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First integrals of the axisymmetric shape equation of lipid membranes*

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The shape equation of lipid membranes is a fourth-order partial differential equation. Under the axisymmetric condition, this equation was transformed into a second-order ordinary differential equation (ODE) by Zheng and Liu (*Phys. Rev. E* **48** 2856 (1993)). Here we try to further reduce this second-order ODE to a first-order ODE. First, we invert the usual process of variational calculus, that is, we construct a Lagrangian for which the ODE is the corresponding Euler–Lagrange equation. Then, we seek symmetries of this Lagrangian according to the Noether theorem. Under a certain restriction on Lie groups of the shape equation, we find that the first integral only exists when the shape equation is identical to the Willmore equation, in which case the symmetry leading to the first integral is scale invariance. We also obtain the mechanical interpretation of the first integral by using the membrane stress tensor.

Keywords: lipid membrane, shape equation, first integral, Noether theorem

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1. Introduction

The elasticity of membranes and shells has drawn much research attention. In 1812, Poisson proposed a functional for the bending energy of a shell

$$F = \frac{k_c}{2}\int_M H^2 \mathrm{d}A, \tag{1}$$

where H represents the mean curvature of the shell surface, M and dA represent the shell surface and its area element, respectively. Equation (1) is known as the Willmore functional in mathematics, and is invariant under conformal transformations of the embedding space. The equilibrium configurations of the shell minimize the Willmore functional, and must satisfy the Willmore equation

$$\nabla^2 H + 2H(H^2 - K) = 0, \tag{2}$$

where K represents the Gauss curvature of the shell surface, and ∇^2 is the Laplace–Beltrami operator on a 2-dimensional surface. For compact surfaces in 3-dimensional Euclidian space, Willmore proved that round spheres and their images under conformal transformations correspond to the least minimum of the Willmore functional. Thus the Willmore functional takes values no less than 4π for all compact surfaces. He further conjectured that the values of the Willmore functional are no less than $2\pi^2$ for compact surfaces of genus one in 3-dimensional Euclidian space. With this topology, the Willmore tori and their images under conformal transformations are the least minimum. The Willmore tori are special tori with the ratio of their two generating radii being $\sqrt{2}$. The Willmore conjecture was recently proved by Marques and Neves via min-max theory.[1] Willmore surfaces can also arise from the mKdV equation with the deformation technique. Tek investigated the Weingarten and Willmore-like surfaces arising from the spectral deformation of the Lax pair for the mKdV equation.[2]

In 1973, Helfrich proposed the spontaneous curvature model of lipid membranes.[3] He supposed that a lipid membrane could be treated as a liquid crystalline thin film. Based on the elastic theory of liquid crystals, Helfrich derived the bending energy of the membrane

$$F_{\mathrm{H}} = \int_M \left[(k_c/2)(2H + c_0)^2 + \bar{k}K\right] \mathrm{d}A, \tag{3}$$

where $k_c > 0$ and $\bar{k}$ are the bending moduli of the membrane, and c_0 is the spontaneous curvature reflecting the difference of environmental factors or chemical constituents between the two leaflets of the membrane. The Helfrich functional prompted a flourishing theory of lipid membrane elasticity.[4–25]

Since the area of a lipid bilayer is almost inextensible and the volume enclosed by a lipid vesicle is hardly compressed, the shape of the lipid vesicle in equilibrium corresponds to the minimal value of Eq. (3) at fixed volume and area. We therefore introduce two Lagrange multipliers to enforce these constraints, and then minimize the following extended functional:

$$F = \int_M \left[(k_c/2)(2H + c_0)^2 + \bar{k}K + \lambda\right] \mathrm{d}A + p\int \mathrm{d}V, \tag{4}$$

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where the Lagrange multipliers λ and p can be interpreted as the surface tension and osmotic pressure, respectively. The first order variation of Eq. (4) leads to the shape equation of lipid membranes[4,5]

$$2k_c\nabla^2 H + k_c(2H + c_0)(2H^2 - 2K - c_0 H) - 2\lambda H + p = 0. \quad (5)$$

This is a fundamental equation in the study of lipid membranes, which gives the various equilibrium shapes of membranes,[11] such as spherical vesicle, Clifford torus, Dupin cyclide, and circular biconcave discoid. It is a nonlinear partial differential equation (PDE), and could degenerate into the Willmore equation when p, λ, and c_0 are vanishing. Equation (5) also takes a common form not only in mathematical biology, but also in many other subjects, such as conformal geometry[26] and the theory of thin elastic shells.[27] Thus understanding the properties of Eq. (5) is an important task. Konopelchenko investigated Eqs. (4) and (5) by introducing the so-called generalized Weierstrass representation.[6] He found that it could be reduced to several solvable equations, such as the Liouville equation or the Schrödinger equation. Following this framework, Landolfi provided a new type of periodic developable surfaces and further discussed the statistical properties of membranes described by the Helfrich energy.[8] Numerical study reveals that the biconcave discoid configuration indeed corresponds to the minimum value of Eq. (4) under appropriate physiological conditions,[9] which may explain the biconcave shape of the normal red blood cells.

One can generate an axisymmetric surface by rotating a planar curve $z = z(\rho)$, with ρ being the radius of revolution around the z-axis. The axisymmetric surface may be expressed as

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = \int \tan\psi(\rho)\mathrm{d}\rho, \quad (6)$$

where ϕ is the azimuthal angle and ψ is the tangent angle of the profile curve as shown in Fig. 1.

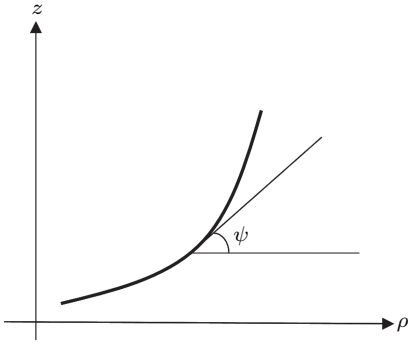


Fig. 1. Schematic of the profile curve.

