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Exact solitary wave solutions of a nonlinear Schrödinger equation model with saturable-like nonlinearities governing modulated waves in a discrete electrical lattice

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In this paper, we introduce and propose exact and explicit analytical solutions to a novel model of the nonlinear Schrödinger (NLS) equation. This model is derived as the equation governing the dynamics of modulated cutoff waves in a discrete nonlinear electrical lattice. It is characterized by the addition of two terms that involve time derivatives to the classical equation. Through those terms, our model is also tantamount to a generalized NLS equation with saturable; which suggests that the discrete electrical transmission lines can potentially be used to experimentally investigate wave propagation in media that are modeled by such type of nonlinearity. We demonstrate that the new terms can enlarge considerably the forms of the solutions as compared to similar NLS-type equations. Sine–Gordon expansion-method is used to derive numerous kink, antikink, dark, and bright soliton solutions.

Keywords: nonlinear Schrödinger equation, nonlinear time derivative terms, saturable nonlinearity, exact solitary solutions

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1. Introduction

Nonlinear equations are widely used as models to describe many important dynamical phenomena in various fields of sciences, particularly in nonlinear optics,[1–4] in plasma physics,[5,6] in biophysics,[7] in Bose–Einstein condensates,[8] in atomic chain,[9,10] in Fermi–Pasta–Ulam lattice,[11,12] in crystals,[13–16] and in discrete electrical transmission lines.[17–24] Among these equations, the famous nonlinear Schrödinger (NLS) equation is usually derived from fundamental principles as a primary approximation. As such, its suitability for the accurate description of the phenomena of interest in these fields can be limited. This has aroused the interest in investigation approaches that relax some of the assumptions that are usually considered in a first approximation. Such approaches often lead to generalized forms of the nonlinear Schrödinger equation rather than the standard one. Consider for instance a discrete nonlinear electrical transmission line (DNLTL) that consists of a number of identical *LC* blocks connected in sequence one to the other. Each block, whose model is presented in Fig. 1, contains two linear inductors L_s and L_p of which one is in the series branch and the other in shunt branch, and a bias-dependent capacitor C_p responsible for the nonlinearity.

We assume that the capacitance–voltage relationships are approximated by $C_p(V_n) = C_{0p}(1 - 2\alpha V_n + 3\beta V_n^2)$, and $C_s(V) = C_{0s}(1 - 2\eta V + 3\lambda V^2)$ respectively for the nonlinear capacitor $C_p(V_n)$ in the parallel branch and for the one in the series branch, $C_s(V)$. Thus, C_{0p} and C_{0s} are the respective limiting values of these capacitances when the voltages across them are infinitesimally small. α and β, and similarly η and λ, represent respectively the quadratic and cubic coefficients of nonlinearity.

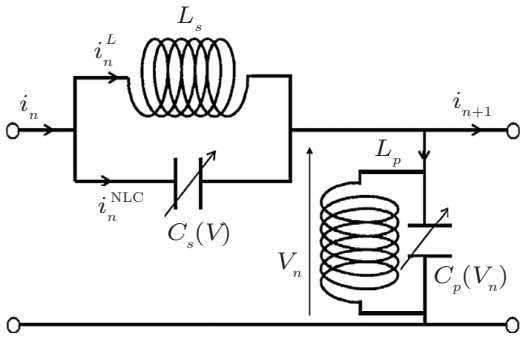


Fig. 1. Schematic representation of a unit block of the nonlinear transmission line.

Applying Kirchhoff laws to this circuit leads to the following set of differential equations governing the dynamics of signals in the network[15]

