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Chiral p-wave pairing of ultracold fermionic atoms due to a quadratic band touching*

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We study the superfuild ground state of ultracold fermions in optical lattices with a quadratic band touching. Examples are a checkerboard lattice around half filling and a kagome lattice above one third filling. Instead of pairing between spin states, here we focus on pairing interactions between different orbital states. We find that our systems have only odd-parity (orbital) pairing instability while the singlet (orbital) pairing instability vanishes thanks to the quadratic band touching. In the mean field level, the ground state is found to be a chiral p-wave pairing superfluid (mixed with finite f-wave pairing order-parameters) which supports Majorana fermions.

Keywords: degenerate cold fermionic atoms, topological states, Majorana fermions

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1. Introduction

Excitations obeying neither Fermi–Dirac nor Bose–Einstein statistics but non-Abelian statistics can emerge in interacting many-fermion systems.[1,2] One known prototype is the vortex excitations in a $p_x \pm ip_y$ superfluid (or superconducting) state.[3] There is a topologically protected zero-energy Majorana bound state in each quantized vortex and the braiding of the Majorana fermions leads to the non-Abelian statistics.[4–6] This phenomenon also exists in other fully-gapped superfluid/superconducting states with odd Chern numbers where the bulk topology protects the Majorana fermion in each vortex. The robustness of topological protection is proposed to be exploited for quantum computation.[2,7–10] Except for a few cases,[11–27] triplet pairing is crucial for the establishment of non-Abelian topological orders. However, in reality triplet pairing is scarce, whereas singlet pairing prevails. One of the reasons is that the attractive interaction between fermions with opposite spins is stronger than that between fermions with the same spin. Because of the Pauli exclusion principle, the latter can only take place between two fermions on different lattice sites whereas the former can happen to those on the same site. The on-site interaction is usually stronger than that between neighbor sites. For spinless (spin-polarized) fermionic systems, if there is only a single orbit in each unit cell, only the triplet pairing is possible.[28] If there are multiple orbits (denoted as pseudo spins), singlet pairing still prevails as the interaction between different orbits in the same unit cell (i.e., between different pseudo-spins) is stronger than that between the same type of orbits (the same pseudo-spin) in adjacent unit cells.

In this work, we propose a scenario that suppresses the singlet (inter-orbit) pairing instability and opens a route toward triplet orbital pairing. This mechanism works well in the weak pairing regime regardless of the relative strength of the interaction between lattice sites with the same pseudo spin and that between sites with different pseudo spins. The system of interest has a quadratic band touching which is protected by time-reversal and space-inversion symmetries.[29,30] The chemical potential is above or below the quadratic band touching point. Such a system can be realized by using advanced techniques in fermionic optical lattices. Particularly, by using optical lattices of checkerboard pattern together with high-orbital states, it was shown that a quadratic band touching is possible in the fermionic optical lattices.[29,31] The unique property of such systems is that the pseudo-spin polarization on the Fermi surface has a winding number $N_w = \pm 2$. Due to such winding as well as the time-reversal symmetry, the $\boldsymbol{k}$ and $-\boldsymbol{k}$ states on the Fermi surface have the same pseudo-spin polarization, i.e., the pseudo-spin winds one period (2π) from $\boldsymbol{k}$ to $-\boldsymbol{k}$. In this way the singlet pairing instability on the Fermi surface is ineffective in the weak pairing regime and the triplet pairing becomes more important. For concreteness, we study two examples: (i) the checkerboard lattice system and (ii) the kagome lattice system. The ground state is found to be a $p_x \pm ip_y$ pairing superfluid state which supports Majorana fermions with non-Abelian statistics. These systems are possibly realized in ultracold fermionic atom or polar-molecule gases in multi-orbit optical lattices[29,30,32–36] with interaction

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tuned by Feshbach resonances[37,38] or other techniques.[39–42]