For axisymmetric surfaces, the shape equation (5) can of course be written as an ordinary differential equation (ODE) rather than a PDE, making it much easier to solve. By using the parameterization in Eq. (6), it becomes[28]

$$-\frac{\cos\psi}{\rho}\left\{\rho\cos\psi\left[\frac{(\rho\sin\psi)'}{\rho}\right]'\right\}' - \frac{1}{2}\left[\frac{(\rho\sin\psi)'}{\rho}\right]^3 + \frac{(\rho\sin\psi)'(\sin^2\psi)'}{\rho^2} - \frac{c_0(\sin^2\psi)'}{\rho} + \frac{\tilde{\lambda}(\rho\sin\psi)'}{\rho} + \tilde{p} = 0, \quad (7)$$

where $\tilde{\lambda} \equiv \lambda/k_c + c_0^2/2$, $\tilde{p} \equiv p/k_c$, $\psi' = \mathrm{d}\psi/\mathrm{d}\rho$, and $\psi'' = \mathrm{d}^2\psi/\mathrm{d}\rho^2$. A first integral of Eq. (7) is found in Ref. [29] and the above third-order ODE is reduced to the following second-order ODE:

$$\rho\cos^3\psi\psi'' - \frac{\rho}{2}\cos^2\psi\sin\psi\psi'^2 + \cos^3\psi\psi' + \frac{\sin^3\psi}{2\rho} + c_0\sin^2\psi - \frac{\tilde{\lambda}\rho^2 + 1}{\rho}\sin\psi - \frac{\tilde{p}}{2}\rho^2 + \Omega_0 = 0, \quad (8)$$

where Ω_0 is the first integral of Eq. (7). This constant can be interpreted as a tension in the membrane along the symmetry axis, and is a result of the axisymmetry of the surface. Researchers have found several special solutions of Eq. (8), including minimal surfaces (catenoid, helicoid, etc.), constant mean curvature surfaces (sphere, cylinder, unduloid,[30,31] etc.), Willmore surfaces (Clifford torus,[32] Dupin cyclide,[33] inverted catenoid,[34] etc.), cylinder-like surfaces,[35–37] and circular biconcave discoid.[38,39]

As equation (8) is a second-order ODE, a natural question is whether we can further transform it into a first-order ODE by constructing another first integral of Eq. (8), which is the main goal of this paper. Usually, a second-order ODE may be regarded as the equation of motion of a mechanical system in mathematics. Thus, we might be able to transform Eq. (8) into a first-order ODE by finding the first integrals for a given mechanical system. In a previous work by Capovilla *et al.*,[15,16] they treated the Helfrich energy (3) as a kind of action describing the motion of a particle, and gave the shape equation by solving the corresponding Hamilton equations. Then equation (8) could emerge naturally from their framework with axial symmetry. Here we start from Eq. (8) directly, and construct the corresponding Lagrangian by solving the inverse problem of the calculus of variations. Our aim is to find the first integrals of Eq. (8). The rest of the paper is organized as follows. In Section 2, we will construct the Lagrangian corresponding to shape equation (8). In Section 3, we will briefly introduce the generalized Noether theorem, which we will use to look for the first integrals of shape equation (8). In the special case of vanishing p, c_0, λ, and Ω_0, we successfully obtain a first integral and the corresponding special solution to the shape equation. In Section 4, we find that this special solution can also be derived from the Hamilton–Jacobi equation. Section 5 generalizes the first integral to cases with $\Omega_0 \neq 0$ and demonstrates that we can also connect the origins of the first integral

and its mechanical interpretation with the membrane's stress tensor. Section 6 contains a summary and discussion of our results and outlook.

2. Lagrangian corresponding to the shape equation

Generally speaking, physicists tend to believe that there is not always a Lagrangian for a given equation of motion, especially for a dissipative system with friction or viscosity. In mathematics, it is usually called the inverse variation problem to find the Lagrangian corresponding to a Euler–Lagrange equation. For the shape equation (8), we indeed successfully constructed its Lagrangian. Next, we will demonstrate the specific procedure.

Assume that we have a Lagrangian with the following very general form:

$$L=\frac{1}{2}\alpha(t,u)\dot{u}^2+\beta(t,u)\dot{u}+\gamma(t,u), \tag{9}$$

and an arbitrary equation of motion

$$a(t,u)\ddot{u}+b(t,u)\dot{u}^2+c(t,u)\dot{u}+d(t,u)=0, \tag{10}$$

where $\ddot{u}=\mathrm{d}^2u/\mathrm{d}t^2$ and $\dot{u}=\mathrm{d}u/\mathrm{d}t$. The Euler–Lagrange equation $\frac{\partial L}{\partial u}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}}\right)=0$ corresponding to Eq. (9) may be expressed as

$$\alpha\ddot{u}+\frac{1}{2}\frac{\partial\alpha}{\partial u}\dot{u}^2+\frac{\partial\alpha}{\partial t}\dot{u}+\frac{\partial\beta}{\partial t}-\frac{\partial\gamma}{\partial u}=0. \tag{11}$$

Equations (10) and (11) have the same solution if there is a non-vanishing function f such that

$$\frac{\alpha}{a}=\frac{\frac{1}{2}\frac{\partial\alpha}{\partial u}}{b}=\frac{\frac{\partial\alpha}{\partial t}}{c}=\frac{\frac{\partial\beta}{\partial t}-\frac{\partial\gamma}{\partial u}}{d}=f. \tag{12}$$

Now compare the shape equation (8) with the equation of motion (10) by identifying t and u with ρ and ψ, respectively. Then from Eq. (12), we derive

$$\begin{aligned}&\alpha=\rho\cos\psi,\quad f=\sec^2\psi,\\&\frac{\partial\beta}{\partial\rho}-\frac{\partial\gamma}{\partial\psi}=\sec^2\psi\left(\frac{\sin^3\psi}{2\rho}+c_0\sin^2\psi\right.\\&\qquad\left.-\frac{\tilde{\lambda}\rho^2+1}{\rho}\sin\psi-\frac{\tilde{p}}{2}\rho^2+\Omega_0\right).\end{aligned} \tag{13}$$

The functions β and γ contain some degrees of freedom which we are free to choose. This is because Lagrangians that differ by a divergence lead to the same equations of motion. Consider two different Lagrangians of the form of Eq. (9):

$$L=\frac{1}{2}\alpha\psi'^2+\beta\psi'+\gamma, \tag{14}$$

$$\tilde{L}=\frac{1}{2}\alpha\psi'^2+\tilde{\beta}\psi'+\tilde{\gamma}. \tag{15}$$

Take the difference between Eqs. (14) and (15),

$$L-\tilde{L}=(\beta-\tilde{\beta})\psi'+(\gamma-\tilde{\gamma}). \tag{16}$$

On the other hand, we have

$$\frac{\partial\beta}{\partial\rho}-\frac{\partial\gamma}{\partial\psi}=\frac{\partial\tilde{\beta}}{\partial\rho}-\frac{\partial\tilde{\gamma}}{\partial\psi}, \tag{17}$$

because equation (13) requires the quantity $\partial\beta/\partial\rho-\partial\gamma/\partial\psi$ to have a specific form. We can rearrange this to give

$$\frac{\partial}{\partial\rho}(\beta-\tilde{\beta})=\frac{\partial}{\partial\psi}(\gamma-\tilde{\gamma}). \tag{18}$$

