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Generalized Chaplygin equations for nonholonomic systems on time scales*

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The generalized Chaplygin equations for nonholonomic systems on time scales are proposed and studied. The Hamilton principle for nonholonomic systems on time scales is established, and the corresponding generalized Chaplygin equations are deduced. The reduced Chaplygin equations are also presented. Two special cases of the generalized Chaplygin equations on time scales, where the time scales are equal to the set of real numbers and the integer set, are discussed. Finally, several examples are given to illustrate the application of the results.

Keywords: nonholonomic system, generalized Chaplygin equations, time scales

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1. Introduction

The calculus on time scales, which is initiated by Hilger in his PhD thesis in 1988,[1] can be used to develop the theory of dynamic equations on time scales in order to unify and extend the usual differential equations and difference equations. The similarities and differences between continuous and discrete systems are revealed. The physical essences of the continuous and discrete systems and the other complex dynamical systems can be expressed more clearly and accurately by the theory on time scales. Therefore, the theories on time scales have been used in many fields such as mathematics, mechanics, economics, etc.[2–11]

The variational problems on time scales were first introduced by Bohner in 2004.[12] After that the theory on time scales was applied to economics by Atici, Biles and Lebedinsky,[7] the Euler–Lagrange equations of variational problems with nabla derivatives were established. Martins and Torres extended the variational problems with delta derivatives to the variational problems with nabla derivatives,[13] and the higher order Euler–Lagrange equations with nabla derivatives were presented. The variational problems with delta and nabla derivatives were further discussed by Torres,[14] and the paper proved that the unification and extension of variational on time scales was not unique. Besides, Bartosiewicz, Martins and Torres presented the second Euler–Lagrange equations on time scales.[15] Moreover, the theories on time scales were extended to optimal control systems by Malinowska and Ammi,[16] and the corresponding Euler–Lagrange equations were derived.

As is well known, the holonomic systems are described by the second Lagrange equations, while the nonholonomic systems are described by more complex differential equations of motion. However, the controversy exists on the exchange relations for nonholonomic systems.[17–19] Historically, there were two views on the form of the exchange relations for nonholonomic systems: one is the Hölder view, the other is the Suslov view.

Recently, the differential equations of motion with multipliers for nonholonomic system on time scales were established by Fu *et al.*[20] Zu and Zhu[21] deduced the Hamilton canonical equations for the nonholonomic system in phase space on time scales. Then, Song and Zhang extended the variational problem for mechanical systems on time scales to Birkhoffian systems.[22] Furthermore, the methods of reduction for a Lagrange system with nabla derivative were discussed by Jin and Zhang.[23] Plenty of important results concerning the differential equations on time scales have been obtained.[24–28] However, the differential equations of motion for Chaplygin systems on time scales have not been investigated yet in the literature.

In this paper, we mainly establish the generalized Chaplygin equations according to the Suslov view on time scales. The rest of this paper is organized as follows. In Section 2, the generalized Chaplygin equations in the Suslov definition on time scales are established. In Section 3, the Chaplygin equations on time scales are deduced. In Section 4, several examples are given to illustrate the applications of the results. Finally, some conclusions are drawn from the present study in Section 5.

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2. Generalized Chaplygin equations on time scales

We mainly discuss the generalized Chaplygin equations according to the Suslov view on time scales in this section. The basic definitions and facts concerning the calculus on time scales used in the following section can be found in Refs. [4] and [5].

Assume that the configuration of a mechanical system is determined by n generalized coordinates q_k $(k=1,2,\ldots,n)$. The motion of the system is subjected to g bilateral ideal constraints of Chetaev type and given as follows:

$$q^{\Delta}_{\varepsilon+\beta}=\varphi_\beta\left(t,q^{\sigma}_{k},q^{\Delta}_{r}\right)\left(\begin{array}{c}\beta=1,2,\ldots,g;\ \varepsilon=n-g\\ r=1,2,\ldots,\varepsilon;\ k=1,2,\ldots,n\end{array}\right), \quad (1)$$

the restriction of constraints exerted on the virtual displacement on time scales can be expressed as

$$\delta q^{\sigma}_{\varepsilon+\beta}=\sum_{r=1}^{\varepsilon}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\left(\sigma(t)\right)\delta q^{\sigma}_{r}. \quad (2)$$

From Eq. (1), we have

$$\delta q^{\Delta}_{\varepsilon+\beta}=\sum_{r=1}^{\varepsilon}\left[\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\delta q^{\Delta}_{r}+\left(\frac{\partial\varphi_\beta}{\partial q^{\sigma}_{r}}+\sum_{\gamma=1}^{g}\frac{\partial\varphi_\beta}{\partial q^{\sigma}_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial q^{\Delta}_{r}}\left(\sigma(t)\right)\right)\delta q^{\sigma}_{r}\right] \quad (3)$$

and

$$\delta q_{\varepsilon+\beta}=\sum_{r=1}^{\varepsilon}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\delta q_r\quad(\beta=1,2,\ldots,g). \quad (4)$$

