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Another look at the moist baroclinic Ertel–Rossby invariant with mass forcing*

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Due to the importance of the mass forcing induced by precipitation and condensation in moist processes, the Lagrangian continuity equation without a source/sink term utilized to prove the Ertel–Rossby invariant (ERI) and its conservation property is re-derived considering the mass forcing. By introducing moist enthalpy and moisture entropy, the baroclinic ERI could be adapted to moist flow. After another look at the moist ERI, it is deployed as the dot product between the generalized velocity and the generalized vorticity in moist flow, which constitutes a kind of generalized helicity. Thus, the baroclinic ERI is further extended to the moist case. Moreover, the derived moist ERI forumla remains formally consistent with the dry version, no matter whether mass forcing is present. By using the Weber transformation and the Lagrangian continuity equation with a source/sink effect, the conservation property of the baroclinic ERI in moist flow is revisited. The presence or absence of mass forcing in the Lagrangian continuity equation determines whether or not the baroclinic ERI in moist flow is materially conserved. In other words, it would be qualified as a quasi-invariant but only being dependent on the circumstances. By another look at the moist baroclinic ERI, it is surely a neat formalism with a simple physical explanation, and the usefulness of its anomaly in diagnosing atmospheric flow is demonstrated by case study.

Keywords: quasi-Ertel–Rossby invariant, generalized helicity, mass forcing, adapting

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1. Introduction

The subject of helicity has been pursued fairly actively in the fluid dynamic and magnetohydrodynamic literature for many years. Moffatt[1] surveyed the significance and treatment of helicity, particularly in magnetohydrodynamics. The term "helicity density", or, for brevity, "helicity", refers to a volume-integrated quantity for the local variable $\boldsymbol{v}_{\rm A}\cdot\boldsymbol{\omega}_{\rm A}$,[2–4] where $\boldsymbol{v}_{\rm A}$ and $\boldsymbol{\omega}_{\rm A}$ ($\boldsymbol{\omega}_{\rm A}=\nabla\times\boldsymbol{v}_{\rm A}$) are the absolute velocity vector and absolute vorticity vector, respectively. It has been shown that helicity in three-dimensional (3D) isentropic flow is a constant.[2–4]

In fact, early in 1949, Ertel and Rossby[5] derived the conservation theorem of hydrodynamics, $\frac{\rm d}{{\rm d}t}[\alpha(\boldsymbol{v}_{\rm A}-\nabla W)\cdot\boldsymbol{\omega}_{\rm A}]=0$, from both Lagrange's equations and Euler's equations in barotropic flow. The form of $\alpha(\boldsymbol{v}_{\rm A}-\nabla W)\cdot\boldsymbol{\omega}_{\rm A}$ is named the Ertel–Rossby invariant (ERI),[5] where

$$W_{\rm A}=\int_0^t\left(\frac{v_{\rm A}^2}{2}-\phi_{\rm a}-h\right){\rm d}t,$$

α is the specific volume (the reciprocal of the density), h is the specific enthalpy, and $\phi_{\rm a}$ is the Newtonian potential of attraction. It is actually an alternative form of helicity and another kind of helicity-conservation law in 3D barotropic flow.

To relax the restrictions, several authors have attempted to look for generalization that is also valid for non-barotropic flow, for whatever helicity or ERI. Mobbs[6] replaced the absolute velocity vector $\boldsymbol{v}_{\rm A}$ by the generalized velocity $\boldsymbol{v}_{\rm A}-\eta\nabla s$, and the absolute vorticity vector $\boldsymbol{\omega}_{\rm A}$ by the generalized vorticity $(\boldsymbol{\omega}_{\rm A}-\nabla\eta\times\nabla s)$, and proposed corresponding non-barotropic generalization of the helicity-conservation law, where η is one of the Clebsch potentials[7–9] and s is the specific entropy. Gaffet[10] also presented a type of generalization of the 3D helicity-conservation law assuming zero potential vorticity, which coincides formally with the non-barotropic generalization formula $((\boldsymbol{v}_{\rm A}-\eta\nabla s)\cdot(\boldsymbol{\omega}_{\rm A}-\nabla\eta\times\nabla s))$ developed by Mobbs.[6] Zdunkowski and Bott[11] revisited the ERI based on Weber transformation and the Lagrangian continuity equation in barotropic atmosphere. According to the above generalization approach to extending to baroclinic flow,[6,10] Zdunkowski and Bott[11] also gave its baroclinic form $(\alpha(\boldsymbol{v}_{\rm A}-\nabla W-\eta\nabla s)\cdot(\boldsymbol{\omega}_{\rm A}-\nabla\eta\times\nabla s))$.

The derivations by the above authors were performed in dry flow, which was used heuristically in Yang *et al.*'s paper[12] to further extend the derivation to moist flow on the basis of the baroclinic ERI (a kind of generalized helicity) given by Zdunkowski and Bott (henceforth referred to as

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ZB03)[11] and they proved its conservation property by beginning from continuity equation without mass sour/sink term. Furthermore, the mass forcing induced by precipitation and condensation in moist processes is significant.[13–16] Therefore, in the present study we try to answer the following questions. 1) Recalling previous derivation of ERI and its conservation property, the continuity equation is utilized, which brings forth the first question. What formula will the continuity equation be transformed into in general coordinate when a source/sink term is considered? 2) Does the quantity keep its usual expression in moist fluid when mass forcing is given? 3) If the Lagrangian continuity equation utilized in the proof of the ERI's conservation property[11] is changed from a no-source to a with-source/sink term due to mass forcing, does the baroclinic ERI have a conservation property in moist flow? To answer these questions, we adapt the continuity equation with a mass source/sink effect in general coordinates in Section 2. Then we revisit the baroclinic ERI, trying to gain an insight into its nature, to improve new understanding by another look at ERI itself, and to bring new knowledge due to mass forcing in moist flow in Section 3. Based on the derivation of the continuity equation with a mass source/sink term in general coordinates in Section 2, the conservation of the baroclinic ERI in moist flow under certain conditions will be proved and discussed in Section 4. Then ERI anomaly will be demonstrated by case study in Section 5. A summary is provided in the final section.

2. Adapt the continuity equation with a mass source/sink term in general coordinates

Recalling previous derivation of ERI, the continuity equation is utilized.[11,12] Specifically, we focus on moist environmental flow, for which many authors have pointed out that the mass forcing induced by precipitation and condensation is significant.[13–16] However, a source/sink term due to mass forcing was not included in the Lagrangian continuity equation during the derivation of the ERI and its conservation property,[11,12] which brings forth the first question outlined in the introduction: what form does the continuity equation take in general coordinate when a source/sink term is considered? To answer this question, the derivation is performed below.