This equation implies that there must be a function $\Phi(\rho,\psi)$ satisfying

$$\frac{\partial\Phi}{\partial\psi}=\beta-\tilde{\beta},\quad \frac{\partial\Phi}{\partial\rho}=\gamma-\tilde{\gamma}, \tag{19}$$

which leads to

$$L-\tilde{L}=\frac{\mathrm{d}}{\mathrm{d}\rho}\Phi(\rho,\psi). \tag{20}$$

Thus the difference between Eqs. (14) and (15) is just a total derivative of some function, and the equations of motion corresponding to the two equations are therefore the same. For the sake of simplicity, we take $\beta=0$, and the Lagrangian is expressed as

$$\begin{aligned}L=&\frac{\rho\cos\psi}{2}\psi'^2+\frac{\tan\psi\sin\psi}{2\rho}+c_0\psi+\tilde{\lambda}\rho\sec\psi\\&-\left(c_0+\Omega_0-\frac{\tilde{p}}{2}\rho^2\right)\tan\psi.\end{aligned} \tag{21}$$

3. Noether theorem and conservation law

In Newtonian mechanics, Noether's theorem explains the connection between symmetries and conservation laws. Since we have the Lagrangian of the equation of motion, we can construct the first integrals of the equation of motion by using Noether's theorem and looking for symmetries of this Lagrangian.

3.1. The generalized Noether theorem

From the generalized Noether theorem, we know that[40] if and only if the value of the variational integral $\int L(t,u,\dot{u})\mathrm{d}u$ is invariant at the extrema under a continuous transformation group with generator

$$X=\xi\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial u}, \tag{22}$$

then the scalar

$$I=\xi L+(\eta-\xi\dot{u})\frac{\partial L}{\partial\dot{u}} \tag{23}$$

provides a first integral for the Euler–Lagrange equation corresponding to L. We can test this invariance by checking that there exists a function F such that

$$X(L)+L\dot{\xi}=F\frac{\delta L}{\delta u}, \tag{24}$$

where

$$\frac{\delta L}{\delta u} \equiv \frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}}\right),$$
$$X(L) = \xi \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial u} + \left(\dot{\eta} - \dot{u}\dot{\xi}\right)\frac{\partial L}{\partial \dot{u}}. \tag{25}$$

We demand that $F \neq \eta - \xi\dot{u}$ because the first integral (23) is otherwise trivial.

Alternatively, the infinitesimal test condition (24) could be replaced by the divergence condition[40]

$$X(L) + L\dot{\xi} = F\frac{\delta L}{\delta u} + \dot{B}. \tag{26}$$

This is equivalent to Eq. (24) modulo a total derivative term in the Lagrangian. By using this condition, the corresponding first integrals become

$$I = \xi L + (\eta - \xi\dot{u})\frac{\partial L}{\partial \dot{u}} - B. \tag{27}$$

3.2. First integral of the axisymmetric shape equation of lipid membranes

For the sake of brevity, we rewrite the Lagrangian (21) in the form

$$L(\rho, \psi, \psi') = \frac{\rho\cos\psi}{2}\psi'^2 - V(\rho, \psi), \tag{28}$$

where we now consider all the terms not containing ψ' as constituting a potential energy

$$V(\rho,\psi) = -\frac{\tan\psi\sin\psi}{2\rho} - c_0\psi - \tilde{\lambda}\rho\sec\psi + \left(c_0 + \Omega_0 - \frac{\tilde{p}}{2}\rho^2\right)\tan\psi. \tag{29}$$

Applying conditions (25) and (26) for invariance of the action to Eq. (28) with $u \to \psi$ and $t \to \rho$, we find that, if the first integral (27) exists, then $B(\rho,\psi,\psi')$, $\eta(\rho,\psi,\psi')$, $\xi(\rho,\psi,\psi')$, and $F(\rho,\psi,\psi')$ satisfy the following equation:

$$\begin{aligned}
&\left[2\rho\cos\psi F - 2B_{\psi'} - \left(\rho\psi'^2\cos\psi + 2V\right)\xi_{\psi'} + 2\rho\psi'\cos\psi\eta_{\psi'}\right]\psi'' \\
&- \rho\cos\psi\xi_\psi\psi'^3 + \left[\cos\psi\left(\xi - \rho\xi_\rho + 2\rho\eta_\psi\right) - \rho\sin\psi(F+\eta)\right]\psi'^2 \\
&+ 2\left[\cos\psi\left(F + \rho\eta_\rho\right) - B_\psi - V\xi_\psi\right]\psi' - 2\xi V_\rho \\
&+ 2(F-\eta)V_\psi - 2B_\rho - 2V\xi_\rho = 0.
\end{aligned} \tag{30}$$

In the above equation, ψ'', ψ', and ψ should be regarded as independent variables. The subscripts ρ, ψ, and ψ' denote the partial derivatives with respect to ρ, ψ, and ψ'. Naturally, we can separate Eq. (30) into two equations according to the order of ψ'',

$$2\rho\cos\psi F - 2B_{\psi'} - \left(\rho\psi'^2\cos\psi + 2V\right)\xi_{\psi'} + 2\rho\psi'\cos\psi\eta_{\psi'} = 0, \tag{31}$$

$$\begin{aligned}
&-\rho\cos\psi\xi_\psi\psi'^3 + \left[-\rho\sin\psi(F+\eta) + \cos\psi\left(\xi - \rho\xi_\rho + 2\rho\eta_\psi\right)\right]\psi'^2 \\
&+ 2\left[\cos\psi\left(F + \rho\eta_\rho\right) - B_\psi - V\xi_\psi\right]\psi' \\
&- 2\xi V_\rho + 2(F-\eta)V_\psi - 2B_\rho - 2V\xi_\rho = 0.
\end{aligned} \tag{32}$$

We have investigated the situation that B, η, ξ, and F do not explicitly contain ψ', and find that the solutions to Eqs. (31) and (32) only exist if the parameters c_0, $\tilde{\lambda}$, $\tilde{p}$, and Ω_0 all vanish. By making these parameters vanish, equation (8) in fact becomes the axisymmetric Willmore equation. The details of these calculations will be shown in the appendix. The final result is

$$B = 0, \quad \eta = 0, \quad \xi = 2\rho. \tag{33}$$

Thus X is a generator of dilations, a subset of the well-known conformal invariance of the Willmore functional. By substituting Eq. (33) into Eq. (27), we obtain its first integral

$$\begin{aligned}
I &= \rho^2\cos\psi\psi'^2 - \frac{\sin^2\psi}{\cos\psi} \\
&= \rho^2\sec\psi\left(\cos^2\psi\psi'^2 - \frac{\sin^2\psi}{\rho^2}\right).
\end{aligned} \tag{34}$$

Note that $\sin\psi/\rho$ and $-\psi'\cos\psi$ are the principal curvatures of the membrane surface. The solution of the ODE (34) can then be obtained by the quadrature

$$\rho = \rho_0 \exp\left(\pm\int \frac{\cos\psi}{\sqrt{I\cos\psi + \sin^2\psi}}\mathrm{d}\psi\right), \tag{35}$$

with ρ_0 being an unknown constant.