From Eq. (4), we have

$$\begin{aligned}\frac{\Delta}{\Delta t}\delta q_{\varepsilon+\beta}&=\frac{\Delta}{\Delta t}\left(\sum_{r=1}^{\varepsilon}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\delta q_r\right)\\&=\sum_{r=1}^{\varepsilon}\left(\frac{\Delta}{\Delta t}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\delta q^{\sigma}_{r}+\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\frac{\Delta}{\Delta t}\delta q_r\right)\\&\quad(\beta=1,2,\ldots,g).\end{aligned} \quad (5)$$

Subtracting Eq. (3) from Eq. (5), we obtain

$$\begin{aligned}&\frac{\Delta}{\Delta t}\delta q_{\varepsilon+\beta}-\delta q^{\Delta}_{\varepsilon+\beta}\\&=\sum_{r=1}^{\varepsilon}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\left(\frac{\Delta}{\Delta t}\delta q_r-\delta q^{\Delta}_{r}\right)\\&\quad+\sum_{r=1}^{\varepsilon}T^{\varepsilon+\beta}_{r}\delta q^{\sigma}_{r}\quad(\beta=1,2,\ldots,g),\end{aligned} \quad (6)$$

where

$$T^{\varepsilon+\beta}_{r}=\frac{\Delta}{\Delta t}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}-\frac{\partial\varphi_\beta}{\partial q^{\sigma}_{r}}-\sum_{\gamma=1}^{g}\frac{\partial\varphi_\beta}{\partial q^{\sigma}_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial q^{\Delta}_{r}}\left(\sigma(\tau)\right). \quad (7)$$

Since the independence of δq^{Δ}_{r}, we have

$$\frac{\Delta}{\Delta t}\delta q_r=\delta q^{\Delta}_{r}\,(r=1,2,\ldots,\varepsilon). \quad (8)$$

Substituting Eq. (8) into Eq. (6), we obtain

$$\frac{\Delta}{\Delta t}\delta q_{\varepsilon+\beta}-\delta q^{\Delta}_{\varepsilon+\beta}=\sum_{r=1}^{\varepsilon}T^{\varepsilon+\beta}_{r}\delta q^{\sigma}_{r}\quad(\beta=1,2,\ldots,g). \quad (9)$$

The D'Alembert–Lagrange principle on time scales can be written as

$$\int_{t_1}^{t_2}\sum_{s=1}^{n}\left(-\frac{\Delta}{\Delta t}\frac{\partial L}{\partial q^{\Delta}_{s}}+\frac{\partial L}{\partial q^{\sigma}_{s}}+Q_s\right)\delta q^{\sigma}_{s}\Delta t=0, \quad (10)$$

where $[t_1,t_2]\subseteq\mathbb{T}$, the Lagrangian is $L=L\left(t,q^{\sigma}_{k},q^{\Delta}_{k}\right)\in C_{rd}$, and the generalized force is $Q_s=Q_s\left(t,q^{\sigma}_{k},q^{\Delta}_{k}\right)\in C_{rd}$. They satisfy the boundary conditions

$$\delta q_s|_{t=t_1}=\delta q_s|_{t=t_2}=0\quad(s=1,2,\ldots,n). \quad (11)$$

Equation (10) can be reduced into

$$\begin{aligned}&\int_{t_1}^{t_2}\sum_{s=1}^{n}\left[\delta L-\frac{\Delta}{\Delta t}\left(\sum_{s=1}^{n}\frac{\partial L}{\partial q^{\Delta}_{s}}\delta q_s\right)\right.\\&\left.+\sum_{s=1}^{n}\frac{\partial L}{\partial q^{\Delta}_{s}}\left(\frac{\Delta}{\Delta t}\delta q_s-\delta q^{\Delta}_{s}\right)+Q_s\delta q^{\sigma}_{s}\right]\Delta t=0.\end{aligned} \quad (12)$$

Considering Eqs. (8) and (11), equation (12) can be expressed as

$$\int_{t_1}^{t_2}\left[\delta L+\sum_{r=1}^{\varepsilon}\left(\sum_{\beta=1}^{g}\frac{\partial L}{\partial q^{\Delta}_{\varepsilon+\beta}}T^{\varepsilon+\beta}_{r}+\tilde{Q}_r\right)\delta q^{\sigma}_{r}\right]\Delta t=0, \quad (13)$$

where

$$\tilde{Q}_r=Q_r+\sum_{\beta=1}^{g}Q_{\varepsilon+\beta}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\left(\sigma(t)\right). \quad (14)$$