2. Quadratic band touching in checkerboard and kagome lattices

The checkerboard lattice is depicted in Fig. 1(a). In each unit cell, there are two sites labeled as red dots (pseudo spin ↑) and blue squares (pseudo spin ↓) in the figure. Allowing one orbit in each site, for spinless (spin-polarized) fermions, the Hamiltonian takes the form $H = H_0 + H_{\rm int}$ with

$$H_0 = -\sum_{\langle i\sigma, j\sigma'\rangle} t_{i\sigma,j\sigma'}(c^\dagger_{i\sigma}c_{j\sigma'} + \text{H.c.}), \tag{1}$$

$$H_{\rm int} = -\frac{1}{2}\sum_{\langle i\sigma, j\sigma'\rangle} V_{\sigma\sigma'} n_{i\sigma} n_{j\sigma'}. \tag{2}$$

Here i and j are the indices of the unit cells, while pseudo-spins ($\sigma, \sigma' = \uparrow, \downarrow$) denote the two different orbits in each unit cell. $\langle i\sigma, j\sigma'\rangle$ restricts the summation to the nearest and the next nearest neighbors. Note that the sign of the interaction Hamiltonian is negative which is chosen to describe the attractive interaction with positive $V_{\sigma\sigma'}n_{i\sigma}$. The system is designed in such a way that the hopping amplitudes in the x and y directions between the ↑ sites are t' (solid links in Fig. 1(a)) and t'' (dotted links), respectively, while for the ↓ sites, they are t'' and t', respectively (see Fig. 1(a)). The hopping amplitude between the ↑ and ↓ sites is t (dashed links). The fermions on such a lattice have the following single-particle Hamiltonian:

$$H_0 = \sum_{\boldsymbol{k}} \psi^\dagger_{\boldsymbol{k}} \mathscr{H}_0(\bar{\boldsymbol{k}}) \psi_{\boldsymbol{k}},$$

with[29,30]

$$\begin{aligned} &\psi_{\boldsymbol{k}} = (c_{\boldsymbol{k}\uparrow}, c_{\boldsymbol{k}\downarrow})^{\rm T}, \\ &\mathscr{H}_0(\bar{\boldsymbol{k}}) = h_0(\bar{\boldsymbol{k}})\sigma_0 + \boldsymbol{h}(\bar{\boldsymbol{k}})\cdot\hat{\boldsymbol{\sigma}}, \end{aligned} \tag{3}$$

where $\hat{\sigma}_0$ denotes the 2×2 identity matrix, $\hat{\boldsymbol{\sigma}}$ is the Pauli matrix vector, and

$$\begin{aligned} h_0(\bar{\boldsymbol{k}}) &= -2t_0(\cos\bar{k}_x + \cos\bar{k}_y), \\ h_z(\bar{\boldsymbol{k}}) &= -2t_z(\cos\bar{k}_x - \cos\bar{k}_y), \\ h_x(\bar{\boldsymbol{k}}) &= 8t_x\cos\frac{\bar{k}_x}{2}\cos\frac{\bar{k}_y}{2}, \quad h_y(\bar{\boldsymbol{k}}) = 0, \end{aligned}$$

with $t_0 = (t' + t'')/2$, $t_z = (t' - t'')/2$, and $t_x = -t/2$.[29] The system has the time-reversal symmetry as well as the space-inversion symmetry. There are two bands and they touch only at $\boldsymbol{K} = (\pi, \pi)$ in the whole Brillouin zone.[29,30] The band touching is quadratic, i.e., in the vicinity of $\boldsymbol{K}$, the Hamiltonian consists of only terms quadratic in momentum. Specifically, in such a region $h_0(\boldsymbol{k}) = t_0 k^2$, $h_z(\boldsymbol{k}) = t_z(k_x^2 - k_y^2)$, $h_x(\boldsymbol{k}) = 2t_x k_x k_y$, and $h_y(\boldsymbol{k}) = 0$, where $\boldsymbol{k} = \bar{\boldsymbol{k}} - \boldsymbol{K}$. The spectrum is