If the source/sink term is added, the continuity equation, $\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\nabla\cdot\boldsymbol{v}_{\mathrm{A}}=0$, can be rewritten in general q^i coordinates[11] as

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\nabla\cdot\boldsymbol{v}+\rho\nabla\cdot\boldsymbol{v}_{\mathrm{p}}=S_r, \tag{1}$$

where $\boldsymbol{v}_{\mathrm{A}}=\boldsymbol{v}+\boldsymbol{v}_{\mathrm{p}}=\boldsymbol{v}+\boldsymbol{v}_{\Omega}+v_{\mathrm{D}}$, $\boldsymbol{v}_{\Omega}$ is the rotational velocity due to the rotation of the earth about its axis, and v_{D} is known as the deformation velocity due to the deformation of the coordinate surfaces $q^i=$ constant, since in an arbitrary time-dependent coordinate system a point fixed on a coordinate surface $q^i=$ constant may perform motions with respect to the corresponding x^i-coordinate of the inertial system. In Eq. (1), $S_r=\nabla\cdot(\rho_r\boldsymbol{V}_r)$ is the mass source/sink term.[13–15,17] Since Schubert *et al.*[15] and Lackmann and Yablonsky[16] pointed out the importance of the mass source/sink term in the moist atmosphere, the source/sink term is included in Eq. (1). The velocity of water condensate relative to air $\boldsymbol{V}_r=-14.164\rho_r^{0.1364}(\rho_{a0}/\rho_a)^{0.5}f_{\mathrm{ice}}$; $\rho_{\mathrm{a0}}=p_0/RT_0$ is the reference density for dry air; $p_0=1000$ hPa is the reference pressure; $T_0=273.15$ K is the reference temperature; R is a gas constant of dry air; ρ_a, ρ_w, and ρ_r are the densities of dry air, water vapor, and airborne condensate, respectively; and

$$f_{\mathrm{ice}}=\begin{cases}0.2+0.8\,\mathrm{sech}\left(\dfrac{T_0-T}{5.0}\right), & \text{if } T<T_0,\\ 1.0, & \text{if } T\geq T_0.\end{cases}$$

In Gao *et al.*'s study,[17] in order to avoid calculating $\boldsymbol{V}_r$, which might be difficult to determine accurately in the real atmosphere, S_r is estimated by Q_m approximately. In their calculations, the condensation rate Q_m is computed by differentiating the density of water vapor with respect to time, from the model outputs (time series of ρ_w fields). We adopt this simple method of calculation herein.

In general q^i coordinates, the del operator and velocity vector are denoted as $\nabla=\boldsymbol{q}^n\frac{\partial}{\partial q^n}$ [Eq. (M3.21) in ZB03], $\boldsymbol{v}=\dot{\boldsymbol{q}}^n\boldsymbol{q}^n$ [Eq. (M3.32) in ZB03], and symbols $\sqrt{g}=\frac{1}{J(q^1,q^2,q^3)}$ and $\frac{\partial}{\partial t}+\dot{q}^n\frac{\partial}{\partial q^n}=\frac{\mathrm{d}}{\mathrm{d}t}$ [Eq. (M3.35) in ZB03] are used, with $J(q^1,q^2,q^3)\neq 0$ herein. Note that the derivation here is similar to that in the paper by Zdunkowski and Bott,[11] but with a mass source/sink term.

By combining with Eq. (M4.36) in ZB03, the following relation is obtained

$$\nabla\cdot\boldsymbol{v}_{\mathrm{p}}=\frac{1}{\sqrt{g}}\left(\frac{\partial\sqrt{g}}{\partial t}\right)_{q^i}=\nabla\cdot\boldsymbol{v}_{\mathrm{D}},$$

where the divergence of $\boldsymbol{v}_{\Omega}$ is zero. Thus, equation (1) is rewritten as

$$\sqrt{g}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\sqrt{g}\nabla\cdot v+\rho\frac{\partial\sqrt{g}}{\partial t}=\sqrt{g}S_r. \tag{2}$$

By utilizing Eq. (M3.56) in ZB03,

$$\nabla\cdot A=\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial q^n}(A^n\sqrt{g})\right)=\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial q^n}(q^n\cdot A\sqrt{g})\right),$$

equation (2) becomes

$$\sqrt{g}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\frac{\partial}{\partial q^n}(\dot{q}^n\sqrt{g})+\rho\frac{\partial\sqrt{g}}{\partial t}=\sqrt{g}S_r. \tag{3}$$

The terms on the left-hand side of Eq. (3) can be written as

$$\sqrt{g}\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\frac{\partial}{\partial q^n}(\dot{q}^n\sqrt{g})+\rho\frac{\partial\sqrt{g}}{\partial t}$$

$$= \sqrt{g}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{\partial\sqrt{g}}{\partial t} + \dot{q}^n\frac{\partial\sqrt{g}}{\partial q^n}\right) + \rho\sqrt{g}\frac{\partial\dot{q}^n}{\partial q^n}$$
$$= \sqrt{g}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\mathrm{d}\sqrt{g}}{\mathrm{d}t} + \rho\sqrt{g}\frac{\partial\dot{q}^n}{\partial q^n}.$$

Therefore, equation (3) becomes

$$\frac{\mathrm{d}(\rho\sqrt{g})}{\mathrm{d}t} + \rho\sqrt{g}\frac{\partial\dot{q}^n}{\partial q^n} = \sqrt{g}S_r. \tag{4}$$

From Eq. (4), we obtain the continuity equation with a mass source/sink term in general q^i coordinates,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha J(q^1,q^2,q^3)] - \alpha J(q^1,q^2,q^3)\frac{\partial\dot{q}^n}{\partial q^n} = -\alpha^2 J(q^1,q^2,q^3)S_r. \tag{5}$$

Based on Eq. (5), the derivations of both moist baroclinic ERI formula itself and its conservation property will be revisited to investigate the change due to mass forcing.

3. Revisiting baroclinic Ertel–Rossby invariant in moist flow

To answer the second question aforementioned in the introduction, the baroclinic ERI is reviewed quickly to explore whether the quantity keeps its usual expression in moist fluid when the mass forcing is given by the derivation below.

3.1. Reviewing and complementarily understanding the derivation of ERI

Starting from the vector relationship of three momentum equations for frictionless flow ($\mathrm{d}\boldsymbol{v}_\mathrm{A}/\mathrm{d}t = -\nabla\phi_\mathrm{a} - \alpha\nabla p$) and the differential form of the definition of equivalent potential temperature

$$\left(\frac{1}{\theta_e}\nabla\theta_e = \frac{1}{\theta}\nabla\theta + \nabla\left(\frac{Lq_s}{C_\mathrm{p}T}\right)\right),$$

the following equation is gained

$$\frac{\mathrm{d}\boldsymbol{v}_\mathrm{A}}{\mathrm{d}t} = -\nabla(\phi_\mathrm{a} + h_m) + T\nabla s_m, \tag{6}$$

where the specific entropy ($s_m = C_\mathrm{p}\ln\theta_e$) and enthalpy ($h_m = C_\mathrm{p}T + Lq_s$) are introduced. The absolute velocity vector $\boldsymbol{v}_\mathrm{A} = \boldsymbol{v} + \boldsymbol{\Omega}\times r$ with $\boldsymbol{v}$, $\boldsymbol{\Omega}$, and $\boldsymbol{r}$ denoting the relative velocity vector, the angular velocity of rotating frame, and the radius of the earth; θ, q_s, T, C_p, and L represent the potential temperature, saturated specific humidity, absolute temperature, the specific heat of dry air at constant pressure p, and the latent heat of vaporization, respectively.

