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An extension of integrable equations related to AKNS and WKI spectral problems and their reductions

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A novel hierarchy of integrable nonlinear evolution equations related to the combined Ablowitz–Kaup–Newell–Segur (AKNS) and Wadati–Konno–Ichikawa (WKI) spectral problems is proposed, from which the Lax pair for a corresponding negative flow and its infinite many conservation laws are obtained. Furthermore, a reduction of this hierarchy is discussed, by which a generalized sinh-Gordon equation is derived on the basis of its negative flow.

Keywords: integrable extension, nonlinear evolution equations, infinite conservation laws

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1. Introduction

A most active field of recent research involved in applied mathematics and theoretical physics is concerned with integrable systems.[1–6] The AKNS system is an important integrable model for leading to a wide class of nonlinear evolution equations such as the KdV, the nonlinear Schrödinger, the mKdV, the sine-Gordon equations, and others.[7] The WKI model, proposed in Ref. [8], could describe the nonlinear oscillation of elastic beams under tension. Various methods have been developed to study integrable systems, for instance, the inverse scattering transformation,[9–11] the Darboux and Bäcklund transformation,[12–14] the algebraic-geometrical approach,[15–17] and others.[18–20] It is discovered that two inverse scattering formalisms of the AKNS and WKI models are connected through a generalized gauge transformation and a transformation of the space and time coordinates depending on a dependent variable.[21]

The main aim of this article is to study the extension of integrable nonlinear evolution equations related to the combined AKNS and WKI spectral problems and their reductions. In Section 2, with the aid of the zero-curvature equation, we obtain a novel hierarchy of integrable nonlinear evolution equations associated with a 2×2 matrix spectral problem. The first nontrivial member is

$$\begin{aligned} u_{t_0} &= \frac{1}{2}\left(\frac{\alpha_0\ell_0^2 u}{\sqrt{1+\ell_0^2uv}}\right)_{xx} + \frac{2\alpha_0 u}{\sqrt{1+\ell_0^2uv}},\\ v_{t_0} &= -\frac{1}{2}\left(\frac{\alpha_0\ell_0^2 v}{\sqrt{1+\ell_0^2uv}}\right)_{xx} - \frac{2\alpha_0 v}{\sqrt{1+\ell_0^2uv}}, \end{aligned} \tag{1}$$

where u and v are two potentials, α_0 and ℓ_0 are constants. A key feature of Eq. (1) is that the nonlinear term appears in the form of $\sqrt{1+\ell_0^2uv}$, which implies that the system is highly nonlinear. Then we deduce a negative flow associated with the same 2×2 matrix spectral problem

$$\begin{aligned} &u_{xt}+\ell_0^2(uw)_{xx}+4uw=0,\\ &v_{xt}+\ell_0^2(vw)_{xx}+4vw=0,\\ &2(1+\ell_0^2uv)w_x+\ell_0^2(uv)_xw+(uv)_t=0, \end{aligned} \tag{2}$$

in which u, v, and w are three potentials, ℓ_0 is a constant. We obtain the infinite many conservation laws of Eq. (2) with the aid of the Riccati forms for the two linear spectral problems. In Section 3, we discuss the reduction of the hierarchy in Section 2 and obtain a novel hierarchy of nonlinear evolution equations, in which the first member is

$$u_{t_0} = \frac{1}{2}\left(\frac{\alpha_0\ell_0^2u}{\sqrt{1+\ell_0^2u^2}}\right)_{xxx} + 2\left(\frac{\alpha_0 u}{\sqrt{1+\ell_0^2u^2}}\right)_x, \tag{3}$$

where u is a potential, α_0 and ℓ_0 are constants. Based on a deep reduction of the negative flow, we deduce a novel integrable generalization of the sinh-Gordon equation

$$u_{xt} = \left(1+\frac{1}{4}\ell_0^2\partial^2\right)\sinh u, \tag{4}$$

which can be reduced to the classical sinh-Gordon equation for $\ell_0=0$. We similarly obtain the Lax pair of Eq. (4) and derive its infinite many conservation laws.