$$\varepsilon_{\boldsymbol{k}\pm} = t_0 k^2 \pm k^2\sqrt{t_z^2\cos^2(2\theta_{\boldsymbol{k}}) + t_x^2\sin^2(2\theta_{\boldsymbol{k}})},$$

with $\theta_{\boldsymbol{k}} = \text{Arg}[k_x + ik_y]$. At half-filling, the chemical potential is at the quadratic band touching point. Away from it only one band crosses the chemical potential (Fig. 1(c)) when $|t_0| \le |t_z|, |t_x|$. Another nontrivial property is that the pseudo-spin polarization on the Fermi surface has a nonzero winding number. It can be written as

$$N_{\rm w} = \frac{1}{2\pi}\oint_{\rm FS} {\rm d}\phi_{\boldsymbol{k}}, \tag{4}$$

where FS stands for the Fermi surface and $\phi_{\boldsymbol{k}} = \text{Arg}[h_z(\boldsymbol{k}) + ih_x(\boldsymbol{k})]$ is the direction of the pseudo-spin polarization in the z–x plane. Note that the winding number $N_{\rm w} = 2\text{sgn}(t_x t_z) = \pm 2$ as well as the time-reversal symmetry guarantees that the pseudo-spin polarization at $\boldsymbol{k}$ is the same as that at $-\boldsymbol{k}$ on the Fermi surface (see Fig. 1(d)).

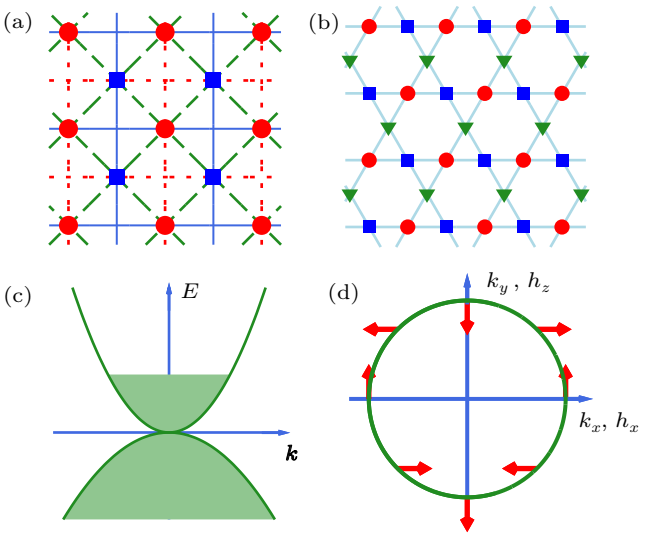


Fig. 1. (color online) (a) Checkerboard and (b) kagome lattices. In the checkerboard lattice, the hopping amplitudes along the blue (solid), red (dotted), and green (dashed) links are t', t'', and t, respectively. (c) Band structure and band filling near half-filling with $t_x = t_z$ in the checkerboard lattice. $\boldsymbol{k}$ is measured from the quadratic band touching point $\boldsymbol{K}$. (d) Direction of the pseudo-spin field (h_x, h_z) (also represents the pseudo-spin polarization direction) on the Fermi surface of a quadratic band touching system.

In the kagome lattice, there are three orbits (sites) in each unit cell, labeled as the red dots, the blue squares, and the green triangles in Fig. 1(b), which we shall denote as 1, 2, and 3. With only the nearest neighbor hopping, the free Hamiltonian can be written as

$$H_0 = \sum_{\boldsymbol{k}} \psi^\dagger_{\boldsymbol{k}} \mathscr{H}_{\rm kgm}(\boldsymbol{k}) \psi_{\boldsymbol{k}},$$