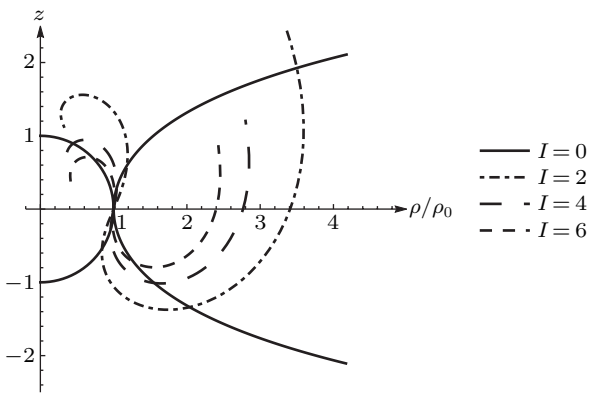


Fig. 2. Profile curves with different I.

In fact, I is the only independent parameter in this solution. For the sake of simplicity, we assume that I is positive. The results are qualitatively similar when I is negative, with the signs switching in Eq. (35). The profile curve of the surface corresponding to different I is depicted in Fig. 2. Choosing the negative sign in the exponent in Eq. (35) leads to solutions with $\rho/\rho_0 \leq 1$, while choosing the positive sign leads to solutions with $\rho/\rho_0 \geq 1$. For example, if we take $I = 0$, we find

two solutions, $\rho = \rho_0 \sin\psi$ and $\rho = \rho_0 \csc\psi$, depending on which branch we choose for the square root in the exponent. The solution $\rho = \rho_0 \sin\psi$ describes a sphere, and is shown as the solid with $\rho/\rho_0 \leq 1$. The solution $\rho = \rho_0 \csc\psi$ describes a catenoid, and is shown as the solid line with $\rho/\rho_0 \geq 1$. This kind of degeneracy suggests that there may be a conformal transformation between these two configurations. When $I \neq 0$, it is obvious that ρ cannot be zero because of the exponential form of Eq. (35). This ensures that the profile curve will not hit the z-axis, which means that the surface will not be closed as the spherical topology. Furthermore, we can derive the tangent components of the profile curve from Eq. (34)

$$\frac{\mathrm{d}\rho}{\mathrm{d}\psi} = \pm\frac{\rho(\psi)\cos\psi}{\sqrt{I\cos\psi+\sin^2\psi}}, \tag{36}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\psi} = \pm\frac{\rho(\psi)\sin\psi}{\sqrt{I\cos\psi+\sin^2\psi}}. \tag{37}$$

Since the expression under the square root must be positive, we find that the domain of ψ is $\psi \in (2k\pi - \arccos(I/2-\sqrt{1+I^2/4}),\ 2k\pi + \arccos(I/2-\sqrt{1+I^2/4}))$, where k is an arbitrary integer. The only effect of the $2k\pi$ term is to rescale the curve. The point where $\psi = 2k\pi \pm \arccos(I/2-\sqrt{1+I^2/4})$ corresponds to an inflection point of the profile curve where the curvature changes sign; these can also be seen as maxima and minima of ψ. Solving Eqs. (36) and (37) over the interval $\psi \in (2k\pi - \arccos(I/2-\sqrt{1+I^2/4}),\ 2k\pi + \arccos(I/2-\sqrt{1+I^2/4}))$ leads to only a segment of the profile curve between two such inflection points. One can assemble the entire surface by tiling together such segments and attaching them at their inflection points. Each segment is the image of the previous one under rescaling, leading to a surface that is self-similar.

Equations (36) and (37) do not admit any closed solutions, with the exception of the sphere. This can be seen by considering the segments mentioned above. From Eq. (35), it follows that the scaling factor between two segments is

$$\lambda = \exp\left(\int_{-\arccos\left(I/2-\sqrt{1+\frac{I^2}{4}}\right)}^{\arccos\left(I/2-\sqrt{1+\frac{I^2}{4}}\right)} \frac{\cos\psi}{\sqrt{I\cos\psi+\sin^2\psi}}\mathrm{d}\psi\right). \tag{38}$$

For a profile curve to close, the scale factor must be unity $\lambda = 1$, so that it maps to itself upon rescaling. However, this occurs only when $I = 0$. This implies that equations (36) and (37) do not permit solutions with, for example, toroidal topology. This may seem paradoxical at first, as the Clifford torus is a well-known solution to the Willmore equation having toroidal topology. However, the Clifford torus is omitted by our discussion at this point, since it is only valid when $\eta \neq 0$.

We can also rewrite Eq. (34) as

$$\rho\cos\psi\psi' \pm \sqrt{I\cos\psi+\sin^2\psi} = 0. \tag{39}$$

By using the fact that $\tan\psi = \mathrm{d}z/\mathrm{d}\rho$, this can in turn be transformed into

$$\frac{\mathrm{d}^2z}{\mathrm{d}\rho^2} = \pm\frac{1}{\rho}\left[\left(\frac{\mathrm{d}z}{\mathrm{d}\rho}\right)^2+1\right]\sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\rho}\right)^2+I\sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\rho}\right)^2+1}}. \tag{40}$$

This form of Willmore equation was studied in some other works[41–43] through the geometric (point) Lie symmetry group analysis. Although we reproduce the same solution, we show that it can be found by constructing the first integral (34). Furthermore, we prove that this is the only first integral, not only of the Willmore equation, but of the more general Helfrich equation, if the divergence function B and the generators ξ and η do not depend on ψ'. This point is illustrated in the appendix.

4. Hamilton–Jacobi equation

For the case of vanishing c_0, $\tilde{\lambda}$, $\tilde{p}$, and Ω_0, our first integral can also be obtained via the Hamilton–Jacobi equation.

As we have constructed the Lagrangian (28), the conjugate momentum to the coordinate ψ is

$$p \equiv \frac{\partial L}{\partial\psi'} = \rho\cos\psi\psi'. \tag{41}$$

We further obtain the Hamiltonian

$$H = p\psi' - L = \frac{p^2}{2\rho\cos\psi} + V(\rho,\psi). \tag{42}$$

This leads to the corresponding Hamilton–Jacobi equation

$$\frac{\partial S}{\partial\rho} + \frac{1}{2\rho\cos\psi}\left(\frac{\partial S}{\partial\psi}\right)^2 + V = 0, \tag{43}$$

where $S = S(\rho,\psi)$ is the principal function.