2. Extension related to the AKNS and WKI spectral problems

In this section, we shall investigate the following 2×2 matrix spectral problem

$$\psi_x = U\psi,\ \ U=\begin{pmatrix} \mathrm{i}\lambda & (1+\ell_0\lambda)u\\ (1-\ell_0\lambda)v & -\mathrm{i}\lambda\end{pmatrix}, \tag{5}$$

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where u and v are two potentials, λ is a constant spectral parameter and $\ell_0 \in \mathbb{C}$ is a constant parameter. For $\ell_0 = 0$, spectral problem (5), which can yield a wide class of nonlinear evolution equations, is just the well-known AKNS spectral problem presented by Ablowitz, Kaup, Newell, and Segur. In order to derive a hierarchy of nonlinear evolution equations associated with spectral problem (5), we solve the stationary zero-curvature equation

$$V_x - [U,V] = 0,$$
$$V = \begin{pmatrix} (1-\ell_0^2\lambda^2)a & (1+\ell_0\lambda)(c+\mathrm{i}\ell_0^2\lambda ua) \\ (1-\ell_0\lambda)(b+\mathrm{i}\ell_0^2\lambda va) & -(1-\ell_0^2\lambda^2)a \end{pmatrix}, \quad (6)$$

where

$$a = \sum_{j\geq 0} a_j\lambda^{-j}, \quad b = \sum_{j\geq 0} b_j\lambda^{-j}, \quad c = \sum_{j\geq 0} c_j\lambda^{-j}, \quad (7)$$

and coefficients a_j, b_j, c_j are functions to be determined. A direct calculation shows that equations (6) and (7) imply the Lenard recursion equation

$$KS_j = \mathrm{i}JS_{j+1}, \quad j \geq 0, \quad JS_0 = 0, \quad (8)$$

with $S_j = (c_j, b_j, a_j)^{\mathrm{T}}$, $j \geq 0$, and the two operators K, J defined as

$$K = \begin{pmatrix} \partial & 0 & 2u \\ 0 & \partial & -2v \\ v & -u & \partial \end{pmatrix}, \quad J = \begin{pmatrix} 2 & 0 & -\ell_0^2\partial u \\ 0 & -2 & -\ell_0^2\partial v \\ v & -u & \partial \end{pmatrix}.$$

It is easy to calculate that

$$a_0 = \frac{\alpha_0}{\sqrt{1+\ell_0^2 uv}}, \quad b_0 = -\frac{1}{2}\ell_0^2\partial v a_0, \quad c_0 = \frac{1}{2}\ell_0^2\partial u a_0, \quad (9)$$

where α_0 is an arbitrary constant. The recursion relation (8) can uniquely determine S_j, $j \geq 1$, if the constants of integration are identified as zero.

Case 1 Let ψ satisfy the spectral problem (5) and the auxiliary problem

$$\psi_{t_m} = V^{(m)}\psi, \quad V^{(m)} = (V_{ij}^{(m)})_{2\times 2}, \quad (10)$$

where each entry $V_{ij}^{(m)} = V_{ij}(a^{(m)}, b^{(m)}, c^{(m)})$,

$$a^{(m)} = \sum_{j=0}^{m} a_j\lambda^{m-j}, \quad b^{(m)} = \sum_{j=0}^{m} b_j\lambda^{m-j},$$
$$c^{(m)} = \sum_{j=0}^{m} c_j\lambda^{m-j}. \quad (11)$$

Then the compatibility condition of Eqs. (5) and (10) yields the zero-curvature equation, $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to a hierarchy of nonlinear evolution equations

$$u_{t_m} = \partial c_m + 2ua_m, \quad v_{t_m} = \partial b_m - 2va_m, \quad m \geq 0. \quad (12)$$

The first nontrivial member in the hierarchy is Eq. (1). Let $\alpha_0 = -2\mathrm{i}$, $v = \pm u^*$, $t_0 = t$, $\ell_0 \in \mathbb{R}$ or $\ell_0 \in i\mathbb{R}$, then equation (1) turns to

$$\mathrm{i}u_t = \left(\frac{\ell_0^2 u}{\sqrt{1\pm\ell_0^2|u|^2}}\right)_{xx} + \frac{4u}{\sqrt{1\pm\ell_0^2|u|^2}}, \quad (13)$$

which is a generalization of the WKI model proposed in Ref. [8].