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$$\frac{\mathrm{d}^2V_n}{\mathrm{d}t^2}+u_0^2\left(2V_n-V_{n+1}-V_{n-1}\right)+\omega_0^2V_n+\beta\frac{\mathrm{d}^2V_n^3}{\mathrm{d}t^2}=\alpha\frac{zd^2V_n^2}{\mathrm{d}t^2}+C_{0r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(V_{n-1}+V_{n+1}-2V_n\right)$$
$$-\eta C_{0r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\left(V_{n-1}-V_n\right)^2-\left(V_n-V_{n+1}\right)^2\right]+\lambda C_{0r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\left(V_{n-1}-V_n\right)^3-\left(V_n-V_{n+1}\right)^3\right],\ n=1,2,\ldots,N \tag{1}$$

with n being the total number of blocks. Here, $C_{0r}=C_{0s}/C_{0p}$ is a dimensionless constat while $\omega_0^2=1/(L_pC_{0p})$ and $u_0^2=1/(L_sC_{0p})$ are characteristic frequencies of the system. Equation (1) has been used to investigate analytically and numerically the peak solitary wave and the modulational instability phenomenon in network.[26] Similarly, it has been used also to describe the dynamics of modulated waves in the specific case of the linear capacitor C_s in the series branch, that is, for $\eta=\lambda=0$.[18,27,28] Other special cases of this equation abound in the study of nonlinear discrete electrical transmission lines.[21,29]

A basic approximation that is commonly made in investigating equations of this type consists in taking their solutions in the form $A_0\exp(\mathrm{i}(kn-\omega t))$ while assuming A_0 to be infinitesimally small. Such solutions are characterized the so-called linear dispersion relation, given in our case by

$$\omega^2=\frac{\omega_0^2+4u_0^2\sin^2(k/2)}{1+4C_{0r}\sin^2(k/2)},$$

where ω (rad/s) and k (rad/cell) are, respectively, the angular frequency and wave number. The graph of the above relation, shown in Fig. 2, presents a lower cutoff mode frequency when $k=0$ and an upper cutoff frequency when $k=\pi$; which indicates that the network is a bandpass waveguide.

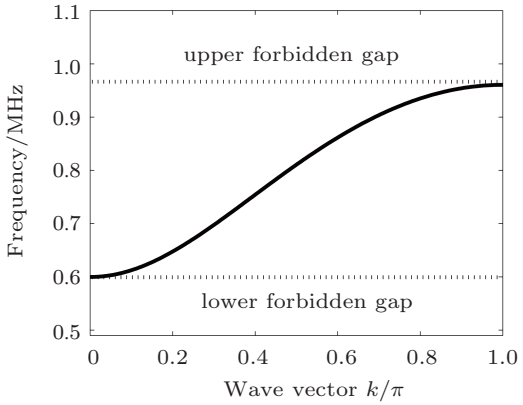


Fig. 2. The linear dispersion curve of the network according of the wave vector k (rad/cell) for $\omega_0=3.77\times10^6$ rad/s, $u_0=2.58\times10^6$ rad/s, and $C_{0r}=0.03$.

Focusing our attention on waves with frequencies approaching the lower cutoff frequency of the lattice but with amplitude of arbitrary magnitude, we look for a solution of Eq. (1) in the form

$$V_n(t)=A(x,T)\exp(\mathrm{i}\theta)+\mathrm{c.c.}, \tag{2}$$

where "c.c." stands for complex conjugate, $\theta=-\omega t$ is the phase variable, $x=\varepsilon n$ and $T=\varepsilon^2 t$ with a dimensionless parameter ε are introduced to account for the slow temporal and spatial variations of the amplitude induced by the mutual coupling between the oscillators. It is obvious from Eq. (3) that the voltage at any site shifted by an integer m from a reference site n is given by

$$V_{n+m}(t)=A(x+\varepsilon m,T)\exp(\mathrm{i}\theta)+\mathrm{c.c.}$$

In order to reduce the mathematical intractability of the manipulations involved in the analysis, we assume that $0<\varepsilon\ll1$ so that Taylor expansions of the following form

$$A(x+\varepsilon m,T)=A(x,T)+\varepsilon m\frac{\partial A}{\partial x}(x,T)+\frac{\varepsilon^2m^2}{2}\frac{\partial^2A}{\partial x^2}(x,T)$$
$$+\frac{\varepsilon^3m^3}{6}\frac{\partial^3A}{\partial x^3}(x,T)+\cdots$$

are warranted. Here the specific values $m=1$ and $m=-1$ are relevant for Eq. (1). The use of the variables θ, x, and T also implies that the time derivative operators are transformed according to

$$\frac{\mathrm{d}}{\mathrm{d}t}=-\omega\frac{\partial}{\partial\theta}+\varepsilon^2\frac{\partial}{\partial T},$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}=\omega^2\frac{\partial^2}{\partial\theta^2}-2\varepsilon^2\omega\frac{\partial^2}{\partial T\partial\theta}+\varepsilon^4\frac{\partial^2}{\partial T^2}.$$