When c_0, $\tilde{\lambda}$, $\tilde{p}$, and Ω_0 vanish, equation (43) becomes

$$\frac{\partial S}{\partial\rho} + \frac{1}{2\rho\cos\psi}\left(\frac{\partial S}{\partial\psi}\right)^2 - \frac{\tan\psi\sin\psi}{2\rho} = 0. \tag{44}$$

This equation may be solved by separation of variables. The solution is given by

$$S = \pm\int\sqrt{\bar{I}\cos\psi+\sin^2\psi}\mathrm{d}\psi - \frac{\bar{I}}{2}\ln\rho, \tag{45}$$

where $\bar{I}$ is a constant.

This method also leads us quickly back to the quadrature for $\psi(\rho)$. The conjugate variable to $\bar{I}$ is

$$J = \frac{\partial S}{\partial\bar{I}} = \frac{1}{2}\left(\pm\int\frac{\cos\psi}{\sqrt{\bar{I}\cos\psi+\sin^2\psi}}\mathrm{d}\psi - \ln\rho\right). \tag{46}$$

Since J is another constant, its total differential vanishes. Taking advantage of this fact, we obtain

$$2\mathrm{d}J = 0 = \pm\frac{\cos\psi}{\sqrt{\bar{I}\cos\psi+\sin^2\psi}}\mathrm{d}\psi - \frac{\mathrm{d}\rho}{\rho}. \tag{47}$$

Identifying $\bar{I} = I$, we see that it agrees with Eq. (39). Thus we may derive the first integral (34) by solving Eq. (47) for $\bar{I}$.

5. Mechanical interpretation

The existence of Lagrangian (21) and the Noether symmetry suggests that there may be a mechanical interpretation for the first integral. We found that we can use the membrane's stress tensor $\boldsymbol{f}^a$ to identify the origins of our first integral as well as its mechanical interpretation. Furthermore, this method gives a coordinate independent expression for the first integral I, and generalizes it to cases when $\Omega_0 \neq 0$.

Generally, the stress tensor of a membrane is given by

$$\begin{aligned} \boldsymbol{f}^a &= f^{ab}\boldsymbol{e}_b + f^a\boldsymbol{n}, && (48)\\ f^a &= -2k_c\nabla^a H, && (49)\\ f^{ab} &= k_c K^{ab}(2H+c_0) - g^{ab}\left[\frac{k_c}{2}(2H+c_0)^2+\lambda\right], && (50) \end{aligned}$$

where K_{ab} is the curvature tensor, and $\boldsymbol{e}_a$ are the surface tangent vectors.[12] In equilibrium, the divergence of the stress tensor balances the local pressure difference across the membrane,

$$\boldsymbol{\nabla}_a \boldsymbol{f}^a = p\boldsymbol{n}, \tag{51}$$

where $\boldsymbol{n}$ is the surface normal vector.

As mentioned above, the Willmore functional is invariant under conformal transformations of the ambient space. Consider the subset of conformal transformations known as dilations, i.e., scale transformations. Following Ref. [44], we can find the Noether current associated with these transformations by considering the energy along with a system of Lagrange multipliers

$$\begin{aligned} E = \int_M \mathrm{d}A\big\{&\mathcal{E}(g_{ab},K_{ab}) - \boldsymbol{f}^a\cdot(\partial_a\boldsymbol{X}-\boldsymbol{e}_a) + \lambda^a_\perp \boldsymbol{e}_a\cdot\boldsymbol{n}\\ &+\lambda_n(\boldsymbol{n}^2-1) + \Lambda^{ab}(K_{ab}-\boldsymbol{e}_a\cdot\partial_b\boldsymbol{n})\\ &+\lambda^{ab}(g_{ab}-\boldsymbol{e}_a\cdot\boldsymbol{e}_b)\big\}, \end{aligned} \tag{52}$$

where $\boldsymbol{X}$ is the embedding of the membrane, and $\mathcal{E}$ is some energy density that depends only on g_{ab} and K_{ab}. The Lagrange multiplier terms allow us to vary $\boldsymbol{X}$, $\boldsymbol{e}_a$, K_{ab}, and g_{ab} independently.

Modulo the Euler–Lagrange equations for the surface, the variation of the energy is

$$\delta E = -\int_M \mathrm{d}A\nabla_a(\boldsymbol{f}^a\cdot\delta\boldsymbol{X} + \Lambda^{ab}\boldsymbol{e}_b\cdot\delta\boldsymbol{n}). \tag{53}$$

Under an infinitesimal scale transformation, the normal vector is unchanged, while the variation of the embedding is $\delta\boldsymbol{X} = \varepsilon\boldsymbol{X}$. We therefore identify $Q^a = \boldsymbol{f}^a\cdot\boldsymbol{X}$ as the Noether current associated with scale invariance. Indeed, taking the divergence of this current, we find

$$\begin{aligned} \nabla_a(\boldsymbol{f}^a\cdot\boldsymbol{X}) &= p\boldsymbol{n}\cdot\boldsymbol{X} + \boldsymbol{f}^a\cdot\nabla_a\boldsymbol{X}\\ &= p\boldsymbol{n}\cdot\boldsymbol{X} + \boldsymbol{f}^a\cdot\boldsymbol{e}_a = p\boldsymbol{n}\cdot\boldsymbol{X} + f^a_a, \end{aligned} \tag{54}$$

where we have used Eq. (48) to find the final equality. For the Willmore functional $p=\lambda=c_0=0$, we see that both terms in $\nabla_a Q^a$ vanish.

This Noether current leads us to a conservation law for axially symmetric membranes as follows. Integrate $\nabla_a(\boldsymbol{f}^a\cdot\boldsymbol{X})$ over a strip Σ of the surface bounded by two horizontal rings, as shown in Fig. 3. We can use the divergence theorem to replace the integral with two line integrals

$$\begin{aligned} &\int_\Sigma \mathrm{d}A\nabla_a(\boldsymbol{f}^a\cdot\boldsymbol{X})\\ &= \int_{\mathcal{C}_1}\mathrm{d}\phi\ \rho\boldsymbol{f}^a\cdot\boldsymbol{X}t_a - \int_{\mathcal{C}_2}\mathrm{d}\phi\ \rho\boldsymbol{f}^a\cdot\boldsymbol{X}t_a = 0, \end{aligned} \tag{55}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the upper and lower boundary curves of the strip, and t_a is the surface vector tangent to the meridians. Since the patch of membrane is chosen arbitrarily, we conclude that the integral

$$\int_{\mathcal{C}}\mathrm{d}\phi\ \rho\boldsymbol{f}^a\cdot\boldsymbol{X}t_a = 2\pi\rho\boldsymbol{f}^a\cdot\boldsymbol{X}t_a \tag{56}$$

is constant for all rings $\mathcal{C}$.