Case 2 Assume that ψ satisfies the spectral problem (5) and the following auxiliary problem

$$\psi_t = V\psi = \begin{pmatrix} (\mathrm{i}\lambda^{-1} - \mathrm{i}\ell_0^2\lambda)w & (1+\ell_0\lambda)(-2\mathrm{i}\lambda^{-1}\partial^{-1}uw - \ell_0^2 uw) \\ (1-\ell_0\lambda)(2\mathrm{i}\lambda^{-1}\partial^{-1}vw - \ell_0^2 vw) & -(\mathrm{i}\lambda^{-1} - i\ell_0^2\lambda)w \end{pmatrix}\psi, \quad (14)$$

where w is a function to be determined, and $\partial^{-1}\partial = \partial\partial^{-1} = 1$. Here we should point out that the position for the evolution of the space variable and the time variable in the Lax pair, i.e., Eqs. (5) and (14), is the same. Therefore, function w can be regarded also as a potential. The element V_{ij} of the matrix V in Eq. (14) contains an item of λ with a negative power, which is different to Case 1. The structure of matrix V ensures that coefficients of λ with the same power are zero in the zero-curvature equation, $U_t - V_x + [U,V] = 0$. Meanwhile, the existence of nontrivial nonlinear evolution equations should also be guaranteed. A direct calculation gives rise to a new interesting nonlinear evolution system

$$u_t = -4\partial^{-1}uw - \ell_0^2(uw)_x,$$
$$v_t = -4\partial^{-1}vw - \ell_0^2(vw)_x,$$
$$w_x = 2u\partial^{-1}vw + 2v\partial^{-1}uw, \quad (15)$$

which is equivalent to Eq. (2). Let $v = \pm u^*$ and $\ell_0 \in \mathbb{R}$. Then equation (2) is reduced to a new nonlinear evolution equation

$$u_{xt} + \ell_0^2(uw)_{xx} + 4uw = 0,$$
$$2(\pm 1 + \ell_0^2|u|^2)w_x + \ell_0^2(|u|^2)_x w + (|u|^2)_t = 0. \quad (16)$$

In what follows, we shall derive conservation laws for Eq. (2). Define $\rho = (\ln\psi_1)_x$, from the spectral problem (5), we arrive at a Riccati-type equation

$$u\rho_x + u\rho^2 - u_x\rho - u^2 v + \mathrm{i}\lambda u_x + \lambda^2 u(1+\ell_0^2 uv) = 0. \quad (17)$$

Let $\lambda = \mathrm{i}\zeta^{-1}$. By inserting the ansatz

$$\rho = \sum_{j=-1}^{\infty} \rho_j\zeta^j \quad (18)$$

into Eq. (17) and comparing the coefficients of ζ with the same power, we obtain ρ_{-1}, ρ_0, and the recursion formulas for ρ_j, $j \geq 1$ as follows:

$$\begin{aligned}
&\rho_{-1} = \sqrt{1+\ell_0^2 uv},\\
&\rho_0 = \frac{u_x}{2u} + \frac{2u_x\sqrt{1+\ell_0^2uv} - \ell_0^2 u(uv)_x}{4u(1+\ell_0^2uv)},\\
&2u\rho_{-1}\rho_1 = u_x\rho_0 - u\rho_{0,x} - u\rho_0^2 + u^2 v,\\
&2u\rho_{-1}\rho_j = u_x\rho_{j-1} - u\rho_{j-1,x} - u\sum_{\substack{m+n=j-1\\ 0\leq m,n\leq j-1}} \rho_m\rho_n
\end{aligned} \tag{19}$$

for $j \geq 2$. On the other hand, denote $\theta = (\ln\psi_1)_t$ and from the auxiliary problem (14), we have

$$\theta = (\mathrm{i}\lambda^{-1} - \mathrm{i}\ell_0^2\lambda)w + \frac{(-2\mathrm{i}\lambda^{-1}\partial^{-1}uw - \ell_0^2 uw)(\rho - \mathrm{i}\lambda)}{u}. \tag{20}$$

Expanding θ into the following series of ζ ($\zeta = \mathrm{i}\lambda^{-1}$)

$$\theta = \sum_{j=-1}^{\infty} \theta_j \zeta^j, \tag{21}$$

we can gain the expressions of θ_j immediately from Eqs. (18 and (20)

$$\begin{aligned}
&\theta_{-1} = -\ell_0^2 w\sqrt{1+\ell_0^2uv}, \quad \theta_0 = -\frac{4\left(1+\sqrt{1+\ell_0^2uv}\right)\partial^{-1}uw + \ell_0^2u_xw}{2u} + \frac{\ell_0^4uw(uv)_x - 2\ell_0^2u_xw\sqrt{1+\ell_0^2uv}}{4u(1+\ell_0^2uv)},\\
&\theta_1 = w - \frac{2\rho_0\partial^{-1}uw + \ell_0^2uw\rho_1}{u}, \quad \theta_j = -\frac{2\rho_{j-1}\partial^{-1}uw + \ell_0^2uw\rho_j}{u}, \quad j \geq 2.
\end{aligned} \tag{22}$$