Considering the above relations, the dynamics of the envelop $A(x,T)$ is governed by the following nonlinear partial differential

$$\mathrm{i}\frac{\partial A}{\partial T}+P\frac{\partial^2A}{\partial x^2}+Q|A|^2A=\mathrm{i}s_1A^2\frac{\partial\overline{A}}{\partial T}+\mathrm{i}s_2|A|^2\frac{\partial A}{\partial T} \tag{3}$$

with $P=(u_0^0-C_{0r}\omega_0^2)/2\omega_0$, $Q=3\omega_0\beta/2$, $s_1=-3\beta$, and $s_2=-6\beta$.

This equation differs from the classical NLS equation by the two terms of cubic nonlinearity in its right-hand-side. It is worth to observe that these terms involve derivatives with respect to the variable attached with the first order derivative in the standard NLS equation. Although the s_2-term can be considered to be common in NLS type equations with saturable nonlinearity, the s_1-term as well as the full right-hand-side of Eq. (3) have, to the best of our knowledge, rarely been considered. Equation (3) has been derived above as governing the transmission of modulated signals with carrier frequency in the lower gap of the spectrum of this system. However, the legendary universality of NLS type equations makes it legitimate to expect that its applications can extend beyond discrete

electrical transmission line. For instance, this equation may also be written in the form

$$\mathrm{i}\frac{\partial A}{\partial T}+\frac{P}{1+s_2|A|^2}\frac{\partial^2 A}{\partial x^2}+\frac{Q|A|^2A}{1+s_2|A|^2}=\mathrm{i}\frac{s_1A^2}{1+s_2|A|^2}\frac{\partial \overline{A}}{\partial T},$$

which shows that it is a generalized NLS equation with saturable type nonlinearities. Accordingly, waves propagation in media modeled by such nonlinearities,[30–33] including metamaterials in particular, can possibly be investigated experimentally using basic nonlinear discrete electrical transmission lines.[34] This further motivates the interest in the solutions of Eq. (3). In the literature, these solutions exist on several forms[4,8,35–42] and are very significant for the description of the phenomena in diverse fields of Physics and Engineering.

Our investigations in this work then aim to determine exact and explicit analytical solutions to this new model of NLS equation. To this end, our paper is organized as follows. The next section begins with the reduction of the extended NLS equation to a single nonlinear ordinary differential equation (ODE). Sine–Gordon expansion-method, which is among the most powerful and effective method devised by mathematicians and physicists to find exact solutions of ODEs is then briefly outlined. In Section 3, the exact solitary wave solutions of this last equation will be given. The last section will be devoted to the conclusion.

2. The methods

2.1. ODE equation

Over the past several decades, many research works have focused on techniques for solving partial differential equations (PDEs). A first step in most of these techniques consists in reducing the PDEs in consideration into ODEs. In this view, we are interested in this paper by the class of solutions of Eq. (3) which can be represented by

$$A(x,T)=\rho(\tau)\exp\left[\mathrm{i}(q_{\mathrm{e}}x-\Omega T+\phi(\tau))\right],\quad \tau=x-v_{\mathrm{e}}T, \tag{4}$$

where q_{e}, q_{e}, and Ω are real constants to be determined as part of solutions while ρ and ϕ and real-valued functions of a single variable. Substituting Eq. (4) into Eq. (3) and separating the real and imaginary part leads to the following system of coupled nonlinear ODEs:

$$\left[(v_{\mathrm{e}}-2Pq_{\mathrm{e}})\rho+v_{\mathrm{e}}(s_1-s_2)\rho^3\right]\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)-P\rho\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2+P\frac{\mathrm{d}^2\rho}{\mathrm{d}\tau^2}+(\Omega-Pq_{\mathrm{e}}^2)\rho+[Q+(s_1+s_2)\Omega]\rho^3=0, \tag{5a}$$

$$\left[2Pq_{\mathrm{e}}-v_{\mathrm{e}}+v_{\mathrm{e}}(s_1+s_2)\rho^2\right]\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+2P\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}+P\rho\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}=0. \tag{5b}$$