Let

$$\tilde{L}=\tilde{L}\left(t,q^{\sigma}_{s},q^{\Delta}_{r}\right)=L\left(t,q^{\sigma}_{s},q^{\Delta}_{r},\varphi_\beta\left(t,q^{\sigma}_{s},q^{\Delta}_{r}\right)\right), \quad (15)$$

then we will have

$$\begin{aligned}\frac{\partial\tilde{L}}{\partial q^{\sigma}_{s}}&=\frac{\partial L}{\partial q^{\sigma}_{s}}+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q^{\Delta}_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial q^{\sigma}_{s}}\\&=\frac{\partial L}{\partial q^{\sigma}_{r}}+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q^{\sigma}_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\left(\sigma(t)\right)\\&\quad+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q^{\Delta}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q^{\sigma}_{r}}+\sum_{\gamma=1}^{g}\frac{\partial\varphi_\beta}{\partial q^{\sigma}_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial q^{\sigma}_{r}}\left(\sigma(t)\right)\right),\end{aligned} \quad (16)$$

$$\frac{\partial\tilde{L}}{\partial q^{\Delta}_{r}}=\frac{\partial L}{\partial q^{\Delta}_{r}}+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q^{\Delta}_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}, \quad (17)$$

$$\begin{aligned}\delta\tilde{L}&=\sum_{s=1}^{n}\frac{\partial\tilde{L}}{\partial q^{\sigma}_{s}}\delta q^{\sigma}_{s}+\sum_{r=1}^{\varepsilon}\frac{\partial\tilde{L}}{\partial q^{\Delta}_{r}}\delta q^{\Delta}_{r}\\&=\sum_{r=1}^{\varepsilon}\left(\frac{\partial L}{\partial q^{\sigma}_{r}}+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q^{\sigma}_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\left(\sigma(t)\right)\right)\delta q^{\sigma}_{r}\\&\quad+\sum_{r=1}^{\varepsilon}\left[\left(\frac{\partial L}{\partial q^{\Delta}_{r}}+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q^{\Delta}_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial q^{\Delta}_{r}}\right)\delta q^{\Delta}_{r}\right.\end{aligned}$$

$$+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}\left(\frac{\partial \varphi_\beta}{\partial q_r^{\sigma}}+\sum_{\gamma=1}^{g}\frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\gamma}^{\sigma}}\frac{\partial \varphi_\gamma}{\partial q_r^{\sigma}}(\sigma(t))\right)\delta q_r^{\sigma}\Bigg]$$
$$=\sum_{s=1}^{n}\frac{\partial L}{\partial q_s^{\sigma}}\delta q_s^{\sigma}+\sum_{s=1}^{n}\frac{\partial L}{\partial q_s^{\Delta}}\delta q_s^{\Delta}$$
$$=\delta L. \tag{18}$$

Equation (13) can be written as

$$\int_{t_1}^{t_2}\left[\delta\tilde{L}+\sum_{r=1}^{\varepsilon}\left(\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}T_r^{\varepsilon+\beta}+\tilde{Q}_r\right)\delta q_r^{\sigma}\right]\Delta t=0, \tag{19}$$

and equation (19) is the Hamilton principle of a nonholonomic Chaplygin system on time scales.

From Eq. (19), we obtain

$$\int_{t_1}^{t_2}\sum_{r=1}^{\varepsilon}\Bigg[\frac{\partial\tilde{L}}{\partial q_r^{\Delta}}\delta q_r^{\Delta}+\left(\frac{\partial\tilde{L}}{\partial q_r^{\sigma}}+\sum_{\beta=1}^{g}\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}^{\sigma}}\frac{\partial\varphi_\beta}{\partial q_r^{\sigma}}(\sigma(t))\right.$$
$$\left.+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}T_r^{\varepsilon+\beta}+\tilde{Q}_r\right)\delta q_r^{\sigma}\Bigg]\Delta t=0. \tag{20}$$

Considering the integrating formula on time scales[4] and the boundary conditions (11), we have

$$\int_{t_1}^{t_2}\left(\frac{\partial\tilde{L}}{\partial q_r^{\sigma}}+\sum_{\beta=1}^{g}\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}^{\sigma}}\frac{\partial\varphi_\beta}{\partial q_r^{\sigma}}(\sigma(t))\right.$$
$$\left.+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}T_r^{\varepsilon+\beta}+\tilde{Q}_r\right)\delta q_r^{\sigma}\Delta t$$
$$=-\int_{t_1}^{t_2}\int_{t_1}^{t}\left(\frac{\partial\tilde{L}}{\partial q_r^{\sigma}}+\sum_{\beta=1}^{g}\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}^{\sigma}}\frac{\partial\varphi_\beta}{\partial q_r^{\sigma}}(\sigma(\tau))\right.$$
$$\left.+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}T_r^{\varepsilon+\beta}+\tilde{Q}_r\right)\Delta\tau\delta q_r^{\Delta}\Delta t. \tag{21}$$