In Cartesian coordinates, the momentum vector, Eq. (6), expressed in terms of the absolute velocity components, can be written in components as

$$\frac{\mathrm{d}^2x^k}{\mathrm{d}t^2} = -\frac{\partial}{\partial x^k}(\phi_\mathrm{a} + h_m) + T\frac{\partial s_m}{\partial x^k}, \quad k = 1,2,3. \tag{7}$$

In Lagrangian coordinates, equation (7) is rewritten as

$$\left(\frac{\partial^2x^k}{\partial t^2}\right)_L = -\frac{\partial}{\partial x^k}(\phi_\mathrm{a} + h_m) + T\frac{\partial s_m}{\partial x^k}, \quad k = 1,2,3, \tag{8}$$

where the subscript L denotes the Lagrangian system. We use the vector $\boldsymbol{a}$ to denote the Lagrangian particle labels or "enumeration coordinates", which we define as the coordinates of the spatial position of the fluid blob at $t = 0$. There is a one-to-one map between the enumeration coordinates and the vector $\boldsymbol{x}(\boldsymbol{a},t)$, which gives the position of the particle at time t whose initial position is $\boldsymbol{a}$. We assume that both $\boldsymbol{x}(\boldsymbol{a},t)$ and its inverse $\boldsymbol{a}(\boldsymbol{x},t)$ are differentiable everywhere. By multiplying both sides of Eq. (8) by $\partial x^k/\partial a^i$ and then summing over k, we obtain

$$\frac{\partial x^n}{\partial a^i}\left(\frac{\partial^2x^n}{\partial t^2}\right)_L = -\frac{\partial x^n}{\partial a^i}\frac{\partial}{\partial x^n}(\phi_\mathrm{a} + h_m) + T\frac{\partial s_m}{\partial x^n}\frac{\partial x^n}{\partial a^i}$$
$$= -\frac{\partial}{\partial a^i}(\phi_\mathrm{a} + h_m) + T\frac{\partial s_m}{\partial a^i}. \tag{9}$$

Equation (9) can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{\partial x^n}{\partial a^i}\frac{\partial x^n}{\partial t}\right)_L = \frac{\partial L_{\mathrm{A}m}}{\partial a^i} + T\frac{\partial s_m}{\partial a^i}, \tag{10}$$

with

$$\frac{\partial}{\partial a^i}\left(\frac{1}{2}\frac{\partial x^n}{\partial t}\frac{\partial x^n}{\partial t}\right)_L = \frac{\partial}{\partial a^i}\left(\frac{\boldsymbol{v}_\mathrm{A}^2}{2}\right).$$

For brevity,

$$L_{\mathrm{A}m} = \frac{\boldsymbol{v}_\mathrm{A}^2}{2} - \phi_\mathrm{a} - h_m$$

is defined as the Lagrangian function in the absolute reference frame.

To derive an auxiliary relation needed to justify Eq. (11) below, we note that at the initial time ($t = 0$), the Lagrangian and Eulerian systems coincide so that $\boldsymbol{x}(\boldsymbol{a},0) = \boldsymbol{a}$. This implies that $\partial x^n/\partial a^i = \partial x^n/\partial x^i$. This is equal to the usual Kronecker delta function δ_{in}, which is defined as one when its two subscripts, i and n, are equal and zero otherwise. Therefore,

$$\left(\frac{\partial x^n}{\partial a^i}\frac{\partial x^n}{\partial t}\right)_{t=0} = \left(\frac{\partial x^n}{\partial x^i}\frac{\partial x^n}{\partial t}\right)_{t=0} = \left(\delta_{in}\frac{\partial x^n}{\partial t}\right)_{t=0}$$
$$= \left(\frac{\partial x^i}{\partial t}\right)_{t=0} = (u_\mathrm{A}^i)_{t=0}.$$

After equation (10) is integrated with respect to time for moist isentropic flow ($\dot{s}_m = 0$), we obtain the following expression:

$$\left(\frac{\partial x^n}{\partial a^i}\frac{\partial x^n}{\partial t}\right)_{L,t} = \left(\frac{\partial x^i}{\partial t}\right)_{t=0} + \frac{\partial W_{\mathrm{A}m}}{\partial a^i} + \eta\frac{\partial s_m}{\partial a^i}, \tag{11}$$

where $\eta = \int_0^t T\,\mathrm{d}t$ and $W_{\mathrm{A}m} = \int_0^t L_{\mathrm{A}m}\,\mathrm{d}t$ are the action integrals of the absolute reference system. By their definition as particle labels, defined as the initial positions of the particles, the enumeration coordinates a do not change with time ($\dot{a}^i = 0$).

Returning to the x^i system by a series of simplification similar to ZB03, then there will be

$$\left(\frac{\partial x^k}{\partial t}\right)_L = \frac{\partial a^n}{\partial x^k}\left(\frac{\partial x^n}{\partial t}\right)_{t=0} + \frac{\partial W_{\mathrm{A}m}}{\partial x^k} + \eta\frac{\partial s_m}{\partial x^k}, \quad (12)$$
$$k = 1,2,3.$$

The above equations are multiplied by the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ for $k = 1,2,3$, respectively. By adding the results and using the definitions

$$\nabla a^k = \boldsymbol{i}\frac{\partial a^k}{\partial x^1} + \boldsymbol{j}\frac{\partial a^k}{\partial x^2} + \boldsymbol{k}\frac{\partial a^k}{\partial x^3}$$

and $\boldsymbol{v}_{\mathrm{A}} = \boldsymbol{i}u_{\mathrm{A}} + \boldsymbol{j}v_{\mathrm{A}} + \boldsymbol{k}w_{\mathrm{A}}$ with $(\partial x^1/\partial t)_L = u_{\mathrm{A}}$, $(\partial x^2/\partial t)_L = v_{\mathrm{A}}$ and $(\partial x^3/\partial t)_L = w_{\mathrm{A}}$, we obtain the absolute velocity

$$\boldsymbol{v}_{\mathrm{A}} = u_{\mathrm{A},0}\nabla a^1 + v_{\mathrm{A},0}\nabla a^2 + w_{\mathrm{A},0}\nabla a^3 + \nabla W_{\mathrm{A}m} + \eta\nabla s_m. \quad (13)$$

The suffix 0 refers to $t = 0$, and the notation $\mathrm{d}()/\mathrm{d}t = (\dot{})$.

From Eq. (13), move the terms $\nabla W_{\mathrm{A}m}$ and $\eta\nabla s_m$ on the right-hand side to the left-hand side of Eq. (13), and define

$$\boldsymbol{B}_{\mathrm{A}} = \boldsymbol{v}_{\mathrm{A}} - \nabla W_{\mathrm{A}m} - \eta\nabla s_m, \quad (14a)$$

which implies

$$\nabla\times\boldsymbol{B}_{\mathrm{A}} = \nabla\times\boldsymbol{v}_{\mathrm{A}} - \nabla\eta\times\nabla s_m, \quad (14b)$$

then we will obtain the expression $\psi_{\mathrm{ER}} = \alpha\boldsymbol{B}_{\mathrm{A}}\cdot\nabla\times\boldsymbol{B}_{\mathrm{A}} = \alpha(\boldsymbol{v}_{\mathrm{A}} - \nabla W_{\mathrm{A}m} - \eta\nabla s_m)\cdot(\nabla\times\boldsymbol{v}_{\mathrm{A}} - \nabla\eta\times\nabla s_m)$, here is dubbed the "baroclinic ERI" in moist flow.