where

$$\begin{aligned} &\psi_{\boldsymbol{k}} = (c_{\boldsymbol{k}1}, c_{\boldsymbol{k}2}, c_{\boldsymbol{k}3})^{\rm T}, \\ &\mathscr{H}_{\rm kgm}(\boldsymbol{k}) = -2t\begin{pmatrix} 0 & \cos(k_{12}/2) & \cos(k_{13}/2) \\ \cos(k_{12}/2) & 0 & \cos(k_{23}/2) \\ \cos(k_{13}/2) & \cos(k_{23}/2) & 0 \end{pmatrix}, \end{aligned}$$

with t being the hopping amplitude. $k_{ij} = \boldsymbol{k}\cdot\boldsymbol{n}_{ij}$ for $i, j = 1, 2, 3$ with $\boldsymbol{n}_{12} = (1, 0)$, $\boldsymbol{n}_{13} = (1/2, \sqrt{3}/2)$, and $\boldsymbol{n}_{23} = \boldsymbol{n}_{13} -$

$\boldsymbol{n}_{12}$ being the unit vectors along the bonds connecting the three types of orbits. The spectrum is

$$\varepsilon_{\boldsymbol{k}0} = 2t,$$
$$\varepsilon_{\boldsymbol{k}\pm} = -t \pm t\sqrt{4\left(\cos^2\frac{k_{12}}{2}+\cos^2\frac{k_{13}}{2}+\cos^2\frac{k_{23}}{2}\right)-3}.$$

In the kagome lattice, the + and 0 bands touch quadratically at $\boldsymbol{K}' = (0,0)$. Around $\boldsymbol{K}'$ the spectrum is approximately $\varepsilon_{\boldsymbol{k}+} = 2t - (1/4)tk^2$, $\varepsilon_{\boldsymbol{k}0} = 2t$, and $\varepsilon_{\boldsymbol{k}-} = -4t + (1/4)tk^2$. For $t < 0$ and not much above 1/3 filling (the results are similar for $t > 0$ and below 2/3 filling), only the + band crosses the Fermi surface. Projecting out the − band which is far above, one obtains an effective Hamiltonian in the form of Eq. (3) with $h_0(\boldsymbol{k}) = (1/8)|t|k^2$, $h_z(\boldsymbol{k}) = (1/8)|t|(k_x^2 - k_y^2)$, $h_x(\boldsymbol{k}) = (1/4)|t|k_xk_y$, and $h_y(\boldsymbol{k}) = 0$, where the pseudo-spin up and down states are defined as $|\uparrow\rangle = (1/\sqrt{2})(1,-1,0)^{\mathrm{T}}$ and $|\downarrow\rangle = (1/\sqrt{6})(1,1,-2)^{\mathrm{T}}$, respectively.

3. Pairing under isotropic attractive interactions

3.1. Absence of singlet pairing instability

We note that in quadratic band touching systems described by Eq. (3), the Bogoliubov–de Gennes (BdG) Hamiltonian with a singlet pairing interaction $H_{\mathrm{int}}^{\mathrm{sp}} = -\frac{1}{2}\sum_{\boldsymbol{k}}\Delta_{\mathrm{s}}(\boldsymbol{k})(c^{\dagger}_{\boldsymbol{k}\uparrow}c^{\dagger}_{-\boldsymbol{k}\downarrow} - c^{\dagger}_{\boldsymbol{k}\downarrow}c^{\dagger}_{-\boldsymbol{k}\uparrow}) + \mathrm{H.c.}$ can be diagonalized exactly. The quasi-particle spectrum is found to be $\pm\left[\varepsilon_{\boldsymbol{k}\pm} - \sqrt{\mu^2 + |\Delta_{\mathrm{s}}(\boldsymbol{k})|^2}\right]$ with $\varepsilon_{\boldsymbol{k}\pm} = h_0(\boldsymbol{k}) \pm \sqrt{h_z(\boldsymbol{k})^2 + h_x(\boldsymbol{k})^2}$ and μ being the chemical potential. This gapless quasi-particle spectrum reveals that there is no singlet pairing instability in quadratic band touching systems. The physical reason is that, due to the Fermi surface topology ($N_{\mathrm{w}} = \pm 2$) and the time-reversal symmetry, the pseudo-spin directions at $\boldsymbol{k}$ and $-\boldsymbol{k}$ on the Fermi surface are the same. Such type of Fermi surface does not have the singlet pairing instability in the weak pairing regime.