Since $(\ln\psi_1)_{tx} = (\ln\psi_1)_{xt}$, we obtain

$$\rho_t = \theta_x. \tag{23}$$

The coefficients ρ_j and θ_j in the expansion of ρ and θ are called conservation densities and currents, respectively. Specially, the third expression of Eq. (2) implies the following conservation law:

$$(\sqrt{1+\ell_0^2uv})_t = (-\ell_0^2w\sqrt{1+\ell_0^2uv})_x, \tag{24}$$

which is just the first conservation law in Eq. (23) for Eq. (2).

3. Reductions of nonlinear evolution equations

In this section, we shall discuss a reduction for the hierarchy of nonlinear evolution equations (12). To this end, we shall reconsider the 2×2 spectral problem (5) in the case of $v = u$, i.e.,

$$\psi_x = U\psi, \quad U = \begin{pmatrix} \mathrm{i}\lambda & (1+\ell_0\lambda)u \\ (1-\ell_0\lambda)u & -\mathrm{i}\lambda \end{pmatrix}, \tag{25}$$

where u is a potential, λ is a constant spectral parameter, and $\ell_0 \in \mathbb{C}$ is a constant parameter. The key problem for the reduction is to find the suitable auxiliary problem of Eq. (25).

Case 1 Let ψ satisfy the spectral problem (25) and the auxiliary problem

$$\psi_{t_m} = V^{(m)}\psi, \quad V^{(m)} = (V_{ij})_{2\times 2}, \tag{26}$$

where

$$\begin{aligned}
&V_{11} = -V_{22} = 2\mathrm{i}\lambda(1-\ell_0^2\lambda^2)a^{(m)},\\
&V_{12} = (1+\ell_0\lambda)\left(\frac{1}{2}b_x^{(m)} + 2ua^{(m)} + \mathrm{i}\lambda b^{(m)} - 2\ell_0^2\lambda^2 ua^{(m)}\right),\\
&V_{21} = (1-\ell_0\lambda)\left(\frac{1}{2}b_x^{(m)} + 2ua^{(m)} - \mathrm{i}\lambda b^{(m)} - 2\ell_0^2\lambda^2 ua^{(m)}\right),\\
&a^{(m)} = \sum_{j=0}^{m} a_j\lambda^{2(m-j)}, \quad b^{(m)} = \sum_{j=0}^{m} b_j\lambda^{2(m-j)}.
\end{aligned}$$

The coefficients a_j and b_j are determined by the following Lenard recursion equation

$$\begin{aligned}
&KS_j = JS_{j+1}, \quad S_j = (b_j, a_j)^{\mathrm{T}}, \quad j \geq 0,\\
&a_0 = \frac{\alpha_0}{\sqrt{1+\ell_0^2u^2}}, \quad b_0 = \left(\frac{\alpha_0\ell_0^2u}{\sqrt{1+\ell_0^2u^2}}\right)_x,
\end{aligned} \tag{27}$$

with α_0 being an arbitrary constant and the two operators K and J defined as

$$K = \begin{pmatrix} \frac{1}{2}\partial^2 & 2\partial u \\ u & \partial \end{pmatrix}, \quad J = \begin{pmatrix} -2 & 2\ell_0^2\partial u \\ u & \partial \end{pmatrix}.$$

If all the constants of integration are identified as zero in the process of recursion calculation by using recursion relation (27), then the sequence S_j, $j \geq 1$, can be uniquely determined. The compatibility condition of Eqs. (25) and (26) yields the zero-curvature equation,

$$U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0,$$

which is equivalent to a hierarchy of nonlinear evolution equations

$$u_{t_m} = \frac{1}{2}\partial^2 b_m + 2\partial ua_m = -2b_{m+1} + 2\ell_0^2\partial ua_{m+1} \tag{28}$$

with $m \geq 0$. The first nontrivial member in the hierarchy is Eq. (3), which is a generalization of the model given in Ref. [22].