Equation (13) reminds us to recall some basic results under the Clebsch transformation.[7–9] According to the Clebsch transformation[7] for the velocity field in barotropic flow ($\boldsymbol{v}_{\mathrm{A}} = \alpha\nabla\beta + \nabla\phi$) and its generalization in non-barotropic flow proposed by Herivel[8] ($\boldsymbol{v}_{\mathrm{A}} = \nabla\phi + \eta\nabla s$), Seliger and Whitham[9] obtained two additional potentials expressed as

$$\boldsymbol{v}_{\mathrm{A}} = \alpha\nabla\beta + \nabla\phi + \eta\nabla S, \quad (15)$$

where $\alpha\nabla\beta$ corresponds with the first three terms on the rhs of Eq. (13), associated with the initial velocity, and $\nabla\phi$ and $\eta\nabla S$ correspond to the last two terms on the right-hand side of Eq. (13). Thus, formula (13) coincides formally with Eq. (15).

Seliger and Whitham[9] indicated that the velocity field could be written as Eq. (15) for the most general flow. Equation (13) derived above is the further generalization of Eq. (15) to moist flow.

3.2. New knowledge about the formula and physical sense of ERI

Compared with our previous study,[12] the present study gives a simple physical explanation of baroclinic ERI in moist flow on the basis of the more in-depth understanding of the formulism. From Eqs. (14a) and (14b) and the definition of the ERI in moist flow in Section 2, the expression of $\psi_{\mathrm{ER}}(=\alpha(\boldsymbol{v}_{\mathrm{A}} - \nabla W_{\mathrm{A}m} - \eta\nabla s_m)\cdot(\nabla\times\boldsymbol{v}_{\mathrm{A}} - \nabla\eta\times\nabla s_m))$ is deployed as the dot product between $\boldsymbol{B}_{\mathrm{A}}(=\boldsymbol{v}_{\mathrm{A}} - \nabla W_{\mathrm{A}m} - \beta\nabla s_m)$ and $\nabla\times\boldsymbol{B}_{\mathrm{A}}(=\nabla\times\boldsymbol{v}_{\mathrm{A}} - \nabla\beta\times\nabla s_m)$. Like the previous name,[6,11,17] $\boldsymbol{B}_{\mathrm{A}}$ and $\nabla\times\boldsymbol{B}_{\mathrm{A}}$ are called the generalized velocity and generalized vorticity in moist flow, respectively. Both $-\eta\nabla s_m$ in $\boldsymbol{B}_{\mathrm{A}}$ and $-\nabla\beta\times\nabla s_m$ in $\nabla\times\boldsymbol{B}_{\mathrm{A}}$ represent the baroclinic effect, which exactly constitutes the difference between the barotropic[5] and baroclinic ERI expressions.[11] Based on the concepts of generalized velocity and generalized vorticity, $\psi_{\mathrm{ER}} = \alpha\boldsymbol{B}_{\mathrm{A}}\cdot(\nabla\times\boldsymbol{B}_{\mathrm{A}})$ is the generalized helicity in moist baroclinic atmosphere. Therefore, by introducing moist enthalpy (h_m in $W_{\mathrm{A}m}$) and moisture entropy (s_m) in the dry ERI formula ($\alpha(\boldsymbol{v}_{\mathrm{A}} - \nabla W - \eta\nabla s)\cdot(\boldsymbol{\omega}_{\mathrm{A}} - \nabla\eta\times\nabla s)$), the moist ERI is derived, and the moist ERI coincides formally with the dry version.

Thus, we have answered questions 1 and 2 mentioned in the introduction: we can further extend the derivation of the continuity equation in general coordinate to moist flow with mass source/sink term, and the quantity of the so-called ERI does keep its usual expression in moist fluid with mass forcing. That is to say, the moist ERI formula (ψ_{ER}) derived here coincides formally with the dry version ($\alpha(\boldsymbol{v}_{\mathrm{A}} - \nabla W - \eta\nabla s)\cdot(\boldsymbol{\omega}_{\mathrm{A}} - \nabla\eta\times\nabla s)$),[5,11] except that specific enthalpy (h in W_{A}) and specific entropy (s) are replaced by moist enthalpy (h_m in $W_{\mathrm{A}m}$) and moisture entropy (s_m) in the present study by considering the moisture effect. To answer question 3, we now prove whether the baroclinic ERI in moist flow remains conserved following the flow given mass forcing.

3.3. Renew of definition of ERI itself considering mass forcing

It seemed that the former two problems aforementioned in the introduction have been solved. A small question should be noted. At a quick glance over the above derivation, whether the mass forcing is ignored or not, the ERI expression keeps its usual form because the continuity equation derived in Section 2 is not utilized therein during the overall derivation in Section 3. However, mass forcing makes ERI no longer conserved (the detailed proof will be given in the next subsection). Thus, from the definition itself, the so-called ERI is no longer an invariant because it is conditionally conserved. Maybe renewing the definition and name is a more appropriate choice. Thus, it should be renamed a quasi-invariant on the premise of conditional conserved. But we still follow the utility of ERI below in order to respect many previous researchers (e.g., Ertel and Rossby;[5] Zdunkowski and Bott[11]).

4. Extended derivation of the conservation of baroclinic ERI in moist flow with a mass source/sink term

First, a reminder of the topic of the present paper: adapt the baroclinic ERI in moist flow, with mass forcing given. Recalling the derivation process of the ERI conservation property, the continuity equation without a source/sink effect is utilized.[11]

Based on the derivation in Section 2, by a transformation from the Cartesian to general coordinates, replacing the total density ρ by the specific volume α, and carrying out the differentiation, the continuity equation can be rewritten as the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha J(q^1,q^2,q^3)]-\alpha J(q^1,q^2,q^3)\frac{\partial \dot{q}^n}{\partial q^n}=-\alpha^2 J(q^1,q^2,q^3)S_r,$$

if the source/sink term is added, where $J(q^1,q^2,q^3)=\mathrm{Det}(\partial q^j/\partial x^i)$. If $M_{i.}^{.j}$ are the adjoints of the matrix $(\partial q^j/\partial x^i)$, there is[11]

$$\begin{pmatrix}\frac{\partial}{\partial q^1}\\ \frac{\partial}{\partial q^2}\\ \frac{\partial}{\partial q^3}\end{pmatrix}=\left(\frac{\partial x^j}{\partial q^i}\right)\begin{pmatrix}\frac{\partial}{\partial x^1}\\ \frac{\partial}{\partial x^2}\\ \frac{\partial}{\partial x^3}\end{pmatrix}=\frac{(M_{i.}^{.j})}{J(q^1,q^2,q^3)}\begin{pmatrix}\frac{\partial}{\partial x^1}\\ \frac{\partial}{\partial x^2}\\ \frac{\partial}{\partial x^3}\end{pmatrix}$$
$$=\frac{1}{J(q^1,q^2,q^3)}\begin{pmatrix}J(,\ q^2,q^3)\\ J(q^1,\ ,q^3)\\ J(q^1,q^2,\ )\end{pmatrix}. \tag{16}$$