3.2. Interaction and triplet pairing

We shall now consider the triplet pairing instability driven by isotropic attractive interactions. Let us first consider the checkerboard lattices. For convenience, we denote the nearest neighbor interaction as $V_{\uparrow\downarrow} \equiv V$ while the next-nearest neighbor interaction is represented by $V_{\uparrow\uparrow} = V_{\downarrow\downarrow} \equiv U$. Writing the interaction Hamiltonian in the $\boldsymbol{k}$-space yields

$$H_{\mathrm{int}} = -\frac{1}{2}\sum_{\boldsymbol{q}}\Big[US_u(\boldsymbol{q})\sum_{\sigma} n_\sigma(\boldsymbol{q})n_\sigma(-\boldsymbol{q}) + 2VS_v(\boldsymbol{q})n_\uparrow(\boldsymbol{q})n_\downarrow(-\boldsymbol{q})\Big], \tag{5}$$

with $n_\sigma(\boldsymbol{q}) = \sum_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}+\boldsymbol{q}\sigma}c_{\boldsymbol{k}\sigma}$. Here $S_{\mathrm{u}}(\boldsymbol{q}) = 2(\cos q_x + \cos q_y)$ and $S_v(\boldsymbol{q}) = 2\left[\cos\frac{1}{2}(q_x+q_y) + \cos\frac{1}{2}(q_x-q_y)\right]$ are the form factors. Using the Hubbard–Stratonovich decoupling, one can write down the interaction Hamiltonian in the mean field form as

$$H_{\mathrm{int}}^{\mathrm{MF}} = -\frac{1}{2}\sum_{\boldsymbol{k}\nu}\Big[\Delta_\nu(\boldsymbol{k})d_\nu(\boldsymbol{k}) + \mathrm{H.c.}\Big] + \frac{1}{2}E_0, \tag{6}$$

where $d_{x/y}(\boldsymbol{k}) = c^{\dagger}_{\boldsymbol{k}\uparrow}c^{\dagger}_{-\boldsymbol{k}\uparrow} \mp c^{\dagger}_{\boldsymbol{k}\downarrow}c^{\dagger}_{-\boldsymbol{k}\downarrow}$, $d_z(\boldsymbol{k}) = c^{\dagger}_{\boldsymbol{k}\uparrow}c^{\dagger}_{-\boldsymbol{k}\downarrow} + c^{\dagger}_{\boldsymbol{k}\downarrow}c^{\dagger}_{-\boldsymbol{k}\uparrow}$, and $E_0 = \sum_{\boldsymbol{k}\nu}\Delta_\nu(\boldsymbol{k})\langle d_\nu(\boldsymbol{k})\rangle$. The triplet pairing order parameter is written as

$$\Delta_\nu(\boldsymbol{k}) = V_\nu\sum_{\boldsymbol{p}} S(\boldsymbol{k},\boldsymbol{p})\langle d^{\dagger}_\nu(\boldsymbol{p})\rangle, \tag{7}$$

where $V_x = V_y = U$, $V_z = V/2$, and $S(\boldsymbol{k},\boldsymbol{p}) \simeq \frac{1}{2}(k_+p_- + k_-p_+)$ for small wavevectors $\boldsymbol{k}$ and $\boldsymbol{p}$. Here $k_\pm = k_x \pm \mathrm{i}k_y$. Thus the dominant contribution to the triplet pairing comes from the p-wave channel.