It is easy to verify that equation (5b) becomes an exact differential if multiplied by ρ. Doing so leads in effect to

$$(2Pq_{\mathrm{e}}-v_{\mathrm{e}})\rho\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+v_{\mathrm{e}}(s_1+s_2)\rho^3\frac{\mathrm{d}\rho}{\mathrm{d}\tau}+P\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\rho^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)=0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{2Pq_{\mathrm{e}}-v_{\mathrm{e}}}{2}\rho^2+\frac{v_{\mathrm{e}}(s_1+s_2)\rho^4}{4}+P\rho^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right]=0.$$

It is then clear that

$$\frac{2Pq_{\mathrm{e}}-v_{\mathrm{e}}}{2}\rho^2+\frac{v_{\mathrm{e}}(s_1+s_2)\rho^4}{4}+P\rho^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}=\mathscr{C}_0,$$

where $\mathscr{C}_0$ is an arbitrary constant. This last equation is a first integral of Eq. (5b). It can be easily solved for $\mathrm{d}\phi/\mathrm{d}\tau$. Considering the particular case where $\mathscr{C}_0=0$, we obtain

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau}=-\frac{v_{\mathrm{e}}(s_1+s_2)\rho^2}{4P}-\frac{2Pq_{\mathrm{e}}-v_{\mathrm{e}}}{2P}. \tag{6}$$

For simplicity, we fix the q_{e} so that the derivative of the phase $\phi(\tau)$ is proportional to ρ^2. Thus

$$q_{\mathrm{e}}=\frac{v_{\mathrm{e}}}{2P}. \tag{7}$$

By introducing Eq. (6) into Eq. (5a) while accounting for Eq. (7), we obtain the following single nonlinear ODE which governs the real amplitude of the wave:

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\tau^2}+\frac{4\Omega P-v_{\mathrm{e}}^2}{4P^2}\rho+\frac{Q+c_1\Omega}{P}\rho^3+\frac{c_2c_3v_{\mathrm{e}}^2}{16P^2}\rho^5=0 \tag{8}$$

with $c_1=s_1-s_2$, $c_2=s_1+s_2$, and $c_3=3s_2-5s_1$. Equation (8) is the celebrated cubic–quintic Duffing oscillator equation whose periodic solutions have been the focus of several research papers.[43–45] It is worth to notice that the quintic term in this equation is induced by the two nonlinear temporal terms of Eq. (3), though they are only of cubic order nonlinearity.

The next step after transforming the PDEs into ODEs is the effective determination of solutions to the latter. A first approach toward this goal for our problem is a direct integration from the first integral

$$E_M=\frac{1}{2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}\tau}\right)^2+V(\rho),$$

which underlines the analogy of Eq. (8) with the Newtonian equation of motion of a point particle of unit mass. Here, the

wave amplitude $\rho(\tau)$ plays the same role as the relative displacement of the particle; which moves freely in the following non-harmonic potential

$$V(\rho)=\frac{4\Omega P-v_{\rm e}^2}{8P^2}\rho^2+\frac{Q+c_1\Omega}{4P}\rho^4+\frac{c_2c_3v_{\rm e}^2}{96P^2}\rho^6.$$

This approach leads to an improper integral which contains a square root of a six-degree polynomial in the denominator[20] and so is quite difficult to evaluate.

The alternative approach preferred herein rely on the variety of methods available in the literature to deal with this task and which avoid the direct integration. Examples are the (G'/G)-expansion method,[46,47] the exp-expansion method,[48] the extended F-expansion method,[49] the bifurcation method of dynamical system,[50,51] the Jacobi elliptic function rational expansion method,[52] the tanh-function method,[53,54] the tan-expansion method,[55] the transformed rational function method,[56,57] the semi-inverse variational principle,[58] the Hirota bilinear method,[59–62] the variable coefficient Jacobian elliptic function method,[63] the sine-Gordon expansion approach,[64–67] and others. For our problem, the sine-Gordon expansion-method has revealed to be very effective. An important advantage of this techniques is that the computations involved can be systematically performed by a computer algebra system such as Maple or Mathematica. Below, it is briefly reviewed and used later to find various forms of solution to Eq. (8).