Substituting Eq. (21) into Eq. (20), from Dubois–Reymond lemma,[4] we have

$$\frac{\partial\tilde{L}}{\partial q_r^{\Delta}}-\int_{t_1}^{t}\left(\frac{\partial\tilde{L}}{\partial q_r^{\sigma}}+\sum_{\beta=1}^{g}\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}^{\sigma}}\frac{\partial\varphi_\beta}{\partial q_r^{\sigma}}(\sigma(\tau))\right.$$
$$\left.+\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}T_r^{\varepsilon+\beta}+\tilde{Q}_r\right)\Delta\tau=c\quad(r=1,2,\ldots,\varepsilon), \tag{22}$$

where c is constant. Taking the delta derivative of Eq. (22) with respect to t, we obtain

$$\frac{\Delta}{\Delta t}\frac{\partial\tilde{L}}{\partial q_r^{\Delta}}-\frac{\partial\tilde{L}}{\partial q_r^{\sigma}}-\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}T_r^{\varepsilon+\beta}$$
$$-\sum_{\beta=1}^{g}\frac{\partial\tilde{L}}{\partial q_{\varepsilon+\beta}^{\sigma}}\frac{\partial\varphi_\beta}{\partial q_r^{\sigma}}(\sigma(t))=\tilde{Q}_r\quad(r=1,2,\ldots,\varepsilon). \tag{23}$$

Equation (23) is also written as

$$\frac{\Delta}{\Delta t}\frac{\partial\tilde{L}}{\partial q_r^{\Delta}}-\frac{\partial\tilde{L}}{\partial q_r^{\sigma}}-\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}\left(\frac{\Delta}{\Delta t}\frac{\partial\varphi_\beta}{\partial q_r^{\Delta}}-\frac{\partial\varphi_\beta}{\partial q_r^{\sigma}}\right)$$
$$-\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\sigma}}\frac{\partial\varphi_\beta}{\partial q_r^{\Delta}}(\sigma(t))=\tilde{Q}_r\quad(r=1,2,\ldots,\varepsilon). \tag{24}$$

Equations (23) and (24) are the generalized Chaplygin equations according to the Suslov view on time scales.

If $\mathbb{T}=\mathbb{R}$, $\sigma(t)=t$, $\mu(t)=0$, then equation (24) is reduced to the classical generalized Chaplygin equation[29,30]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_r}-\frac{\partial\tilde{L}}{\partial q_r}-\sum_{\beta=1}^{g}\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_r}-\frac{\partial\varphi_\beta}{\partial q_r}\right)$$
$$-\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_r}=\tilde{Q}_r\quad(r=1,2,\ldots,\varepsilon). \tag{25}$$

If $\mathbb{T}=h\mathbb{Z}$, $\sigma(t)=t+h$, $\mu(t)=h$, then equation (24) is reduced to the discrete generalized Chaplygin equation

$$\sum_{r=1}^{\varepsilon}\Bigg[\Delta_t\frac{\partial\tilde{L}}{\partial\Delta_t q_r}-\frac{\partial\tilde{L}}{\partial q_r(t+h)}$$
$$-\sum_{\beta=1}^{g}\frac{\partial L}{\partial\Delta_t q_{\varepsilon+\beta}}\left(\Delta_t\frac{\partial\varphi_\beta}{\partial\Delta_t q_r}-\frac{\partial\varphi_\beta}{\partial q_r(t+h)}\right)$$
$$-\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}(t+h)}\frac{\partial\varphi_\beta}{\partial\Delta_t q_r}(t+h)-\tilde{Q}_r\Bigg]=0, \tag{26}$$

where the continuous derivatives are substituted by $\Delta_t q_s=(1/h)(q_s(t)-q_s(t-h))$ (see Ref. [31]).

The general Chaplygin equations (24) are applied to the nonholonomic constraints system, in which the constraints are first order linear. The numbers of Eq. (24) are $n-g$ degrees of freedom.

3. Chaplygin equations on time scales

In this section, we present the Chaplygin equations on time scales. The generalized Chaplygin equations (24) are reduced to the Chaplygin equations on time scales, when the constraints of the system are first order linear homogeneous stationarity, and the cycle coordinates are explicitly contained in the system on time scales.

In general, if neither the cycle coordinates $q_{\varepsilon+1}^{\sigma}$, $q_{\varepsilon+2}^{\sigma}$, $\ldots,q_n^{\sigma}$ nor the constraint (2) is contained in the system , and $\tilde{Q}_r=0$, then equations (24) are reduced to

$$\frac{\Delta}{\Delta t}\frac{\partial\tilde{L}}{\partial q_r^{\Delta}}-\frac{\partial\tilde{L}}{\partial q_r^{\sigma}}-\sum_{\beta=1}^{g}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}\left(\frac{\Delta}{\Delta t}\frac{\partial\varphi_\beta}{\partial q_r^{\Delta}}-\frac{\partial\varphi_\beta}{\partial q_r^{\sigma}}\right)=0$$
$$(r=1,2,\ldots,\varepsilon), \tag{27}$$

where $\tilde{L}=\tilde{L}\left(t,q_l^{\sigma},q_l^{\Delta}\right)=L\left(t,q_r^{\sigma},q_r^{\Delta},\varphi_\beta\left(t,q_r^{\sigma},q_r^{\Delta}\right)\right)$.

