0\neq 0$ is satisfied.

When $p=2$, we have $\rho=1$ and $u_0^2=-6h/f\neq 0$. So the condition $u_0\neq 0$ is satisfied also.

Summarizing the above discussion, we obtain a necessary condition of integrability as follows:

Theorem 1 Suppose that $f=f(t)\neq 0$ is an arbitrary analytic function, p is a positive integer, $g=g(t)$, $h=h(t)$, and $l=l(t)$ are arbitrary analytic functions.

(i) If equation (1) with $h=0$ and $g\neq 0$ is integrable, then $p=1$;

(ii) If equation (1) with $h\neq 0$ is integrable, then $p=1$ or $p=2$. □

In what follows, under the integrable conditions, we perform the Painlevé analysis completely, so the Painlevé integrability classification of Eq. (1) is obtained.

2.1. Painlevé test for Eq. (1) in the case $h=h(t)=0$

In this case, equation (1) becomes Eq. (3). Furthermore, if it is integrable, then $p=1$.

Thus, substituting Eq. (2) into Eq. (3), we have

$$j=0,\quad u_0=\frac{2g}{f}, \tag{6a}$$

$$j=1,\quad u_1=-\frac{1}{f}\psi', \tag{6b}$$

$$j=2,\quad u_0'+lu_0=0, \tag{6c}$$

$$j=3,\quad u_3=-\frac{1}{4g}(u_1'+lu_1). \tag{6d}$$

By using Eqs. (6a), (6b), and (6d), we can obtain u_j $(j=0,1,3)$ in a unique manner. But from Eq. (6c), we cannot obtain u_2 definitely. In general, we have the recursion relations for Eq. (3) as follows:

$$(j+1)(j-2)gu_j=-u_{j-2}'-lu_{j-2}-(j-2)u_{j-1}\psi'-f\sum_{k=1}^{j-1}(j-k-1)u_ku_{j-k} \tag{7}$$

for all $j\geq 2$.

In view of Eq. (7), we can see that the resonances occur at $j=-1,2$. In particular, if $j=2$, then from Eq. (7) we have Eq. (6c), that is, u_2 is arbitrary.

Thus, let $u_2=0$, the other coefficients u_j $(j=3,4,\ldots)$ can be determined successively from Eq. (7) in a unique manner. This implies that for Eq. (3), there exists a Laurent series (2) with the coefficients given by Eq. (7). Moreover, setting $u_3=0$, that is,

$$u_1'+lu_1=0. \tag{8}$$

Then, by the induction method, it is easy to see that $u_j=0$, for all $j=4,5,\ldots$.

So, we can say that equation (3) possesses the Painlevé property (PP) under the conditions $p=1$ and Eqs. (6c).

Thus, we have the following result.

Theorem 2 The vc-GBE (3) is Painlevé-integrable if and only if $p=1$ and equation (6c) are satisfied. □

Furthermore, the complete classification in terms of Eq. (6c) is as follows:

Case 1 $l=l(t)=0$. In this case, from Eq. (6c), we have $u_0'=0$, that is, $u_0\neq 0$ is a constant.

Substituting it into Eq. (6a), we obtain $f=2c_1g$ ($c_1=1/u_0\neq 0$ is an arbitrary constant).

On the other hand, from Eq. (2) we have

$$u=u_0\phi^{-1}+u_1 \tag{9}$$

is a solution of Eq. (3), where $p=1$, equations (6c) and (8) are satisfied.

Then, from Eq. (8) and $l=0$, we obtain $u_1'=0$, that is, $u_1=c_2$ is a constant. Thus, from Eq. (6b), we have $\psi'=-c_2f$, that is $\psi(t)=-c_2\int f(t)\mathrm{d}t+c_3$. Substituting it into Eq. (9),

we obtain the exact solution of Eq. (3) as follows:

$$u(x,t)=\frac{1}{c_1\left(x-c_2\int f(t)\,\mathrm{d}t+c_3\right)}+c_2, \tag{10}$$

where $c_1\neq 0$, and c_2, c_3 are arbitrary constants.