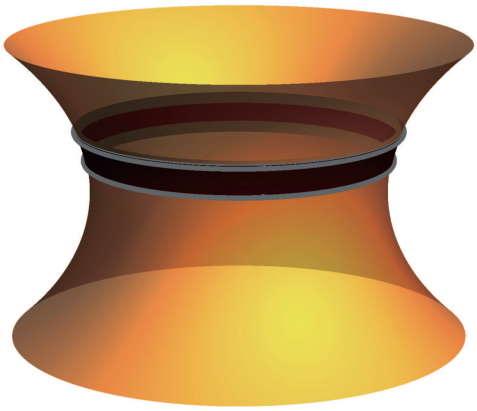


Fig. 3. (color online) An axisymmetric strip Σ (shown in red, with its boundary curves in gray) on a catenoid.

Inserting our parameterization of the surface $\boldsymbol{X} = (\rho\cos\phi, \rho\sin\phi, z)^{\mathrm{T}}$, we find

$$\begin{aligned} &2\pi\rho\boldsymbol{f}^a\cdot\boldsymbol{X}t_a\\ &= \pi k_c\left[\rho^2\psi'^2\cos\psi - \tan\psi(\sin\psi - 2\rho\Omega_0) - 2\Omega_0 z\right]\\ &= \pi k_c I + 2\pi k_c\Omega_0(\rho\tan\psi - z), \end{aligned} \tag{57}$$

where we have used the first integral in Eq. (8) to eliminate ψ'' terms in the stress tensor. Note that this constant does not require Ω_0 to vanish. One explanation for why it is necessary in the framework of the constructed Lagrangian to set $\Omega_0 = 0$ is the presence of the term proportional to $z = \int \mathrm{d}\rho\tan\psi$. At the level of the Lagrangian for $\psi(\rho)$, there is no closed form for $z(\rho)$.

6. Conclusion

We have found the first integral of the second-order shape equation of axisymmetric lipid membranes in a special case from a new mathematical perspective. Our calculations suggest that the existence of this integral requires the vanishing of c_0, $\tilde{\lambda}$, Ω_0, and $\tilde{p}$. In the rest of the paper, we also exhibit how to construct this first integral briefly by Hamilton–Jacobi equation. We further obtain the mechanical interpretation of the first integral by using the membrane stress tensor as in Ref. [12]. We find that the first integral I is proportional to $\boldsymbol{f}^a \cdot \boldsymbol{X} t_a$ when $\Omega_0 = 0$, where t_a is the surface vector tangent to the meridians of the surface.

These results provide us an insight into some limits of the possibility of additional symmetries in the shape equation. In the case of vanished c_0, $\tilde{\lambda}$, Ω_0, and $\tilde{p}$, equation (8) becomes the Willmore equation, which is known to be invariant under conformal transformations. Indeed, we have identified the first integral discussed here as arising from scale invariance; as a result, taking any of those parameters to be non-vanishing, such as c_0, would break the symmetry, invalidating our conservation law. So for the more general form of the axisymmetric Willmore equation with a non-zero Ω_0, we still could not construct its first integral. But we should emphasize that this result depends on the assumption that generator functions ξ and η, or the divergence function B or F, do not explicitly contain ψ'. Only within this restriction on Lie groups of the shape equation, equation (8) has no additional first integrals except for the case when the tension, pressure, and spontaneous curvature vanish. Although the scale invariance will be broken by the parameters in this case, there may be a brand new symmetry that we have not noticed if we consider ψ' in the generator functions, which could lead to a new first integral of Eq. (8) within c_0, $\tilde{\lambda}$, Ω_0, or $\tilde{p}$. It is still an interesting challenge, and we plan to consider such a case in our future work.

Appendix A: Detailed calculations for construction of first integrals

If η, ξ, and B do not explicitly depend on ψ', then Eq. (31) will be

$$\rho \cos\psi F = 0, \tag{A1}$$

which means $F = 0$. Also, equation (32) is transformed into

$$\begin{aligned}&-\rho\cos\psi\xi_\psi\psi'^3 + \left[\cos\psi\left(\xi - \rho\xi_\rho + 2\rho\eta_\psi\right) - \rho\sin\psi\eta\right]\psi'^2\\&+\left[2\rho\cos\psi\eta_\rho - 2\left(B_\psi + V\xi_\psi\right)\right]\psi'\\&-2\left(V\xi_\rho + V_\rho\xi + V_\psi\eta + B_\rho\right) = 0.\end{aligned} \tag{A2}$$

We can separate the above equation into five independent equations according to the different orders of ψ':

$$\rho\cos\psi\xi_\psi = 0, \tag{A3}$$

$$\cos\psi\left(\xi - \rho\xi_\rho + 2\rho\eta_\psi\right) - \rho\sin\psi\eta = 0, \tag{A4}$$

$$\rho\cos\psi\eta_\rho - B_\psi - V\xi_\psi = 0, \tag{A5}$$

$$V\xi_\rho + V_\rho\xi + V_\psi\eta + B_\rho = 0. \tag{A6}$$

From Eqs. (A3) and (A4), we obtain

$$\eta(\rho,\psi) = \frac{C_1(\rho)}{\sqrt{\cos\psi}}\int\sqrt{\cos\psi}\mathrm{d}\psi + \frac{1}{\sqrt{\cos\psi}}C_2(\rho), \tag{A7}$$

$$\xi(\rho) = C_3\rho + 2\rho\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho. \tag{A8}$$

Then from Eqs. (A5), (A6), (29) and $B_{\psi\rho} = B_{\rho\psi}$, we have

$$\begin{aligned}&\frac{1}{2}\{[35 + 56\tilde{\lambda}\rho^2 - 4\left(1 + 6\tilde{\lambda}\rho^2\right)\cos 2\psi + \cos 4\psi\\&+8\rho\sin\psi\left(-9c_0 - 10\Omega_0 + 5\tilde{p}\rho^2 + c_0\cos 2\psi\right)]C_1(\rho)\\&-32\rho\cos^4\psi\left(C_1' + \rho C_1''\right)\}\int\sqrt{\cos\psi}\mathrm{d}\psi\\&+\left[8\rho\left(C_3 + 2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho\right)\right.\\&\left.\left(-c_0 - 2\Omega_0 + 3\tilde{p}\rho^2 + c_0\cos 2\psi + 4\tilde{\lambda}\rho\sin\psi\right)\right.\\&\left.+6\left(-4c_0\rho - 8\Omega_0\rho + 4\tilde{p}\rho^3 + 4c_0\rho\cos 2\psi + 5\sin\psi\right.\right.\\&\left.\left.+8\tilde{\lambda}\rho^2\sin\psi + \sin 3\psi\right)C_1(\rho)\right]\left(1 - \sin^2\psi\right)\sqrt{\sec\psi}\\&+\frac{C_2(\rho)}{2}\left[35 + 56\tilde{\lambda}\rho^2 - 4\left(1 + 6\tilde{\lambda}\rho^2\right)\cos 2\psi + \cos 4\psi\right.\\&\left.+8\rho\left(-9c_0 - 10\Omega_0 + 5\tilde{p}\rho^2 + c_0\cos 2\psi\right)\sin\psi\right]\\&-16\rho\cos^4\psi\left(C_2' + \rho C_2''\right) = 0.\end{aligned} \tag{A9}$$