By using Eq. (16), we find directly that

$$\begin{pmatrix}\frac{\partial \dot{q}^1}{\partial q^1}\\ \frac{\partial \dot{q}^2}{\partial q^2}\\ \frac{\partial \dot{q}^3}{\partial q^3}\end{pmatrix}=\frac{1}{J(q^1,q^2,q^3)}\begin{pmatrix}J(\dot{q}^1,q^2,q^3)\\ J(q^1,\dot{q}^2,q^3)\\ J(q^1,q^2,\dot{q}^3)\end{pmatrix}. \tag{17}$$

Substitution of this expression into Eq. (5) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[\alpha J(q^1,q^2,q^3)]\\ =\ &\alpha[J(\dot{q}^1,q^2,q^3)+J(q^1,\dot{q}^2,q^3)+J(q^1,q^2,\dot{q}^3)]\\ &-\alpha^2 J(q^1,q^2,q^3)S_r,\end{aligned} \tag{18}$$

where all spatial derivatives are taken with respect to the Cartesian coordinates. Equation (18) is similar to the version of the continuity equation in Ertel,[18] and is of great advantage whenever conservative quantities, i.e., invariant field functions, are involved. For example, there is an arbitrary invariant ψ^i $(i=1,2,3)$, which satisfies the relation $\mathrm{d}\psi^i/\mathrm{d}t=\dot{\psi}^i=0$. Then, the term on the left-hand side of Eq. (18) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha J(\psi^1,\psi^2,\psi^3)]=\frac{\mathrm{d}}{\mathrm{d}t}(\alpha\nabla\psi^1\cdot\nabla\psi^2\times\nabla\psi^3)$$

and the sum of terms on the right-hand side of Eq. (18) is equal to the mass source/sink term $(-\alpha^2 J(\psi^1,\psi^2,\psi^3)S_r)$. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\alpha J(\psi^1,\psi^2,\psi^3)]\ &=\ \frac{\mathrm{d}}{\mathrm{d}t}(\alpha\nabla\psi^1\cdot\nabla\psi^2\times\nabla\psi^3)\\ &=\ -\alpha^2 J(\psi^1,\psi^2,\psi^3)S_r.\end{aligned} \tag{19}$$

Equations (14a) and (14b) in Section 3 imply

$$\boldsymbol{B}_{\mathrm{A}}=u_{\mathrm{A},0}\nabla a^1+v_{\mathrm{A},0}\nabla a^2+w_{\mathrm{A},0}\nabla a^3\Rightarrow; \tag{20a}$$
$$\nabla\times\boldsymbol{B}_{\mathrm{A}}=\nabla u_{\mathrm{A},0}\times\nabla a^1+\nabla v_{\mathrm{A},0}\times\nabla a^2+\nabla w_{\mathrm{A},0}\times\nabla a^3. \tag{20b}$$

From Eq. (20), we can obtain the following equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\psi_{\mathrm{ER}}]\ &=\ \frac{\mathrm{d}}{\mathrm{d}t}(\alpha\boldsymbol{B}_{\mathrm{A}}\cdot\nabla\times\boldsymbol{B}_{\mathrm{A}})\\ &=\ \frac{\mathrm{d}}{\mathrm{d}t}[\alpha(u_{\mathrm{A},0}\nabla a^1+v_{\mathrm{A},0}\nabla a^2+w_{\mathrm{A},0}\nabla a^3)\\ &\quad\cdot(\nabla u_{\mathrm{A},0}\times\nabla a^1+\nabla v_{\mathrm{A},0}\times\nabla a^2+\nabla w_{\mathrm{A},0}\times\nabla a^3)].\end{aligned} \tag{21}$$

Note that the Lagrangian particle labels (a^1, a^2, a^3) and the initial values of the velocities, ($u_{\mathrm{A},0}$, $v_{\mathrm{A},0}$, $w_{\mathrm{A},0}$) are all independent of time by definition, and thus of course have zero time derivatives. Therefore, for any term on the right-hand side of Eq. (21), e.g.,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(u_{\mathrm{A},0}\nabla a^1)\cdot(\nabla v_{\mathrm{A},0}\times\nabla a^2)]$$

herein, in combination with Eqs. (18) and (19), we obtain the following equation:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[\alpha(u_{\mathrm{A},0}\nabla a^1)\cdot(\nabla v_{\mathrm{A},0}\times\nabla a^2)]\\ =\ &\frac{\mathrm{d}}{\mathrm{d}t}[\alpha u_{\mathrm{A},0}J(a^1,v_{\mathrm{A},0},a^2)]\\ =\ &\alpha u_{\mathrm{A},0}J(\dot{a}^1,v_{\mathrm{A},0},a^2)+\alpha u_{\mathrm{A},0}J(a^1,\dot{v}_{\mathrm{A},0},a^2)\\ &+\alpha u_{\mathrm{A},0}J(a^1,v_{\mathrm{A},0},\dot{a}^2)-\alpha^2 u_{\mathrm{A},0}J(a^1,v_{\mathrm{A},0},a^2)S_r\\ =\ &-\alpha^2 u_{\mathrm{A},0}J(a^1,v_{\mathrm{A},0},a^2)S_r.\end{aligned}$$

Similarly, other terms on the right-hand side of Eq. (21) are derived. Therefore, equation (21) turns into the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\psi_{\mathrm{ER}}]\ &=\ \frac{\mathrm{d}}{\mathrm{d}t}(\alpha\boldsymbol{B}_{\mathrm{A}}\cdot\nabla\times\boldsymbol{B}_{\mathrm{A}})\\ &=\ -\alpha^2\{u_{\mathrm{A},0}[J(a^1,v_{\mathrm{A},0},a^2)+J(a^1,w_{\mathrm{A},0},a^3)]\\ &\quad+v_{\mathrm{A},0}[J(a^2,u_{\mathrm{A},0},a^1)+J(a^2,w_{\mathrm{A},0},a^3)]\\ &\quad+w_{\mathrm{A},0}[J(a^3,u_{\mathrm{A},0},a^1)+J(a^3,v_{\mathrm{A},0},a^2)]\}S_r.\end{aligned} \tag{22}$$

That is to say, if the mass source/sink term is ignored ($S_{\mathrm{r}}=0$), ψ_{ER} is a conserved invariant ($\frac{\mathrm{d}}{\mathrm{d}t}[\psi_{\mathrm{ER}}]=0$) in the moist flow. The presence or absence of mass forcing in the Lagrangian continuity equation directly determines whether or not the baroclinic ERI in moist flow is materially conserved.