If the constraints of the nonholonomic system on time scales are subjected to first order linear homogeneous stationarity, and the coordinates $q_{\varepsilon+1}^{\sigma},q_{\varepsilon+2}^{\sigma},\ldots,q_n^{\sigma}$ are not explicitly contained in the system. The constraints can be expressed as

$$q_{\varepsilon+\beta}^{\Delta}=\sum_{r=1}^{\varepsilon}B_{\varepsilon+\beta,r}q_r^{\Delta}\quad(\beta=1,2,\ldots,g;\ \varepsilon=n-g), \tag{28}$$

where the coefficients $B_{\varepsilon+\beta,s}$ are functions of coordinates $q_r^{\sigma}\,(r=1,2,\ldots,\varepsilon)$. Then equation (27) is reduced to

$$\frac{\Delta}{\Delta t}\frac{\partial \tilde{L}}{\partial q_r^{\Delta}}-\frac{\partial \tilde{L}}{\partial q_r^{\sigma}}-\sum_{\beta=1}^{g}\sum_{l=1}^{\varepsilon}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}\times\left(\frac{\partial B_{\varepsilon+\beta,r}}{\Delta q_l^{\sigma}}q_l^{\sigma\Delta}-\frac{\partial B_{\varepsilon+\beta,l}}{\partial q_r^{\sigma}}q_l^{\Delta}\right)=0\quad(r=1,2,\ldots,\varepsilon),\tag{29}$$

where

$$\tilde{L}=\tilde{L}\left(t,q_l^{\sigma},q_l^{\Delta}\right)=L\left(t,q_r^{\sigma},q_r^{\Delta},\sum_{r=1}^{\varepsilon}B_{\varepsilon+\beta,r}q_r^{\Delta}\right).$$

Equation (29) can be written as

$$\frac{\Delta}{\Delta t}\frac{\partial \tilde{L}}{\partial q_r^{\Delta}}-\frac{\partial \tilde{L}}{\partial q_r^{\sigma}}-\sum_{\beta=1}^{g}\sum_{l=1}^{\varepsilon}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}\left(\frac{\partial B_{\varepsilon+\beta,r}}{\Delta q_l^{\sigma}}-\frac{\partial B_{\varepsilon+\beta,l}}{\partial q_r^{\sigma}}\right)q_l^{\Delta}-\sum_{\beta=1}^{g}\sum_{l=1}^{\varepsilon}\frac{\partial L}{\partial q_{\varepsilon+\beta}^{\Delta}}\frac{\partial B_{\varepsilon+\beta,r}}{\Delta q_l^{\sigma}}\left(\mu q_l^{\Delta}\right)^{\Delta}=0\quad(r=1,2,\ldots,\varepsilon).\tag{30}$$

Equations (29) and (30) are Chaplygin equations on time scales.

If $\mathbb{T}=\mathbb{R}$, $\sigma(t)=t$, and $\mu(t)=0$, then equation (30) is reduced to the classical Chaplygin equation as follows:[29]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{q}_r}-\frac{\partial \tilde{L}}{\partial q_r}-\sum_{\beta=1}^{g}\sum_{l=1}^{\varepsilon}\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\times\left(\frac{\partial B_{\varepsilon+\beta,r}}{\partial q_l}-\frac{\partial B_{\varepsilon+\beta,l}}{\partial q_r}\right)\dot{q}_l=0\quad(r=1,2,\ldots,\varepsilon).\tag{31}$$

If $\mathbb{T}=h\mathbb{Z}$, $\sigma(t)=t+h$, $\mu(t)=h$, then equation (30) is reduced to the discrete Chaplygin equations as follows:

$$\sum_{r=1}^{\varepsilon}\left[\Delta_t\frac{\partial \tilde{L}}{\partial \Delta_t q_r}-\frac{\partial \tilde{L}}{\partial q_r(t+h)}-\sum_{\beta=1}^{g}\sum_{l=1}^{\varepsilon}\frac{\partial L}{\partial \Delta_t q_{\varepsilon+\beta}}\times\left(\frac{\partial B_{\varepsilon+\beta,r}}{\partial q_l(t+h)}\Delta_t q_l(t+h)-\frac{\partial B_{\varepsilon+\beta,l}}{\partial q_r(t+h)}\Delta_t q_l\right)\right]=0.\tag{32}$$

In general, the Chaplygin equations (30) are applied to the nonholonomic constraint system, in which the constraints are first order linear homogeneous stationarity, and the cycle coordinates are explicitly contained in the system on time scales. The numbers of Chaplygin equations (30) are $n-g$ degrees of freedom.

4. Examples

Example 1 Appell-Hamel problem[32] on time scales

The Lagrangian and constraints of the system on time scales are

$$L=\frac{m}{2}\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2+\left(z^{\Delta}\right)^2\right]-mgz^{\sigma},\tag{33}$$

$$z^{\Delta}=\frac{b}{a}\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{1/2},\tag{34}$$

where m, a, and b are constant. Now we come to establish the Chaplygin equations for the system.