2.2. The sine-Gordon expansion-method

In this subsection we present the sine-Gordon expansion method which will later be utilized to investigate the solitary waves that can propagate through our discrete electric transmission line. This method is based on sine-Gordon equation and the traveling wave transformation.[68–70]

In fact, consider a nonlinear partial differential equations (NLPDEs) with two independent variables x and t,

$$H(U_t,U_x,U_{xx},U_{tt},\ldots)=0. \tag{9}$$

In general, the left-hand side of Eq. (9) is a polynomial in U and its various derivatives. Assume that $U(x,t)=U(\xi)$, $\xi=kx-\upsilon_c t$, where k and υ_c are constants to be determined later. Then equation (9) is reduced to an ordinary differential equation

$$G\left(U,\frac{{\rm d}U}{{\rm d}\xi},\frac{{\rm d}^2U}{{\rm d}\xi^2},\ldots\right)=0, \tag{10}$$

where G is a polynomial of U and its various derivatives. We consider that the sought solution can be expressed as follows:

$$U(\xi)=\sum_{i=1}^{M}\tanh^{i-1}(\xi)\left(a_i\,{\rm sech}(\xi)+b_i\tanh(\xi)\right)+a_0. \tag{11}$$

According to Refs. [65]–[67], equation (11) can be rewritten as

$$U(w)=\sum_{i=1}^{M}\cos^{i-1}(w)\left(a_i\sin(w)+b_i\cos(w)\right)+a_0, \tag{12}$$

where the function $w(\xi)=U(\xi)/2$ satisfies the following relation

$$w'=\sin(w). \tag{13}$$

Equation (13) is variables separable. We obtain the following two significant equations upon solving it

$$\begin{aligned}\sin(w)&=\sin(w(\xi))=\left.\frac{2p\exp(\xi)}{p^2\exp(2\xi)+1}\right|_{p=1}\\&={\rm sech}(\xi),\end{aligned} \tag{14}$$

$$\begin{aligned}\cos(w)&=\cos(w(\xi))=\left.\frac{p^2\exp(2\xi)-1}{p^2\exp(2\xi)+1}\right|_{p=1}\\&=\tanh(\xi).\end{aligned} \tag{15}$$

The parameter M is, in most cases, a positive integer that can be determined by considering the homogeneous balance between the highest order derivative and the highest order nonlinear terms appearing in ODE (10). Collecting and setting the coefficients of terms of the form $\sin^i(w)\cos^j(w)$ to zero yields a system of coupled nonlinear algebraic equations. Solving this system by using Maple gives the values of a_i, b_i, a_0, k_1, and υ_c. Finally, substituting these values of Eq. (11), we obtain the traveling wave solutions to Eq. (10).

3. Exact solutions

We organize our search of solutions according to whether the coefficient of the quintic term is nonzero or not.

3.1. The full cubic–quintic Duffing oscillator equation

We deal first with the most general situation where the coefficient of the quintic term is nonzero, *i.e.*, $c_2c_3v_{\rm e}^2\neq 0$. This is the case when $s_1\neq s_2$ and $3s_2\neq 5s_1$ and $v_{\rm e}\neq 0$. Considering the sine-Gordon expansion-method, the homogenous balance process gives $M=1/2$. Then we make the change of dependent variable $\rho(\tau)=\sqrt{u(\mu\tau)}$ and equation (8) becomes

$$2\mu^2uu''-\mu^2u'^2+\frac{4\Omega P-v_{\rm e}^2}{P^2}u^2+\frac{4(Q+c_1\Omega)}{P}u^3+\frac{c_2c_3v_{\rm e}^2}{4P^2}u^4=0. \tag{16}$$