For the convenience of comparing with previous traditional concept, Ertel potential vorticity herein, equation (22) is further transformed. From Eq. (22), the coefficient before S_r depends on initial condition and does not change with time. Therefore, from Eq. (19), this coefficient could be rewritten as

$$\begin{aligned}&-\alpha^2\{u_{\mathrm{A},0}[\nabla a^1\cdot(\nabla v_{\mathrm{A},0}\times\nabla a^2)+\nabla a^1\cdot(\nabla w_{\mathrm{A},0}\times\nabla a^3)]\\&+v_{\mathrm{A},0}[\nabla a^2\cdot(\nabla u_{\mathrm{A},0}\times\nabla a^1)+\nabla a^2\cdot(\nabla w_{\mathrm{A},0}\times\nabla a^3)]\\&+w_{\mathrm{A},0}[\nabla a^3\cdot(\nabla u_{\mathrm{A},0}\times\nabla a^1)+\nabla a^3\cdot(\nabla v_{\mathrm{A},0}\times\nabla a^2)]\}. \quad (23)\end{aligned}$$

Also combined with Eq. (21), expression (23) is an equivalence with the ERI itself (ψ_{ER}), except for an $-\alpha$ factor endowed in Eq. (23). Thus, a new equation (24) is derived.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[\psi_{\mathrm{ER}}] &= \frac{\mathrm{d}}{\mathrm{d}t}(\alpha\boldsymbol{B}_{\mathrm{A}}\cdot\nabla\times\boldsymbol{B}_{\mathrm{A}})\\&= -\alpha\{\alpha\boldsymbol{B}_{\mathrm{A}}\cdot\nabla\times\boldsymbol{B}_{\mathrm{A}}\}S_r. \quad (24)\end{aligned}$$

It is actually an alternative form of ERI equation with mass forcing. Retrospecting the MPV equation with source/sink effect (i.e., Eq. (11) in Ref. [17] by Gao *et al.*), the mass forcing term responsible for the MPV tendency is $-\alpha\{\alpha\omega_{\mathrm{A}}\cdot\nabla\theta_e\}Q_m$. Thus, the exactly equivalent form of mass forcing effect is derived, for whatever ERI or MPV, expressed as the co-action between $-\alpha S_r$ and ERI (or MPV) itself.

5. Case study

In 2008, Brennan *et al.* used PV anomalies to diagnose the development of extratropical cyclone and low-level jet, as a utility of PV nonconservation.[19] Similarly, we diagnose the evolutions of moist ERI anomalies near typhoons in this section, which may be a possible application of ERI nonconservation to real moist atmosphere for fast-manifold flow. Using the simulation data for a triple-typhoon case, the evolutions of the moist ERI anomalies are analyzed during their landing on the Chinese main land.

5.1. General description and numerical simulation of the triple-typhoon system

Due to the interaction among several typhoons, multi-typhoon systems are complicated. The tracks and landing points are more difficult to capture. A triple-typhoon system comprised of Typhoons "Lionrock", "Kompasu", and "Namtheun" which attacked China together in 2010 is studied here. "Lionrock" began to be named and numbered at 06 UTC on 28 August 2010, formed as a tropical cyclone above the South China Sea at 06 UTC on 28 August 2010, then moved northwards and intensified. After a "question mark"-form path, it landed at ZhangPu county in Fujian province with a maximum wind speed of 23 $\mathrm{m\cdot s^{-1}}$ and a minimum pressure of 990 hPa at about 23 UTC on 1 September 2010. It weakened gradually during its westward movement, but had a long lifetime, lasting about 6 days as an identifiable tropical storm. Compared with "Lionrock", Typhoon "Kompasu" was a relatively fast-moving typhoon that traveled far during its lifetime of only four days. Typhoon "Namtheun" lived less than 2 days. It had a track that was shorter than "Kompasu" and not so complicated as "Lionrock". The long-term stagnation of "Lionrock" and the long-range travel of "Kompasu" brought especially long-lasting, disastrously heavy precipitation over a wide area, which caused enormous casualties and economic loss.

To resolve tropical cyclones and be suitable for a fast manifold analysis, a 3D non-hydrostatic weather research and forecast (WRF) model (V3.7.1) is used to generate a set of meso-scale meteorological fields for the fast manifold analysis. The detailed parameterization scheme and microphysical process are similar to those in previous studies.[20,21] The WRF is configured covering the wide area (111° E–145° E, 10° N–50° N) with a horizontal resolution of 3 km. For the model domain, 6 separate WRF simulations are performed with each one starting at 00 UTC and lasting 30 hours. The first 6 h of each run is discarded as a spin-up run, i.e., each simulation overlaps the previous one to account for model spin-up time. Thus a set of mesoscale meteorological databases with 1-hour interval model outputs is obtained, spanning from 06 UTC 28 August to 06 UTC 3 September 2010. Note that, 6 separate WRF simulations are performed and each simulation overlaps the previous one herein. This design scheme is to avoid the larger and larger simulation errors producing with time, so that the simulated results do not depart from the observation too much.

The comparison between the WRF simulations and the observations is performed firstly. Figure 1 shows the comparisons of the sea-level pressure and the maximal wind speed between simulations and observations for three typhoons during their individual lifetimes. The intensity and evolution of sea-level pressure of "Namtheun" (green solid line in Fig. 1(a)) are simulated better than those of "Lionrock" and "Kompasu" because of its short lifetime and simple track. Due to complex track and long lifetime of the "Lionrock", both the observed and the simulated sea-level pressure evolutions experience several fluctuations accompanied with the increase or decrease of typhoon intensity (blue solid line in Fig. 1(a)). They have a maximum derivation (no more than 10 hPa), and the simulation sea-level pressure is slightly smoother than the observed one. Generally speaking, both the observation and simulation have the synchronous increase or decrease trend with time. For typhoon "Kompasu", the simulated sea-level pressure evolves also consistently with the observed one, descends then increases during its lifetime (orange solid line in

Fig. 1(a)). The simulated maximal wind speeds for three typhoons experience basically consistent evolutions with the observations (Fig. 1(b)). There are not too large gaps between the simulation and the observation. Like the sea-level pressure shown in Fig. 1(a), the maximal wind speed for "Lionrock" also has several waves accompanied with the fluctuations of typhoon intensity.