Let

$$\tilde{L}=\frac{m}{2}\left(1+\frac{b^2}{a^2}\right)\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]-mgz^{\sigma},\tag{35}$$

then we will have

$$\frac{\partial \tilde{L}}{\partial x^{\Delta}}=m\left(1+\frac{b^2}{a^2}\right)x^{\Delta},\tag{36}$$

$$\frac{\partial \tilde{L}}{\partial y^{\Delta}}=m\left(1+\frac{b^2}{a^2}\right)y^{\Delta},\tag{37}$$

$$\frac{\partial z^{\Delta}}{\partial x^{\Delta}}=\frac{b}{a}x^{\Delta}\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{-1/2},\tag{38}$$

$$\frac{\partial z^{\Delta}}{\partial y^{\Delta}}=\frac{b}{a}y^{\Delta}\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{-1/2}.\tag{39}$$

From Eq. (24), we obtain

$$\begin{aligned}&m\left(1+\frac{b^2}{a^2}\right)x^{\Delta\Delta}-\frac{mb^2}{a^2}\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{1/2}\times\frac{\Delta}{\Delta t}\left\{x^{\Delta}\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{-1/2}\right\}\\&+mg\frac{b}{a}\frac{x^{\Delta\sigma}}{\left[\left(x^{\Delta\sigma}\right)^2+\left(y^{\Delta\sigma}\right)^2\right]^{1/2}}=0,\\&m\left(1+\frac{b^2}{a^2}\right)y^{\Delta\Delta}-\frac{mb^2}{a^2}\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{-1/2}\times\frac{\Delta}{\Delta t}\left\{y^{\Delta}\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{-1/2}\right\}\\&+mg\frac{b}{a}\frac{y^{\Delta\sigma}}{\left[\left(x^{\Delta\sigma}\right)^2+\left(y^{\Delta\sigma}\right)^2\right]^{1/2}}=0.\end{aligned}\tag{40}$$

From Eq. (40), we obtain

$$\begin{aligned}&x^{\Delta\Delta}+\frac{b^2}{2a^2}\frac{x^{\Delta\sigma}\left[x^{\Delta\Delta}\left(x^{\Delta}+x^{\Delta\sigma}\right)+y^{\Delta\Delta}\left(y^{\Delta}+y^{\Delta\sigma}\right)\right]}{\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{1/2}\left[\left(x^{\Delta\sigma}\right)^2+\left(y^{\Delta\sigma}\right)^2\right]^{1/2}}+\frac{gb}{a}\frac{x^{\Delta\sigma}}{\left[\left(x^{\Delta\sigma}\right)^2+\left(y^{\Delta\sigma}\right)^2\right]^{1/2}}=0,\\&y^{\Delta\Delta}+\frac{b^2}{2a^2}\frac{y^{\Delta\sigma}\left[x^{\Delta\Delta}\left(x^{\Delta}+x^{\Delta\sigma}\right)+y^{\Delta\Delta}\left(y^{\Delta}+y^{\Delta\sigma}\right)\right]}{\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]^{1/2}\left[\left(x^{\Delta\sigma}\right)^2+\left(y^{\Delta\sigma}\right)^2\right]^{1/2}}+\frac{gb}{a}\frac{y^{\Delta\sigma}}{\left[\left(x^{\Delta\sigma}\right)^2+\left(y^{\Delta\sigma}\right)^2\right]^{1/2}}=0.\end{aligned}\tag{41}$$

Equations (41) are generalized Chaplygin equations of the system.

If $\mathbb{T}=\mathbb{R}$, $\sigma(t)=t$, then equation (41) is reduced to the classical generalized Chaplygin equation as follows:[29]

$$\ddot{x}+\frac{b^2}{a^2}\frac{\dot{x}}{\dot{x}^2+\dot{y}^2}\left(\dot{x}\ddot{x}+\dot{y}\ddot{y}\right)+\frac{gb}{a}\dot{x}\left(\dot{x}^2+\dot{y}^2\right)^{-1/2}=0,$$

$$\ddot{y}+\frac{b^2}{a^2}\frac{\dot{y}}{\dot{x}^2+\dot{y}^2}\left(\dot{x}\ddot{x}+\dot{y}\ddot{y}\right)+\frac{gb}{a}\dot{y}\left(\dot{x}^2+\dot{y}^2\right)^{-1/2}=0. \quad (42)$$

If $\mathbb{T}=h\mathbb{Z}$ and $\sigma(t)=t+h$, then equation (41) is reduced to the discrete generalized Chaplygin equations as follows:

$$\begin{aligned}&\Delta_t\left(\Delta_t x\right)+\Delta_t\left(\Delta_t y\right)+\frac{b^2}{a^2}\frac{\Delta_t x+\Delta_t y}{\left(\Delta_t x\right)^2+\left(\Delta_t y\right)^2}\\&\times\left[\Delta_t x\Delta_t\left(\Delta_t x\right)+\Delta_t y\Delta_t\left(\Delta_t y\right)\right]+\frac{gb}{a}\left(\Delta_t x+\Delta_t y\right)\\&\times\left[\left(\Delta_t x(t+h)\right)^2+\left(\Delta_t y(t+h)\right)^2\right]^{-1/2}=0. \quad (43)\end{aligned}$$