Case 2 Assume that ψ satisfies the spectral problem

$$\psi_x = U\psi, \quad U = \begin{pmatrix} \mathrm{i}\lambda & \frac{1}{2}(1+\ell_0\lambda)u_x \\ \frac{1}{2}(1-\ell_0\lambda)u_x & -\mathrm{i}\lambda \end{pmatrix} \tag{29}$$

and the auxiliary problem

$$\psi_t = V\psi, \quad V = \begin{pmatrix} 2\mathrm{i}(\lambda^{-1} - \ell_0^2\lambda)a & (1+\ell_0\lambda)(2\mathrm{i}\lambda^{-1}b - \ell_0^2 u_x a) \\ (1-\ell_0\lambda)(-2\mathrm{i}\lambda^{-1}b - \ell_0^2 u_x a) & -2\mathrm{i}(\lambda^{-1} - \ell_0^2\lambda)a \end{pmatrix}, \tag{30}$$

where $a = -(1/8)\cosh u$, $b = (1/8)\sinh u$. Then the zero-curvature equation, $U_t - V_x + [U,V] = 0$, generates the novel integrable generalization of the sinh-Gordon equation (4), which is similar to the generalization of the sine-Gordon equation.[23] Here the solution V of the zero-curvature equation in Eq.(30) can be obtained by assuming that V is the linear polynomial of λ and λ^{-1}, and substituting into the zero-curvature equation.

In the following, we shall derive infinitely many conservation laws of Eq. (4) through the same process in Section 2. Let $\rho = (\ln\psi_1)_x$, $\theta = (\ln\psi_1)_t$. From the spectral problem of Eqs. (29) and (30), we obtain the following expressions related to ρ and θ

$$2u_x\rho_x + 2u_x\rho^2 - 2u_{xx}\rho - \frac{1}{2}u_x^3 + 2\mathrm{i}\lambda u_{xx} + \frac{1}{2}\lambda^2 u_x(4 + \ell_0^2 u_x^2) = 0, \tag{31}$$

$$\theta = 2\mathrm{i}(\lambda^{-1} - \ell_0^2\lambda)a + \frac{2(2\mathrm{i}\lambda^{-1}b - \ell_0^2 u_x a)(\rho - \mathrm{i}\lambda)}{u_x}. \tag{32}$$

In Eqs. (31) and (32), functions ρ and θ can be written as a series in powers of ζ ($\lambda = \mathrm{i}\zeta^{-1}$):

$$\rho = \sum_{j=-1}^{\infty} \rho_j \zeta^j, \quad \theta = \sum_{j=-1}^{\infty} \theta_j \zeta^j. \tag{33}$$

Substituting expressions (33) into Eqs. (31) and (32) and comparing powers of ζ with the same power yield two infinite sequences of conserved quantities for Eq. (4)

$$\rho_{-1} = \sqrt{1 + \frac{1}{4}\ell_0^2 u_x^2}, \quad \rho_0 = \frac{u_{xx}\left(1 + \sqrt{1 + (1/4)\ell_0^2 u_x^2}\right)}{2u_x(1 + (1/4)\ell_0^2 u_x^2)},$$

$$4u_x\rho_{-1}\rho_1 = 2u_{xx}\rho_0 - 2u_x\rho_{0,x} - 2u_x\rho_0^2 + \frac{1}{2}u_x^3,$$

$$2u_x\rho_{-1}\rho_j = u_{xx}\rho_{j-1} - u_x\rho_{j-1,x} - u_x \sum_{\substack{m+n=j-1 \\ 0\le m,n\le j-1}} \rho_m\rho_n \tag{34}$$

with $j \ge 2$, and

$$\theta_{-1} = \frac{1}{4}\ell_0^2\sqrt{1 + \frac{1}{4}\ell_0^2 u_x^2}\cosh u,$$

$$\theta_0 = \left(1 + \sqrt{1 + \frac{1}{4}\ell_0^2 u_x^2}\right)\left(\frac{\sinh u}{2u_x} + \frac{\ell_0^2 u_{xx}\cosh u}{8u_x(1 + (1/4)\ell_0^2 u_x^2)}\right),$$

$$\theta_1 = 2a + \frac{4b\rho_0}{u_x} - 2\ell_0^2 a\rho_1,$$

$$\theta_j = \frac{4b\rho_{j-1}}{u_x} - 2\ell_0^2 a\rho_j, \quad j \ge 2. \tag{35}$$

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