For the kagome lattices with isotropic nearest neighbor attractive interaction, following the same procedure, one finds a mean field Hamiltonian of the form Eq. (6) with

$$\Delta_y = -\frac{1}{3}(\Delta_{12} + \Delta_{13} + \Delta_{23}), \tag{8}$$
$$\Delta_x = -\frac{1}{3}(2\Delta_{12} - \Delta_{13} - \Delta_{23}), \tag{9}$$
$$\Delta_z = \frac{1}{\sqrt{3}}(\Delta_{23} - \Delta_{13}). \tag{10}$$

Here $\Delta_{ij} = \frac{V}{4}\sum_{\boldsymbol{p}}\langle c_{-\boldsymbol{p}i}c_{\boldsymbol{p}j} + c_{-\boldsymbol{p}j}c_{\boldsymbol{p}i}\rangle p_{ij}k_{ij}$ for $i,j = 1,2,3$, where V is the interaction between the nearest neighbor sites. The triplet pairing order parameter is again of the p-wave nature.

4. Topological superfluid state

4.1. Projected BdG Hamiltonian and Chern number

We restrict the discussions in the weak pairing regime, where $|\mu|$ is much larger than $|\Delta_\nu(\boldsymbol{k})|$. In such a regime, one can safely ignore pairing interaction between states separated with energy difference $\geq |\mu|$. Consider, say, $\mu > 0$ where we can project the BdG Hamiltonian into the subspace with only the + band that crosses the chemical potential. For simplicity, we consider the situation with $t_x = t_z$ for the checkerboard lattice. The single particle spectrum without interaction is then given by $\varepsilon_{\boldsymbol{k}\pm} = k^2(t_0 \pm t_z)$ and $\varepsilon_{\boldsymbol{k}\pm} = (1/8)|t|k^2(1\pm 1)$ in checkerboard and kagome lattices, respectively. Projecting into the + band the BdG Hamiltonian can be written as $H_{\mathrm{PBdG}} = (1/2)\sum_{\boldsymbol{k}}\Psi_{\mathrm{P}}^{\dagger}(\boldsymbol{k})\mathscr{H}_{\boldsymbol{k}}^{\mathrm{P}}\Psi_{\mathrm{P}}(\boldsymbol{k}) + (1/2)E_0$ with $\Psi_{\mathrm{P}}(\boldsymbol{k}) = (c_{\boldsymbol{k}+}, c^{\dagger}_{-\boldsymbol{k}+})^{\mathrm{T}}$ and

$$\mathscr{H}_{\boldsymbol{k}}^{\mathrm{P}} = \begin{bmatrix} \varepsilon_{\boldsymbol{k}+} - \mu & \Delta_{\mathrm{eff}}(\boldsymbol{k}) \\ \Delta_{\mathrm{eff}}^*(\boldsymbol{k}) & -\varepsilon_{\boldsymbol{k}+} + \mu \end{bmatrix}, \tag{11}$$

where

$$\Delta_{\mathrm{eff}}(\boldsymbol{k}) = \Delta_y(\boldsymbol{k}) + \sum_{\pm}\frac{1}{2}\mathrm{e}^{\pm 2\mathrm{i}\theta_{\boldsymbol{k}}}\left[\Delta_x(\boldsymbol{k}) \mp \mathrm{i}\Delta_z(\boldsymbol{k})\right]. \tag{12}$$

The quasi-particle spectra are obtained as $\pm E_{\boldsymbol{k}}$ with $E_{\boldsymbol{k}} = \sqrt{\xi_{\boldsymbol{k}}^2 + |\Delta_{\mathrm{eff}}(\boldsymbol{k})|^2}$ and $\xi_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}+} - \mu$. The Chern number of

the pairing states with complete quasi-particle gap is found in Ref. [25] as

$$N_{\rm C} = \frac{1}{2\pi}\int_0^{2\pi} {\rm d}\theta_{\boldsymbol{k}}\partial_{\theta_{\boldsymbol{k}}}\theta_{\Delta}(\boldsymbol{k})\Big|_{\rm FS}. \tag{13}$$