Example 2 Motion of a particle on a plane on $\mathbb{T}=\{2^n,n\in\mathbb{N}\}$

The Lagrangian and the generalized forces of the system are

$$\begin{aligned}&L=\frac{1}{2}m\left[\left(q_1^{\Delta}\right)^2+\left(q_2^{\Delta}\right)^2\right],\\&Q_1=mq_2^{\sigma},Q_2=0, \quad (44)\end{aligned}$$

where m is constant. The constraint of the system is

$$\left(q_2^{\Delta}\right)^2=\left(q_1^{\Delta}\right)^2+2q_2^{\sigma}. \quad (45)$$

Let

$$\tilde{L}=m\left(q_1^{\Delta}\right)^2+mq_2^{\sigma}, \quad (46)$$

then we will have

$$\frac{\partial\tilde{L}}{\partial q_1^{\Delta}}=2mq_1^{\Delta}, \quad (47)$$

$$\frac{\partial L}{\partial q_2^{\sigma}}=0, \quad (48)$$

$$\frac{\partial q_2^{\Delta}}{\partial q_1^{\Delta}}=q_1^{\Delta}\left[\left(q_1^{\Delta}\right)^2+2q_2^{\sigma}\right]^{-1/2}. \quad (49)$$

Substituting Eqs. (44), (47), (48), and (49) into Eq. (24), we obtain the generalized Chaplygin equation as follows:

$$\begin{aligned}&\left[\left(q_1^{\Delta}\right)^2+2q_2^{\sigma}+\frac{1}{2}q_1^{\Delta\sigma}\left(q_1^{\Delta}+q_1^{\Delta\sigma}\right)\right]q_1^{\Delta\Delta}\\&+\sigma^{\Delta}(t)\,q_1^{\Delta\sigma}q_2^{\Delta\sigma}-q_2^{\sigma}\left[\left(q_1^{\Delta}\right)^2+2q_2^{\sigma}\right]=0. \quad (50)\end{aligned}$$

Since the time scales

$$\mathbb{T}=\{2^n,n\in\mathbb{N}\},\ \ \sigma(t)=2t,$$

then equation (50) can be written as

$$\begin{aligned}&\left[\left(q_1^{\Delta}\right)^2+2q_2(2t)+\frac{1}{2}q_1^{\Delta}(2t)\left(q_1^{\Delta}+q_1^{\Delta}(2t)\right)\right]q_1^{\Delta\Delta}\\&+2q_1^{\Delta}(2t)\,q_2^{\Delta}(2t)-q_2(2t)\left[\left(q_1^{\Delta}\right)^2+2q_2(2t)\right]=0. \quad (51)\end{aligned}$$

Example 3 Sled sliding at a horizontal plane[32] on time scales

The Lagrangian of the system is

$$L=\frac{1}{2}J\left(\theta^{\Delta}\right)^2+\frac{1}{2}m\left[\left(x^{\Delta}\right)^2+\left(y^{\Delta}\right)^2\right]-\frac{1}{2}k^2(\theta^{\sigma})^2, \quad (52)$$

where m, J, and k are constants. The constraint of the system on time scales is

$$y^{\Delta}=x^{\Delta}\tan\theta^{\sigma}. \quad (53)$$

We come to discuss the differential equations of motion for the system.

Since the coordinate y^{σ} is not contained in Eq. (52) nor Eq. (53), the system is a Chaplygin system. Let

$$\tilde{L}=\frac{1}{2}J\left(\theta^{\Delta}\right)^2+\frac{1}{2}m\left(x^{\Delta}\right)^2\left(1+\tan^2\theta^{\sigma}\right)-\frac{1}{2}k^2(\theta^{\sigma})^2, \quad (54)$$

Then we will have

$$\frac{\partial\tilde{L}}{\partial\theta^{\Delta}}=J\theta^{\Delta}, \quad (55)$$

$$\frac{\partial\tilde{L}}{\partial x^{\Delta}}=m\left(1+\tan^2\theta^{\sigma}\right), \quad (56)$$

$$\frac{\partial\tilde{L}}{\partial\theta^{\sigma}}=m\left(x^{\Delta}\right)^2\tan\theta^{\sigma}\sec^2\theta^{\sigma}-k^2\theta^{\sigma}. \quad (57)$$

From Eq. (53), we obtain

$$B_{y\theta}=0,\quad B_{yx}=\tan\theta^{\sigma}. \quad (58)$$

From Eq. (30), we obtain the Chaplygin equations

$$\begin{aligned}&J\theta^{\Delta\Delta}+k^2\theta^{\sigma}=0,\\&x^{\Delta\Delta}+\left(\tan\theta^{\sigma}+\tan\theta^{\sigma\sigma}\right)x^{\Delta\sigma}\theta^{\Delta\sigma}\sigma^{\Delta}\\&\quad-\tan\theta^{\sigma}x^{\Delta}\left[\theta^{\Delta}+\left(\mu\theta^{\Delta}\right)^{\Delta}\right]=0. \quad (59)\end{aligned}$$

If $\mathbb{T}=\mathbb{R}$, $\sigma(t)=t$, and $\mu(t)=0$, then equation (59) can be written as

$$\begin{aligned}&J\ddot{\theta}+k^2\theta=0,\\&\ddot{x}+\dot{x}\dot{\theta}\tan\theta=0. \quad (60)\end{aligned}$$