Next, we will employ the AKNS approach[4–6,12,21] to construct the Lax pair (LP) of Eq. (1) in this case. In fact, the linear eigenvalue problems can be expressed as

$$\Phi_x=M\Phi,\quad \Phi_t=N\Phi, \tag{11}$$

where $\Phi=(\phi_1,\phi_2)^{\mathrm{T}}$, T denotes the transpose of the matrix, while M and N are presented in the following forms

$$M=\begin{pmatrix}\lambda & q(x,t)\\ r(x,t) & -\lambda\end{pmatrix},\quad N=\begin{pmatrix}A(x,t,\lambda) & B(x,t,\lambda)\\ C(x,t,\lambda) & -A(x,t,\lambda)\end{pmatrix}, \tag{12}$$

where

$$\begin{aligned}
&q(x,t)=\exp\left[\int u(x,t)\,\mathrm{d}x\right],\\
&r(x,t)=u_x\exp\left[-\int u(x,t)\,\mathrm{d}x\right],\\
&A(x,t,\lambda)=2g\lambda^2-gu_x,\\
&B(x,t,\lambda)=g\exp\left[\int u(x,t)\,\mathrm{d}x\right](2\lambda+u),\\
&C(x,t,\lambda)=\exp\left[-\int u(x,t)\,\mathrm{d}x\right]\left[2gu_x\lambda-gu_{xx}+guu_x\right],
\end{aligned}$$

here λ is a parameter independent of x and t, $u=u(x,t)$ denotes the unknown function, the function $g=g(t)$ is an arbitrary analytic function.

It is easy to prove that the compatibility condition $M_t-N_x+[M,N]=0$ gives rise to Eq. (1), which can show that equation (11) is the Lax pair of the vc-KdV types of Eq (1) in the Case 1 (where $c_1=-1$).

Summarizing the above discussion, we have

Corollary 1 Under the condition $l=l(t)=0$, the vc-GBE (3) is integrable if and only if $p=1$ and $f=2c_1g$. In this case, this equation has an exact solution (10). Moreover, the Lax pair of the equation is given by Eqs. (11) and (12) (for $c_1=-1$, i.e., $f=-2g$). □

Case 2 $l=l(t)\neq 0$. In this case, from Eq. (6c), we have

$$u_0=2c_1\exp\left[-\int l(t)\,\mathrm{d}t\right]$$

with $c_1\neq 0$ being a constant.

Substituting it into Eq. (6a), we obtain

$$g=c_1f\exp\left[-\int l(t)\,\mathrm{d}t\right].$$

Then, from Eq. (8), we obtain

$$u_1=c_2\exp\left[-\int l(t)\,\mathrm{d}t\right].$$

Thus, from Eq. (6b), we have

$$\psi'=-c_2f\exp\left[-\int l(t)\,\mathrm{d}t\right],$$

that is

$$\psi(t)=-c_2\int f(t)\exp\left[-\int l(t)\,\mathrm{d}t\right]\mathrm{d}t+c_3.$$

Substituting it into Eq. (9), we obtain the exact solution to Eq. (3) as follows:

$$u(x,t)=\exp\left[-\int l(t)\,\mathrm{d}t\right]\left\{\frac{2c_1}{x-c_2\int f(t)\exp\left[-\int l(t)\,\mathrm{d}t\right]\mathrm{d}t+c_3}+c_2\right\}, \tag{13}$$

where $c_1\neq 0$ and c_2, c_3 are arbitrary constants.

To sum up, we have

Corollary 2 If $l=l(t)\neq 0$, then the vc-GBE (3) is integrable if and only if $p=1$ and

$$g=c_1f\exp\left[-\int l(t)\,\mathrm{d}t\right].$$

In this case, this equation has the exact solution (13). □

2.2. Painlevé test for Eq. (1) in the case $h=h(t)\neq 0$

From Eq. (5), we have to discuss the following two cases, respectively.