Here $\theta_{\Delta}(\boldsymbol{k}) = {\rm Arg}[\Delta_{\rm eff}(\boldsymbol{k})]$, which is exactly the winding number of the effective pairing $\Delta_{\rm eff}(\boldsymbol{k})$ at the Fermi surface. An important observation is that the Chern number must be odd, as only the triplet pairing order parameters contribute to the effective pairing $\Delta_{\rm eff}(\boldsymbol{k})$. The ground state has a complete band gap with an odd Chern number. Thus it is a topological superfluid state which supports Majorana fermions with non-Abelian statistics.

4.2. Topological ground states

The ground state is determined by minimizing the mean field free energy

$$F = -k_{\rm B}T\sum_{\boldsymbol{k}}\ln\left(2\cosh\frac{E_{\boldsymbol{k}}}{2k_{\rm B}T}\right) + \frac{1}{2}E_0 \tag{14}$$

at $T = 0$. To facilitate the discussion, we introduce $g_{\nu\beta} = \sqrt{V_\nu/2}\sum_{\boldsymbol{p}} p_{-\beta}\langle d_\nu^\dagger(\boldsymbol{p})\rangle$ for $\beta = \pm$ for the checkerboard lattice systems. This enables us to write $E_0 = g^2$, where $g = \sqrt{\sum_{\nu,\beta}|g_{\nu\beta}|^2}$ is the pairing amplitude. Using this, we find

$$\Delta_{\rm eff}(\boldsymbol{k}) = \frac{k\sqrt{U}}{2\sqrt{2}}\sum_n \Phi_n {\rm e}^{{\rm i}n\theta_{\boldsymbol{k}}}, \tag{15}$$

with $n = \pm1, \pm3$, where

$$\begin{aligned}\Phi_{\pm1} &= 2g_{y\pm} + g_{x\mp} \mp {\rm i}\sqrt{\frac{V}{2U}}g_{z\mp},\\ \Phi_{\pm3} &= g_{x\pm} \mp {\rm i}\sqrt{\frac{V}{2U}}g_{z\pm}.\end{aligned} \tag{16}$$

By minimizing the free energy numerically, we find that there are two possible ground states related by time-reversal operations. In one of them, only Φ_1 and Φ_{-3} are finite, whereas for the other one, only Φ_{-1} and Φ_3 are nonzero. It is found that in the whole range of V/U, the p wave order parameter $|\Phi_1|$ (or $|\Phi_{-1}|$) is larger than the f wave one $|\Phi_{-3}|$ (or $|\Phi_3|$) (see Fig. 2(a)). Therefore the Chern number of the ground state is $N_{\rm C} = \pm1$. There are two chiral edge states localized at the left and right boundaries separately as shown in Fig. 2(b).

At a special point $V/U = 2$, the f wave order parameter $\Phi_{\pm3}$ vanishes. With this condition, we find that

$$\Delta_y : \Delta_x : \Delta_z = \frac{2}{\sqrt{6}} : \frac{1}{\sqrt{6}} : {\rm i}\frac{1}{\sqrt{6}}$$

and the self-consistent equation can be written as

$$1 = \frac{3}{8}\sum_{\boldsymbol{k}}\tanh\left(\frac{E_{\boldsymbol{k}}}{2k_{\rm B}T}\right)E_{\boldsymbol{k}}^{-1}k^2U, \tag{17}$$

with $E_{\boldsymbol{k}} = \sqrt{\xi_{\boldsymbol{k}}^2 + \frac{3}{4}k^2Ug^2}$. From this equation, we obtain the transition temperature

$$k_{\rm B}T_{\rm c} \simeq \frac{2\gamma}{\pi}\Lambda\exp\left(-\frac{16\pi t_z^2}{3U\Lambda}\right),$$