If $\mathbb{T}=h\mathbb{Z}$, $\sigma(t)=t+h$, and $\mu(t)=h$, then equation (59) can be written as

$$\begin{aligned}&J\Delta_t\left(\Delta_t\theta\right)+k^2\theta(t+h)+\Delta_t\left(\Delta_t x\right)\\&\quad+\Delta_t x\Delta_t\theta(t+h)\tan\theta(t+h)=0. \quad (61)\end{aligned}$$

5. Conclusions

Time scales are an arbitrary nonempty closed subset of the set of real numbers. The study of differential equations on time scales, which has been created in order to unify and extend the study of differential and difference equations, is an area of science and engineering, which has recently gained a lot of attention.

The paper presents the generalized Chaplygin equations on time scales. The Hamilton principles for nonholonomic systems on time scales are established, and the corresponding generalized Chaplygin equation (see Eq. (24)) is deduced (see Eq. (19)). The generalized Chaplygin equations (see Eq. (24)) are reduced to Chaplygin equations (see Eq. (30)), when the constraints of the nonholonomic system on time scales are subjected to first order linear homogeneous stationary, and the coordinates $q_{\varepsilon+1}^{\sigma}, q_{\varepsilon+2}^{\sigma}, \ldots, q_n^{\sigma}$ are not explicitly contained in the systems. When the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$, two special cases of the Chaplygin equations on time scales are presented. The number of the Chaplygin equations on time scales is $n - g$. We can extend the present work to Birkhoffian systems on time scales. It is also available for optimal control systems on time scales, etc.

References

[1] Hilger S 1988 "Ein M$\alpha\beta$kettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten", Ph. D. thesis (Universit Würzburg)
[2] Ahlbrand D, Bohner M and Ridenhour J 2000 *J. Math. Anal. Appl.* **250** 561
[3] Bohner M and Peterson A 2001 *Dynamic Equations on Time Scale: An Introduction with Applications* (Boston: Birkhäuser)
[4] Bohner M and Peterson A 2003 *Advances in Dynamic Equations on Time Scales* (Boston: Birkhäuser)
[5] Agarwal R, Martin B, *et al.* 2004 *J. Comput. Appl. Math.* **141** 1
[6] Bohner M and Guseinov G S H 2004 *Dyn. Sys. Appl.* **13** 351
[7] MAtici F, Biles D C and Lebedinsky A 2006 *Math. Comput. Definition* **43** 718
[8] Zhang H T and Li Y K 2009 *Commun. Nonlinear Sci. Numer. Simul.* **14** 19
[9] Abdeljawad T, Jarad F and Baleanu D 2009 *Adv. Differ. Eqs.* **30** 840386
[10] Hilscher R Š and Zeidan V 2012 *Nonlinear Anal.* **75** 932
[11] Ferreira R A C and Torres D F M 2007 *Math. Control Theory Fin.* **136** 145
[12] Bohner M 2004 *Dyn. Sys. Appl.* **13** 339
[13] Martins N and Torres D F M 2009 *Nonlinear Anal.* **71** e763
[14] Torres D F M 2010 *Int. J. Sim. Mult. Des. Optim.* **4** 11
[15] Bartosiewicz Z, Martins N and Torres D F M 2010 *Eur. J. Control* **17** 9
[16] Malinowska A B and Ammi M R S 2014 *Int. J. Differ. Eqs.* **9** 87
[17] Guo Z H and Gao P Y 1990 *Acta Mech. Sin.* **22** 185 (in Chinese)
[18] Chen B 1991 *Acta Mech. Sin.* **23** 379 (in Chinese)
[19] Guo Z H and Gao P Y 1992 *Acta Mech. Sin.* **24** 253 (in Chinese)
[20] Cai P P, Fu J L and Guo Y X 2013 *Sci. China G-Phys. Mech. Astron.* **56** 1017
[21] Zu Q H and Zhu J Q 2016 *J. Math. Phys.* **57** 082701
[22] Song C J and Zhang Y 2015 *J. Math. Phys.* **56** 102701
[23] Jin S X and Zhang Y 2017 *Chin. Phys. B* **26** 014501
[24] Martins N and Torres D F M 2010 *Appl. Math. Lett.* **23** 1432
[25] Malinowska A B and Martins N 2013 *Abstr. Appl. Anal.* **2013** 1728
[26] Carlson D A 2015 *J. Optim. Theory Appl.* **166** 351
[27] Zhang Y 2016 *Chin. Quar. Mech.* **37** 214 (in Chinese)
[28] Jin S X and Zhang Y 2017 *Chin. Quar. Mech.* **38** 447 (in Chinese)
[29] Mei F X 2013 *Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) (in Chinese)
[30] Chen X W and Mei F X 2016 *Mech. Res. Commun.* **76** 91
[31] Guo H Y, Li Y Q, *et al.* 2002 *Commun. Theor. Phys.* **37** 1
[32] Mei F X, Liu D and Luo Y 1991 *Advanced Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) (in Chinese)