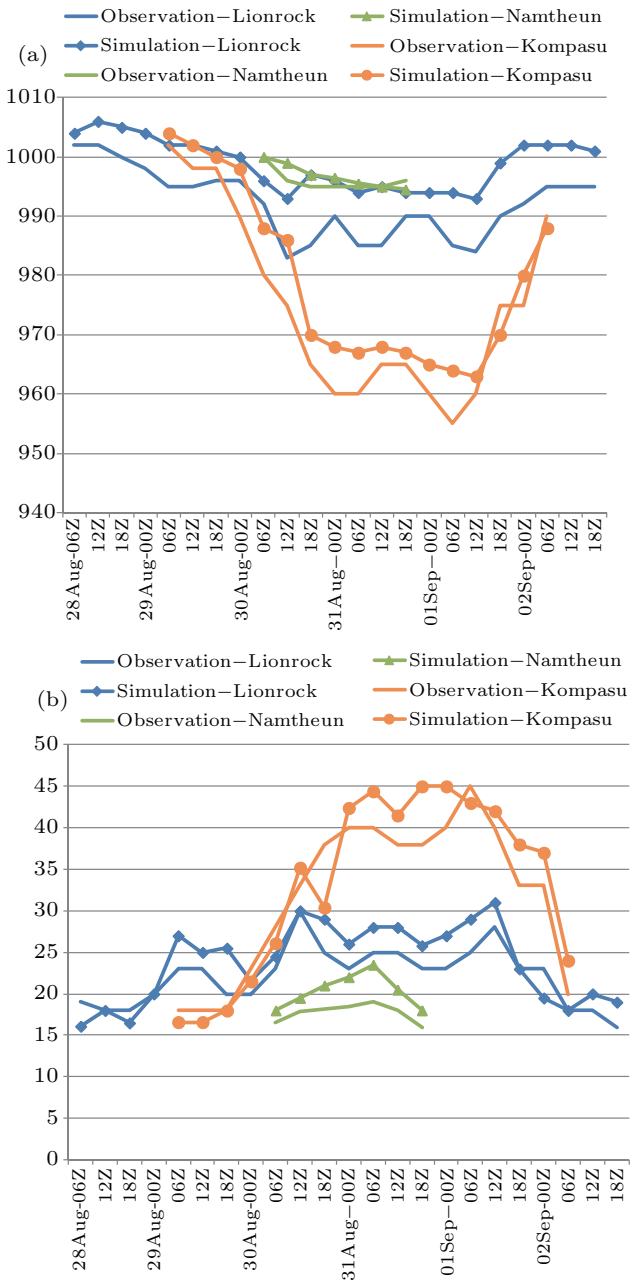


Fig. 1. (color online) Observed (solid line) and simulated (solid line with different symbols marked) evolutions of (a) sea-level pressure (hPa) and (b) maximal wind speed ($m \cdot s^{-1}$) for three typhoon "Lionrock" (blue), "Namtheun" (green), and "Kompasu" (orange).

Figure 2 shows the simulated tracks (black solid line with dots) of three Typhoons "Lionrock", "Namtheun", and "Kompasu" from 06 UTC 28 August to 18 UTC 2 September 2010. The simulated typhoons paths of "Namtheun", and "Kompasu" are very close to the observations (http://typhoon.zjwater.gov.cn/default.aspx). Even for the complex moving path of Typhoons "Lionrock", the simulation does not depart from the observation too far. The purpose of this simulation is to create an in-physics consistent dataset with what would be needed to resolve tropical cyclone and to be suitable for a fast manifold analysis. The simulation result presented above should basically satisfy our main objective.

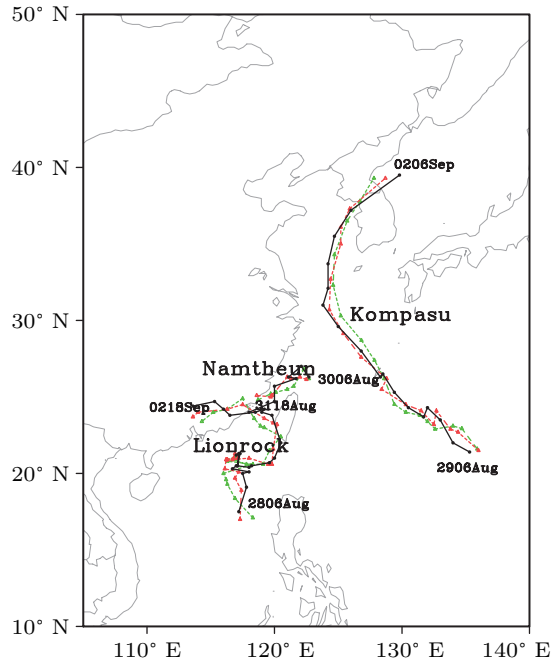


Fig. 2. (color online) Tracks of moist ERI and MPV anomaly centers (red and green dash lines with triangles), superposed with simulated (black solid line with dots) tracks of three Typhoons "Lionrock", "Namtheun", and "Kompasu" from 06 UTC 28 August to 18 UTC 2 September 2010.

5.2. Application of moist ERI anomalies' evolution to a triple-typhoon system

Using the simulation data, the evolutions of the moist ERI anomaly and the MPV anomaly are analyzed for this triple-typhoon case and their landing on the Chinese main land. At 700 hPa, the coherences between the cyclone center and the strong signal of moist ERI anomaly are clearly shown in Figs. 3(a) and 3(b). Note that the moist ERI anomalies shown in Fig. 3 might be produced by the source/sink term in Eq. (22), which may be a possible application of ERI nonconservation to real moist atmosphere for fast-manifold flow. At 12 UTC 30 August (Fig. 3(a)), "Namtheun" was located in the north of Taiwan island at (122.6° E, 26.1° N) with minimum pressure 996 hPa and maximum wind speed 18 $m \cdot s^{-1}$, "Lionrock" was located at (116.9° E, 21.1° N) with an intensity of (980 hPa, 30 $m \cdot s^{-1}$), and "Kompasu" was located at (131.4° E, 23.8° N) with a stronger intensity of (975 hPa, 33 $m \cdot s^{-1}$). The three typhoon locations determine a triangle distributed in the south of 30° N along the southeast coastline of China. The centers of moist ERI anomalies (shaded) are superposed with the centers of three typhoons with stronger signals in "Kompasu" than in "Lionrock" and "Namtheun". After 18 UTC 31 August 2010 (Fig. 3(b)), Typhoon "Namtheun" disappeared

gradually and the triple-typhoon system tended to evolve into a binary-typhoon (BT) system comprised of Typhoons "Lionrock" and "Kompasu". The moist ERI anomalies evolved correspondingly. Firstly, the centers of three moist ERI anomalies became two and moved following the BT system. Secondly, both their intensities identically varied. For example, the moist ERI anomaly was strengthened following the intensification of Typhoon "Kompasu" (965 hPa, 38 m·s^{-1}), and the moist ERI anomaly was weakened following the weakening Typhoon "Lionrock" (990 hPa, 23 m·s^{-1}). At the same time, the strongest Typhoon "Kompasu" still corresponded with the largest moist ERI anomaly.

For comparison, the MPV anomalies at 700 hPa are also computed (Figs. 3(c)–3(d)). In contrast with the moist ERI at 12 UTC 30 August (Fig. 3(a)), which was coherent and well-correlated with the typhoons, the MPV distributions for the triple-typhoon system were scattered (Fig. 3(c)). Their maximum values were located in the north of 40° N. After 18 UTC 31 August, MPV anomalies did not concentrate on typhoons (Fig. 3(d)). The contrasting behaviors of moist ERI and MPV at 700 hPa indicate that the moist ERI in lower troposphere is a better diagnostic tool for typhoons because it concentrates around the centers of typhoons and can show the relative intensity variations of several coexisting typhoons. Moist ERI anomaly can track the movement and development of multi-typhoon systems as well as individual storms.