As we have

$$\sqrt{\sec\psi} = \left(1 - \sin^2\psi\right)^{-\frac{1}{4}}, \tag{A10}$$

$$\int\sqrt{\cos\psi}\mathrm{d}\psi = \int\left(1 - \sin^2\psi\right)^{-\frac{1}{4}}\mathrm{d}(\sin\psi), \tag{A11}$$

we can expand $\sqrt{\sec\psi}$ and $\int\sqrt{\cos\psi}\mathrm{d}\psi$ in terms of $\sin\psi$

$$\sqrt{\sec\psi} = \sum_{n=0}^{\infty}\frac{\Gamma(n+1/4)}{\Gamma(1/4)\Gamma(n+1)}\sin^{2n}\psi, \tag{A12}$$

$$\int\sqrt{\cos\psi}\mathrm{d}\psi = G + \sum_{n=0}^{\infty}\frac{\Gamma(n+1/4)}{(2n+1)\Gamma(1/4)\Gamma(n+1)}\sin^{2n+1}\psi, \tag{A13}$$

where G in Eq. (A13) is an integral constant. Then we can write Eq. (A9) in terms of $\sin\psi$

$$\sum_{n=0}^{4}\theta_n\sin^n\psi + \sum_{n=0}^{\infty}\Theta_n\sin^{2n+5}\psi + \sum_{n=0}^{\infty}\Lambda_n\sin^{2n+6}\psi = 0, \tag{A14}$$

where

$$\begin{aligned}\theta_0 = 8\rho&\left[\left(-2\Omega_0 + 3\tilde{p}\rho^2\right)\left(C_3 + 2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho\right)\right.\\&\left.+3\left(-2\Omega + \tilde{p}\rho^2\right)C_1(\rho)\right]\\&+16G[(1+\tilde{\lambda}\rho^2)C_1(\rho) - \rho(C_1' + \rho C_1'')]\\&+16[(1+\tilde{\lambda}\rho^2)C_2(\rho) - \rho(C_2' + \rho C_2'')],\end{aligned} \tag{A15}$$

$$\theta_1 = 64\tilde{\lambda}\rho^2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho + [64(1+\tilde{\lambda}\rho^2)+4G\rho (-8c_0-10\eta_0+5\tilde{p}\rho^2)]C_1(\rho)+4\rho[8C_3\tilde{\lambda}\rho +(-8c_0-10\Omega_0+5\tilde{p}\rho^2)C_2(\rho)-4C_1'-4\rho C_1''], \quad (A16)$$

$$\theta_2 = -2\rho\Bigg[8C_3c_0-6C_3\Omega_0+9C_3\tilde{p}\rho^2+2(8c_0-6\Omega_0+9\tilde{p}\rho^2) \times\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho+(40c_0+2\Omega_0-12G\tilde{\lambda}\rho-\tilde{p}\rho^2)C_1(\rho) -16G(C_1'+\rho C_1'')-16C_2'-4\rho(3\tilde{\lambda}C_2(\rho)+4C_2'')\Bigg], \quad (A17)$$

$$\theta_3 = -\frac{4}{3}\Bigg[18C_3\tilde{\lambda}\rho^2+44C_1(\rho) +\rho\Bigg(36\tilde{\lambda}\rho\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho+6c_0C_2(\rho) +8\tilde{\lambda}\rho C_1(\rho)+6c_0GC_1(\rho)-23C_1'-23\rho C_1''\Bigg)\Bigg], \quad (A18)$$

$$\theta_4 = \frac{9}{12}C_3\rho(16c_0+2\Omega_0-3\tilde{p}\rho^2)+4C_2(\rho) +\frac{3}{2}\rho(16c_0+2\Omega_0-3\tilde{p}\rho^2)\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho +\frac{\rho}{12}(304c_0+14\Omega_0-7\tilde{p}\rho^2)C_1(\rho)+4GC_1(\rho) -16G\rho(C_1'+\rho C_1'')-16\rho(C_2'+\rho C_2''), \quad (A19)$$

$$\Theta_n = \frac{1}{\Gamma(1/4)\Gamma(n+3)}\Bigg\{-48\tilde{\lambda}\Gamma\left(n+\frac{5}{4}\right)\rho^2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho +[-6C_3\tilde{\lambda}(2n+1)(2n+3)(2n+5)(4n+1)\rho^2 -4(-135-348n-172n^2+24n^3+16n^4)C_1(\rho) -4(n+3)(2n+1)(4n+1)(4n+5)\tilde{\lambda}\rho^2C_1(\rho) -\rho(415+792n+308n^2)(C_1'+\rho C_1'')] \frac{\Gamma(n+1/4)}{(2n+1)(2n+3)(2n+5)}\Bigg\}, \quad (A20)$$

$$\Lambda_n = \frac{\Gamma(n+5/4)\rho}{2\Gamma(1/4)\Gamma(n+4)}\Bigg\{6\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho[8c_0(n+3) +(4n+5)(2\Omega-3\tilde{p}\rho^2)]+[3C_3(2n+3)(2n+5) (8c_0(n+3)+(4n+5)(2\Omega_0-3\tilde{p}\rho^2)) +(8c_0(n+3)(85+82n+16n^2) +(2n+3)(4n+5)(8n+15)(2\Omega-\tilde{p}\rho^2))C_1(\rho)] \times\frac{1}{(2n+3)(2n+5)}\Bigg\}. \quad (A21)$$

Equation (A14) is equivalent to seven independent equations according to the order of $\sin\psi$:

$$\theta_0=\theta_1=\theta_2=\theta_3=\theta_4=\Theta_n=\Lambda_n=0, \quad (A22)$$

which leads to the following equations:

$$\rho\Bigg[(-2\Omega_0+3\tilde{p}\rho^2)\left(C_3+2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho\right) +3(-2\Omega_0+\tilde{p}\rho^2)C_1(\rho)\Bigg] +2G[(1+\tilde{\lambda}\rho^2)C_1(\rho)-\rho(C_1'+\rho C_1'')] +2[(1+\tilde{\lambda}\rho^2)C_2(\rho)-\rho(C_2'+\rho C_2'')]=0, \quad (A23)$$

$$16\tilde{\lambda}\rho^2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho+[16(1+\tilde{\lambda}\rho^2)+G\rho (-8c_0-10\eta_0+5\tilde{p}\rho^2)]C_1(\rho) +8C_3\tilde{\lambda}\rho^2+\rho(-8c_0-10\Omega_0+5\tilde{p}\rho^2)C_2(\rho) -4\rho C_1'-4\rho^2C_1''=0, \quad (A24)$$