Case I $p=1$. In this case, equation (1) becomes the following vc-NLEE:

$$u_t+f(t)uu_x+g(t)u_{xx}+h(t)u_{xxx}+l(t)u=0, \tag{14}$$

where $fh\neq 0$, g and l are arbitrary analytic functions of t.

Thus, by substituting Eq. (2) into Eq. (14), we have

$$j=0,\quad u_0=-\frac{12h}{f}, \tag{15a}$$

$$j=1,\quad u_1=\frac{12g}{5f}, \tag{15b}$$

$$j=2,\quad u_2=\frac{g^2}{5fh}-\frac{1}{f}\psi', \tag{15c}$$

$$j=3,\quad u_3=\frac{g^3}{25fh^2}-\frac{1}{12h}(u_0'+lu_0), \tag{15d}$$

$$j=4,\quad u_1'+lu_1=0, \tag{15e}$$

$$j=5,\quad u_5=\frac{1}{6h}(u_2'+lu_2)+\frac{g^2}{30h}u_3+\frac{11g}{15h}u_4, \tag{15f}$$

$$j=6,\quad u_3'+lu_3+fu_3^2+\frac{2g^2}{5h}u_4+\frac{54}{5}gu_5=0. \tag{15g}$$

From Eqs. (15a), (15b), (15c), and (15d), we can obtain u_j $(j=0,1,2,3)$ in a unique manner. But from Eqs. (15e) and (15g), we cannot obtain u_4 and u_6 definitely. In general, we have the recursion relations for Eq. (14) as follows:

$$(j+1)(j-4)(j-6)hu_j=-u_{j-3}'-lu_{j-3}-(j-4)u_{j-2}\psi'-f\sum_{k=1}^{j-1}(j-k-2)u_ku_{j-k}-(j-3)(j-4)gu_{j-1} \tag{16}$$

for all $j\geq 3$.

From (16), we can obtain all the coefficients u_j $(j=3,4,\ldots)$ for Eq. (2) except u_4 and u_6. For example, if $j=3,5$, then the coefficients u_3 and u_5 are given by Eqs. (15d) and (15f), respectively. If $j=4$, and from Eq. (15e), then we have

$$0u_4=-u_1'-lu_1=0. \tag{17}$$

So, u_4 is arbitrary.

Furthermore, if $j=6$, and from Eq. (15g), then we obtain

$$0u_6=-u_3'-lu_3-fu_3^2-\frac{2g^2}{5h}u_4-\frac{54}{5}gu_5=0, \tag{18}$$

so u_6 is also arbitrary.

Setting $u_4=u_6=0$ and $u_3=0$, from Eqs. (15d) and (15g), we have

$$25fh(u_0'+lu_0)-12g^3=0, \tag{19}$$

and $gu_5=0$. So we have two subcases as follows.

Subcase I-1 $g\neq 0$, from $gu_5=0$, we have $u_5=0$. In this subcase, from Eq. (15f), we have

$$u_2'+lu_2=0, \tag{20}$$

where $l=l(t)$ is an arbitrary function.

Moreover, by the induction method, it is easy to see that $u_j=0$ for all $j\geq 3$.

Therefore, under the conditions (15e), (15g), and (19), from Eq. (2), we have

$$u=u_0\varphi^{-2}+u_1\varphi^{-1}+u_2, \tag{21}$$

where $\varphi=x+\psi(t)$, u_0, u_1, and u_2 are given by Eqs. (15a), (15b), and (15c), respectively.