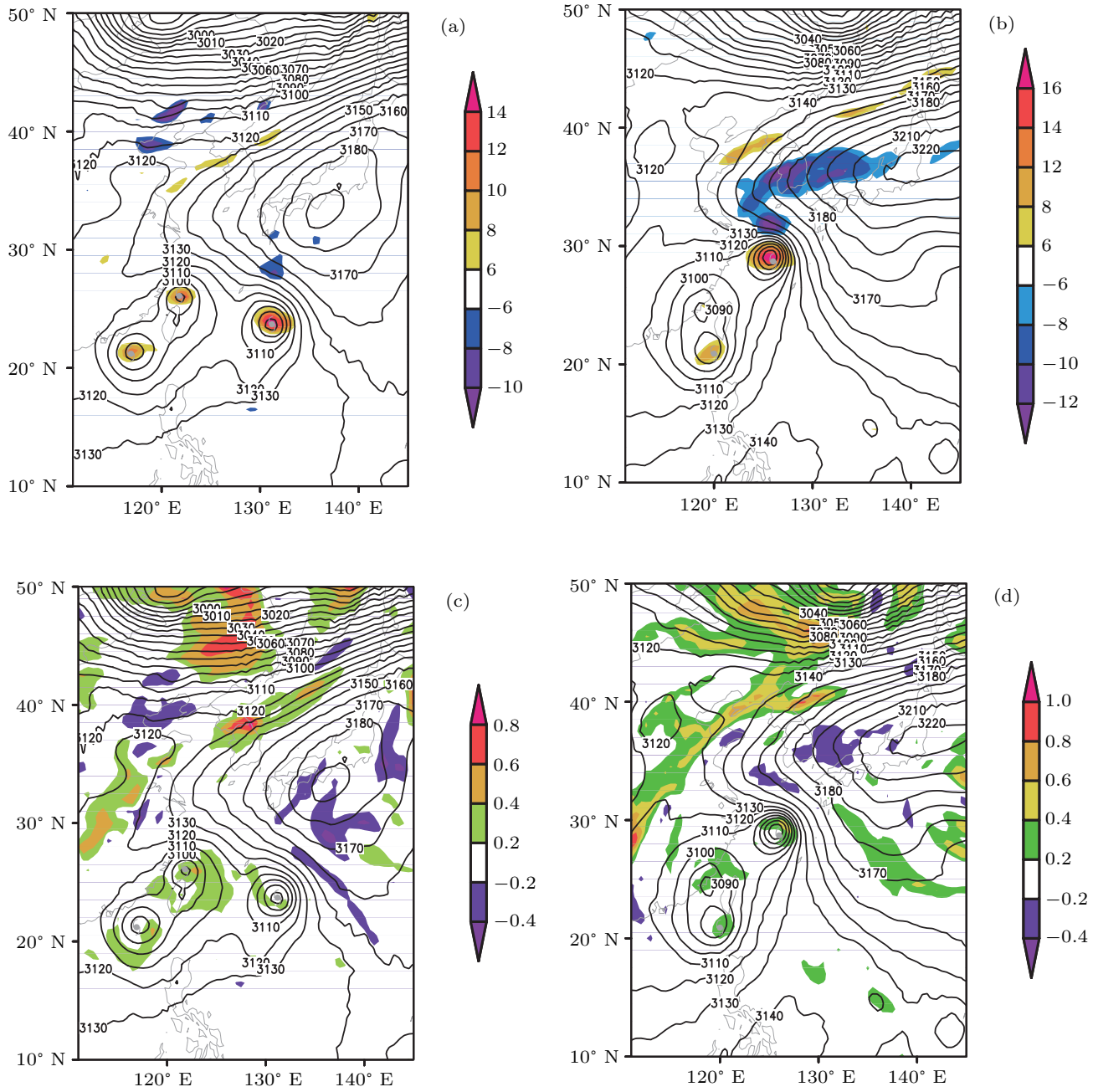


Fig. 3. (color online) Horizontal distributions of (a) moist ERI anomaly (shaded, 10^{-4} m^4·s^{-2}·kg^{-1}·s^{-1}) at 12 UTC 30, (b) moist ERI anomaly at 18 UTC 31, (c) MPV anomaly (shaded, 10^{-4} PVU·s^{-1}) at 12 UTC 30, and (d) MPV anomaly at 18 UTC 31 August 2010; superposed with geopotential height (isolines, dgpm) at 700 hPa.

Returning to Fig. 2, the tracks of the moist ERI and MPV anomaly centers (red and green dashed lines with triangles) are depicted, superposed with simulated (black solid line with dots) tracks of three typhoons during their lifetimes ("Lionrock" from 06 UTC 28 August to 18 UTC 2 September, "Namtheun" from 06 UTC 30 to 18 UTC 31 August, "Kompasu" from 06 UTC 29 August to 06 UTC 2 September, 2010). It shows that the tracks of moist ERI anomaly centers are similar to those of the simulated typhoon centers. Although the MPV anomalies only in the vicinity of centers of typhoons are considered (cf. the large MPV anomaly in the high-latitude region is excluded because it is far from typhoons), ERI anomaly centers at many times are located between the MPV and the simulated typhoon centers. That is to say, the tracks of two equivalences, between ERI and MPV anomalies, are compared. The better collocation between simulated track and ERI anomaly center than that between simulated track and MPV anomaly does mean an improvement. The moist ERI formula focusing on the fast-manifold effect might explain this phenomenon. The moist ERI anomaly centers always are at or ahead of the typhoon centers except that there is a little deviation at the beginning and also after landfall. Even for the fast, long-ranging Typhoon "Kompasu" and the complicated path of Typhoon "Lionrock", the moist ERI anomaly centers still approach to the typhoon centers, and the landfall positions of the moist ERI anomalies and the corresponding typhoons are close. So the moist ERI could be used as a tracer of both the typhoons tracks and their changing intensities.

6. Summary

Based on the generalization of the ERI valid from barotropic to baroclinic flows, the baroclinic ERI is further adapted to moist flow. In doing so, the moist ERI coincides formally with the dry version, except that specific enthalpy and specific entropy are replaced by moist enthalpy and moisture entropy in the present study by considering the moisture effect.

Considering the significance of the mass forcing induced by precipitation and condensation in moist processes, the Lagrangian continuity equation utilized in the proving of the ERI conservation property is re-derived from a no-source to a with-source/sink term due to mass forcing. Then, the conservation property of the baroclinic ERI in moist flow is revisited. It is proved that the baroclinic ERI in moist flow, introduced here, is conserved following the motion if the mass sink is ignored; that is, its total, particle-following time derivative is zero. Therefore, the mass forcing in moist process causes the ERI to no longer be an "invariant", i.e., it actually becomes a "variant".

To respect Ertel and Rossby's achievement, their names are retained in the term introduced in this study: the moist "Ertel–Rossby invariant". Case study also proves moist ERI anomaly to be a better tool to diagnose the movements and intensity variations of several coexisting typhoons. In the near future, more cases of identifiable tropical cyclones, even not yet of hurricane strength, will be selected as the most promising candidates for non-typhoon applications of moist baroclinic ERI anomaly to severe weather that includes fast dynamics.

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