$$8C_3c_0-6C_3\Omega_0+9C_3\tilde{p}\rho^2 +2(8c_0-6\Omega_0+9\tilde{p}\rho^2)\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho +(40c_0+2\Omega_0-12G\tilde{\lambda}\rho-\tilde{p}\rho^2)C_1(\rho)-16G(C_1'+\rho C_1'') -16C_2'-4\rho(3\tilde{\lambda}C_2(\rho)+4C_2'')=0, \quad (A25)$$

$$18C_3\tilde{\lambda}\rho^2+44C_1(\rho)+\rho\Bigg(36\tilde{\lambda}\rho\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho+6c_0C_2(\rho) +8\tilde{\lambda}\rho C_1(\rho)+6c_0GC_1(\rho)-23C_1'-23\rho C_1''\Bigg)=0, \quad (A26)$$

$$9C_3\rho(16c_0+2\Omega_0-3\tilde{p}\rho^2)+48C_2(\rho) +18\rho(16c_0+2\Omega_0-3\tilde{p}\rho^2)\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho +\rho(304c_0+14\Omega_0-7\tilde{p}\rho^2)C_1(\rho)+48GC_1(\rho) -192G\rho(C_1'+\rho C_1'')-192\rho(C_2'+\rho C_2'')=0, \quad (A27)$$

$$-12\tilde{\lambda}(4n+1)(2n+1)(2n+3)(2n+5)\rho^2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho -6C_3\tilde{\lambda}(2n+1)(2n+3)(2n+5)(4n+1)\rho^2 -4[-135-348n-172n^2+24n^3+16n^4 +(n+3)(2n+1)(4n+1)(4n+5)\tilde{\lambda}\rho^2]C_1(\rho) -\rho(415+792n+308n^2)(C_1'+\rho C_1'')=0, \quad (A28)$$

$$6(2n+3)(2n+5)[8c_0(n+3)+(4n+5) (2\Omega_0-3\tilde{p}\rho^2)]\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho+3C_3(2n+3)(2n+5) [8c_0(n+3)+(4n+5)(2\Omega_0-3\tilde{p}\rho^2)] +[8c_0(n+3)(85+82n+16n^2) +(2n+3)(4n+5)(8n+15)(2\Omega_0-\tilde{p}\rho^2)]C_1(\rho)=0. \quad (A29)$$

We note that equations (A28) and (A29) are valid for any non-negative integer n, and expressions of $C_1(\rho)$ and $C_2(\rho)$ do not contain n. Then we investigate the coefficients of n^3 and n^4 in Eq. (A28), which should be zero separately, and lead to two equations

$$-912\tilde{\lambda}\rho^2\left(C_3+2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho\right) -32(3+20\tilde{\lambda}\rho^2)C_1(\rho)=0, \quad (A30)$$

$$-192\tilde{\lambda}\rho^2\left(C_3+2\int\frac{C_1(\rho)}{\rho}\mathrm{d}\rho\right) -64(1+2\tilde{\lambda}\rho^2)C_1(\rho)=0. \quad (A31)$$

Canceling $C_3+2\int C_1(\rho)/\rho\mathrm{d}\rho$ in both Eqs. (A30) and (A31),

we have

$$(2\tilde{\lambda}\rho^2 - 13)C_1(\rho) = 0, \tag{A32}$$

which leaves us no choice but $C_1(\rho) = 0$, and hence $C_3\tilde{\lambda} = 0$. Then equations (A23)–(A27) become

$$\begin{aligned}
&2(1+\tilde{\lambda}\rho^2)C_2(\rho)\\
&+\rho(3C_3\tilde{p}\rho^2 - 2C_3\Omega_0 - 2C_2' - 2\rho C_2'') = 0, &\text{(A33)}\\
&(5\tilde{p}\rho^2 - 8c_0 - 10\Omega_0)C_2(\rho) = 0, &\text{(A34)}\\
&8C_3c_0 - 6C_3\Omega_0 + 9C_3\tilde{p}\rho^2\\
&- 12\tilde{\lambda}\rho C_2(\rho) - 16(C_2' + \rho C_2'') = 0, &\text{(A35)}\\
&c_0C_2(\rho) = 0, &\text{(A36)}\\
&3C_3\rho(16c_0 + 2\Omega_0 - 3\tilde{p}\rho^2)\\
&+16C_2(\rho) - 64\rho(C_2' + \rho C_2'') = 0. &\text{(A37)}
\end{aligned}$$

Canceling $\rho(C_2' + \rho C_2'')$ in both Eqs. (A33) and (A35), we can solve $C_2(\rho)$ as

$$C_2(\rho) = \frac{C_3\rho(8c_0 + 10\Omega_0 - 15\tilde{p}\rho^2)}{4(4+7\tilde{\lambda}\rho^2)}. \tag{A38}$$

Making the same cancellation in Eqs. (A33) and (A37), we can solve $C_2(\rho)$ as

$$C_2(\rho) = \frac{C_3\rho(48c_0 + 70\Omega_0 - 105\tilde{p}\rho^2)}{16(3+4\tilde{\lambda}\rho^2)}. \tag{A39}$$

If $C_3 = 0$, then $C_2(\rho) = 0$, which leads to a trivial solution of the first integral. So we should assume $C_3 \neq 0$, and hence $\tilde{\lambda} = 0$, in our discussion. By combining Eqs. (A38) and (A39), we find

$$6c_0 + 10\Omega_0 - 15\tilde{p}\rho^2 = 0, \tag{A40}$$

which leads to

$$\tilde{p} = 0, \quad \Omega_0 = -\frac{3}{5}c_0. \tag{A41}$$

Then we have

$$C_2(\rho) = \frac{c_0C_3}{8}\rho. \tag{A42}$$

From Eq. (A36), we see

$$c_0C_2(\rho) = \frac{c_0^2C_3}{8}\rho = 0, \tag{A43}$$

which requires

$$c_0 = \Omega_0 = \tilde{p} = C_2(\rho) = 0. \tag{A44}$$

It is exactly the solution of Eq. (A9). We could check this by substituting $C_1(\rho) = C_2(\rho) = c_0 = \tilde{\lambda} = \tilde{p} = \Omega_0 = 0$ into Eq. (A9).

Thus a non-trivial solution for Eq. (A9) only exists in the case of vanishing c_0, $\tilde{\lambda}$, $\tilde{p}$, and Ω_0:

$$C_1(\rho) = C_2(\rho) = 0, \quad C_3 \neq 0, \tag{A45}$$

which leaves the only solution for Eq. (A2)

$$B = 0, \quad \eta = 0, \quad \xi = C_3\rho. \tag{A46}$$

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