By substituting Eq. (15c) into Eq. (20), we obtain

$$5fh^2\psi''-5(f'h^2-fh^2l)\psi'-fg^2hl+fg^2h'+f'g^2h-2fgg'h=0, \tag{22}$$

where $\psi'=\mathrm{d}\psi/\mathrm{d}t$, $f=f(t)$, $g=g(t)$, $h=h(t)$, and $l=l(t)$ are arbitrary functions. Thus, from Eq. (21), we obtain the exact solution of Eq. (14) as follows:

$$u(x,t)=-\frac{12h}{f}(x+\psi)^{-2}+\frac{12g}{5f}(x+\psi)^{-1}+\frac{g^2}{5fh}-\frac{1}{f}\psi', \tag{23}$$

where $\psi=\psi(t)$ is given by Eq. (22).

Subcase I-2 $g=0$, from Eqs. (15b) and (19), we obtain that equations (15e) and (15g) are identity. In this subcase, from Eqs. (15b), (15c), and (15f), we have $u_1=0$, $u_2=-\psi'/f$, and $u_5=(1/6h)(u_2'+lu_2)$. Moreover, from Eq. (15d), we obtain $u_0'+lu_0=0$, i.e.,

$$fh'-f'h+fhl=0, \tag{24}$$

where $f=f(t)$, $h=h(t)$, and $l=l(t)$ are arbitrary functions.

In addition, under the condition (20), we have $u_5=0$. Then, by the induction method, it is easy to see that $u_j=0$ for all $j\geq 3$.

Thus, under the conditions (15e), (15g), and (20), from Eq. (2), we have

$$u=u_0\varphi^{-2}+u_2, \tag{25}$$

where $\varphi=x+\psi(t)$, u_0 and u_2 are given by Eqs. (15a) and (15c), respectively.

By substituting Eq. (15c) with $g=0$ into Eq. (20), we obtain

$$\psi=c_1\int f(t)\exp\left[-\int l(t)\,\mathrm{d}t\right]\mathrm{d}t+c_2.$$

Thus, from Eq. (25), we obtain the exact solution of Eq. (14) with $g=0$ as follows:

$$u(x,t)=-\frac{12h}{f}\left\{x+c_1\int f(t)\exp\left[-\int l(t)\,\mathrm{d}t\right]\mathrm{d}t+c_2\right\}^{-2}-c_1\exp\left[-\int l(t)\,\mathrm{d}t\right], \tag{26}$$

where c_1 and c_2 are arbitrary constants, and equation (24) is satisfied.

Now, we give the Lax pair of Eq. (1) in this subcase under the condition (24). In fact, the linear eigenvalue problems can be expressed as

$$\Phi_x = M\Phi, \quad \Phi_t = N\Phi, \tag{27}$$

where $\Phi = (\phi_1, \phi_2)^{\mathrm{T}}$, T denotes the transpose of the matrix, while M and N are presented in the following forms

$$M = \begin{pmatrix} \lambda & q(x,t) \\ r(x,t) & -\lambda \end{pmatrix}, \quad N = \begin{pmatrix} A(x,t,\lambda) & B(x,t,\lambda) \\ C(x,t,\lambda) & -A(x,t,\lambda) \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned} &q(x,t) = u, \quad r(x,t) = -\frac{f}{6h}, \\ &A(x,t,\lambda) = -4h\lambda^3 - \frac{1}{3}fu\lambda - \frac{1}{6}fu_x - \frac{1}{2}l, \\ &B(x,t,\lambda) = -4hu\lambda^2 - 2hu_x\lambda - hu_{xx} - \frac{1}{3}fu^2, \\ &C(x,t,\lambda) = \frac{2}{3}f\lambda^2 + \frac{f^2}{18h}u, \end{aligned}$$

here λ is a parameter independent of x and t, $u = u(x,t)$ denotes the unknown function, while $f = f(t)$, $h = h(t)$, and $l = l(t)$ are given arbitrary functions.

It is easy to prove that the compatibility condition $M_t - N_x + [M,N] = 0$ gives rise to Eq. (1) in this subcase, which can show that equation (27) is the Lax pair of the vc-KdV types of Eq. (1) in the Subcase I-2 under the condition (24).