when $t_0 = 0$. Here γ is the Euler constant and Λ is the high energy cut-off depending on the bandwidth. In two dimensions, the superconductor/superfluid phase transition is determined by the Kosterlitz–Thouless transition. It was shown that in the weak coupling regime, the Kosterlitz–Thouless transition temperature is close to the $T_{\rm c}$ estimated here.[43,44] For the case of kagome lattice, we find that the ground state is also a $p_x \pm ip_y$ pairing superfluid state with a complete quasi-particle gap which has a nontrivial Chern number $N_{\rm C} = \pm1$. Note that for the kagome lattice there is a flat band (labeled as "0"). When the chemical potential is in the flat band, strong interaction effects can appear even for a small interaction strength as the kinetic energy is suppressed. Our discussions are thus not applicable to such regime. We restrict the studies here to the situation when the chemical potential is in the dispersive "+" band, i.e., the chemical potential is above the quadratic band touching point. Calculation indicates that in this topological ground state, the pairing order parameters satisfy $\Delta_{12} : \Delta_{13} : \Delta_{23} = 1 : {\rm e}^{{\rm i}\pi/3} : {\rm e}^{{\rm i}2\pi/3}$ if $N_{\rm C} = 1$ or $\Delta_{12} : \Delta_{13} : \Delta_{23} = 1 : {\rm e}^{-{\rm i}\pi/3} : {\rm e}^{-{\rm i}2\pi/3}$ for its time reversal partner with $N_{\rm C} = -1$. That is, in real space the pairing order parameter winds $\pm\pi$ in a unit cell.

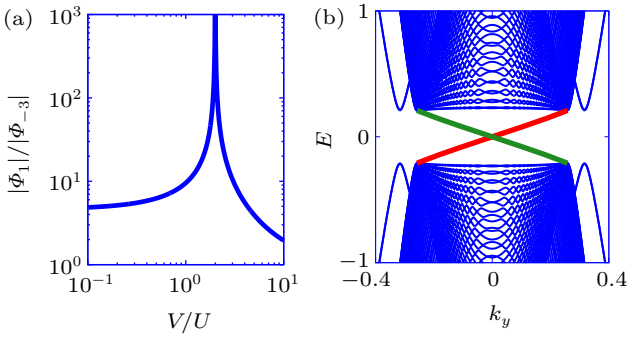


Fig. 2. (color online) Properties of the superfluid ground state of the checkerboard lattice systems. (a) The ratio of the order parameters $|\Phi_1|/|\Phi_{-3}|$ as a function of the ratio of the nearest neighbor interaction to the next-nearest neighbor interaction V/U. (b) Bogoliubov quasi-particle spectra as a function of k_y in a stripe with periodic (open) boundary condition along the y (x) direction. The parameters are $t_0 = 10$, $t_z = t_x = 20$, $\mu = 2$, $V/2 = U = 1$, and $g = 0.95$. The width of the stripe along the x direction is $N_x = 501$ unit cells.

5. Conclusion and discussion

We have studied the superfluid ground states of ultracold fermions in optical lattices under isotropic attractive interactions. Particularly, we focus on the optical lattices with multiple orbits in an unit cell which have a quadratic band touching point. We restrict our discussions to pairing between different orbital states (denoted as pseudo-spin) instead of spin (or hyperfine) degree of freedom. Specifically, we study two kinds

of such lattices: the checkerboard lattice around half filling and the kagome lattice above one third filling where spinless (or spin-polarized) fermions fill the lattice. We develop a mean field theory to study the ground state of the systems. By minimizing the mean field free energy, we find that the ground state is a $p_x \pm ip_y$ pairing (with mixed f-wave pairing) superfluid state for both cases. Such a ground state supports vortex excitations in each of which there is a topologically protected Majorana fermion. The emergent topological order can be useful for quantum information processing.[10] Finally we would like to point out that repulsive interaction driven phase transitions for spinless and spinful fermions in checkerboard and kagome lattices when the Fermi surface is at the quadratic band touching point have been studied in Refs. [29], [30], [45], and [46].

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