Summarizing the above discussions, we achieve the following result.

Theorem 3 The vc-NLEE (1) is Painlevé-integrable if and only if $p = 1$ and equations (15e), (15g) are satisfied. Moreover, if $g \neq 0$, then the exact solution is given by Eq.(23). If $g = 0$, then the exact solution is given by Eq. (26), the Lax pair is given by Eqs. (27) and (28) under the condition (24).

Case II $p = 2$. In this case, equation (1) becomes the following equation

$$u_t + f(t)u^2u_x + g(t)u_{xx} + h(t)u_{xxx} + l(t)u = 0, \tag{29}$$

where $fh \neq 0$, g and l are arbitrary analytic functions of t.

Thus, by substituting Eq. (2) into Eq. (29), we have

$$j = 0,\ u_0^2 = -\frac{6h}{f}, \tag{30a}$$

$$j = 1,\ u_1 = \frac{g}{fu_0}, \tag{30b}$$

$$j = 2,\ u_2 = \frac{1}{fu_0}\left(\frac{g^2}{6h} - \psi'\right), \tag{30c}$$

$$j = 3,\ u_0' + lu_0 = 0, \tag{30d}$$

$$j = 4,\ u_1' + lu_1 + \psi'u_2 + fu_0u_2^2 + fu_1^2u_2 + 4gu_3 = 0, \tag{30e}$$

$$\begin{aligned} j = 5,\ &u_2' + lu_2 - 2(fu_1^2 + \psi')u_3 \\ &+ 2fu_1u_2^2 + 10gu_4 + 12hu_5 = 0. \end{aligned} \tag{30f}$$

From Eqs. (30a), (30b), and (30c), we can obtain u_j $(j = 0,1,2)$ in a unique manner. But from Eqs. (30d) and (30e), we cannot obtain u_3 and u_4 definitely. In general, we have the recursion relations for Eq. (29) as follows:

$$\begin{aligned} (j+1)(j-3)(j-4)hu_j = &-u_{j-3}' - lu_{j-3} - (j-3)u_{j-2}\psi' - (j-2)(j-3)gu_{j-1} \\ &+ f\sum_{k=1}^{j-1} u_0u_ku_{j-k} - f\sum_{k=1}^{j-1}\left[(j-k-1)\sum_{i=0}^{k} u_iu_{k-i}u_{j-k}\right] \end{aligned} \tag{31}$$

for all $j \geq 3$.

From Eqs. (30a)–(30f) and (31), we can obtain all the coefficients u_j $(j = 0,1,2,\ldots)$ for Eq. (2) except u_3 and u_4. For example, if $j = 3$, then from Eq. (30d) we have

$$j = 3, \quad 0u_3 = -u_0' - lu_0 = 0. \tag{32}$$

If $j = 4$, then from Eq. (30e) we have

$$j = 4, \quad 0u_4 = -u_1' - lu_1 - \psi'u_2 - fu_0u_2^2 - fu_1^2u_2 - 4gu_3 = 0. \tag{33}$$

So, u_3 and u_4 are arbitrary.

Setting $u_2 = u_3 = u_4 = 0$, then from Eq. (30f), we have $u_5 = 0$. Generally, by the induction method, it is easy to show that $u_j = 0$, for all $j \geq 3$.

On the other hand, from $u_2 = 0$ and Eq. (30c), we obtain

$$6h\psi' - g^2 = 0. \tag{34}$$

Then, from $u_2 = u_3 = 0$ and Eq. (30e), we have Eq. (8) as well. Furthermore, from Eqs. (30a) and (30d), we have

$$fh' - f'h + 2fhl = 0, \tag{35}$$

where $f = f(t)$, $h = h(t)$ and $l = l(t)$ are arbitrary functions.

Therefore, under the conditions (30d), (30e), and (34), the Laurent series (2) is truncated as follows:

$$u = u_0\varphi^{-1} + u_1, \tag{36}$$

where $\varphi = x + \psi(t)$, u_0 and u_1 are given by Eqs. (30a) and (30b), respectively.

From Eq. (34), we have

$$\psi = \frac{1}{6}\int \frac{g^2}{h}\,\mathrm{d}t + c.$$

Substituting it into Eq. (35), we obtain the exact solution of Eq. (29) as follows:

$$u(x,t) = \pm\sqrt{-\frac{6h}{f}}\left[\left(x+\frac{1}{6}\int\frac{g^2}{h}\,\mathrm{d}t+c\right)^{-1} - \frac{g}{6h}\right], \quad (37)$$

where c is an arbitrary constant, and equation (35) is satisfied.

In what follows, we give the Lax pair of Eq. (1) in this case under the condition (35). In fact, the linear eigenvalue problems can be expressed as

$$\Phi_x = M\Phi, \quad \Phi_t = N\Phi, \quad (38)$$

where $\Phi = (\phi_1, \phi_2)^{\mathrm{T}}$, T denotes the transpose of the matrix, while M and N are presented in the following forms

$$M = \begin{pmatrix} \lambda & q(x,t) \\ r(x,t) & -\lambda \end{pmatrix}, \quad N = \begin{pmatrix} A(x,t,\lambda) & B(x,t,\lambda) \\ C(x,t,\lambda) & -A(x,t,\lambda) \end{pmatrix}, \quad (39)$$

where

$$\begin{aligned} &q(x,t) = u, \quad r(x,t) = -\frac{f}{6h}u, \\ &A(x,t,\lambda) = -4h\lambda^3 - \frac{1}{3}fu^2\lambda - \frac{1}{2}l, \\ &B(x,t,\lambda) = -4hu\lambda^2 - 2hu_x\lambda - hu_{xx} - \frac{1}{3}fu^3, \\ &C(x,t,\lambda) = \frac{2}{3}fu\lambda^2 - \frac{1}{3}fu_x\lambda + \frac{1}{6}fu_{xx} + \frac{f^2}{18h}u^3, \end{aligned}$$

here λ is a parameter independent of x and t, $u = u(x,t)$ denotes the unknown function, while $f = f(t)$, $h = h(t)$, and $l = l(t)$ are given arbitrary functions.

It is easy to prove that the compatibility condition $M_t - N_x + [M,N] = 0$ gives rise to Eq. (1) in this case, which can show that equation (38) is the Lax pair of the vc-KdV types of Eq. (1) in the Case II under the condition (35).

Summarizing the above argument, we achieve the following result.

Theorem 4 The vc-NLEE (29) is integrable if and only if $p = 2$, equations (30d) and (30e) are satisfied. Moreover, the exact solution of Eq. (29) is given by Eq. (37). In particular, if $g = 0$, then the Lax pair of the equation is given by Eqs. (38) and (39) under the condition (35).

3. Conclusion and remarks

In this paper, the complete integrability classification of the generalized variable-coefficient nonlinear evolution equation is investigated by the Painlevé analysis method. The integrable conditions for the vc-NLEEs are obtained in the sense of simplified Painlevé test, and the exact explicit solutions are obtained by the truncated expansion method. Then the Lax pairs of the equations are given under the integrable conditions. Moreover, for the integrable equations, the other properties, such as Bäcklund transformations, infinite generalized symmetries, and conservation laws (CLs) can be considered through other methods, the details are omitted in this paper.

Remark 1 In fact, the Painlevé analysis is a powerful method for dealing with intrinsic properties and exact solutions to the nonlinear equations. But the simplified Painlevé test differs from the general Painlevé test[1–3,19] for dealing with integrable properties and exact solutions sometimes.

Acknowledgements

The authors would like to thank the anonymous reviewers and editors for their valuable comments and suggestions.

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