Repeating the homogeneous balance for this new equation, we now obtain $M = 1$. Thus, it will be used in the continuation to determine the solutions of Eq. (3). Equation (12) for the value $M = 1$ obtained in the balancing process becomes

$$u(w) = b_1 \sin(w) + a_1 \cos(w) + a_0, \tag{17}$$

$$u'(w) = b_1 \sin(w)\cos(w) - a_1 \sin(w)^2, \tag{18}$$

$$u''(w) = b_1 \cos^2(w)\sin(w) - b_1 \sin^3(w) - 2a_1 \cos(w)\sin^2(w). \tag{19}$$

Substituting Eqs. (17), (18), and (19) into Eq. (16), and equating the coefficients of each power of $\sin^i(w)\cos^j(w)$ to zero, we obtain a system of algebraic equations for the parameters a_0, a_1, b_1, v_e, Ω, and μ, namely:

Constant:

$$-4v_e{}^2b_1{}^2 - 4v_e{}^2a_0{}^2 + 16\Omega P a_0{}^2 + 16Q a_0{}^3 P + c_2c_3v_e{}^2a_0{}^4 + 16c_1\Omega a_0{}^3 P + 16\Omega P b_1{}^2 - 4\mu^2 a_1{}^2 P^2 - 8\mu^2 b_1{}^2 P^2 + 6c_2c_3v_e{}^2a_0{}^2b_1{}^2 + 48Q a_0 b_1{}^2 P + 48c_1\Omega a_0 b_1{}^2 P + c_2c_3v_e{}^2 b_1{}^4 = 0,$$

$\sin(w)$:

$$-8v_e{}^2a_0b_1 + 32\Omega P a_0 b_1 + 4c_2c_3v_e{}^2a_0{}^3b_1 + 48Q a_0{}^2 b_1 P + 48c_1\Omega a_0{}^2 b_1 P - 8\mu^2 a_0 b_1 P^2 + 4c_2c_3v_e{}^2 a_0 b_1{}^3 + 16Q b_1{}^3 P + 16c_1\Omega b_1{}^3 P = 0,$$

$\cos(w)$:

$$-8v_e{}^2a_0a_1 + 32\Omega P a_0 a_1 + 4c_2c_3v_e{}^2a_0{}^3a_1 + 48Q a_0{}^2 a_1 P + 48c_1\Omega a_0{}^2 a_1 P + 12c_2c_3v_e{}^2a_0a_1b_1{}^2 - 16\mu^2 a_0 a_1 P^2 + 48Q a_1 b_1{}^2 P + 48c_1\Omega a_1 b_1{}^2 P = 0,$$

$\sin(w)\cos(w)$:

$$32\Omega P a_1 b_1 - 8v_e{}^2 a_1 b_1 + 12c_2c_3v_e{}^2a_0{}^2a_1b_1 + 96Q a_0 a_1 b_1 P + 96c_1\Omega a_0 a_1 b_1 P + 4c_2c_3v_e{}^2 a_1 b_1{}^3 - 16\mu^2 a_1 b_1 P^2 = 0,$$

$\cos^2(w)$:

$$4v_e{}^2b_1{}^2 + 16\Omega P a_1{}^2 - 4v_e{}^2a_1{}^2 + 6c_2c_3v_e{}^2a_0{}^2a_1{}^2 + 48Q a_0 a_1{}^2 P - 16\Omega P b_1{}^2 + 48c_1\Omega a_0 a_1{}^2 P - 8\mu^2 a_1{}^2 P^2 + 20\mu^2 b_1{}^2 P^2 + 6c_2c_3v_e{}^2a_1{}^2b_1{}^2 - 6c_2c_3v_e{}^2a_0{}^2b_1{}^2 - 48Q a_0 b_1{}^2 P - 48c_1\Omega a_0 b_1{}^2 P - 2c_2c_3v_e{}^2 b_1{}^4 = 0, \tag{20}$$

$\sin(w)\cos^2(w)$:

$$12c_2c_3v_e{}^2a_0a_1{}^2b_1 + 16\mu^2 a_0 b_1 P^2 + 48Q a_1{}^2 b_1 P + 48c_1\Omega a_1{}^2 b_1 P - 4c_2c_3v_e{}^2a_0b_1{}^3 - 16Q b_1{}^3 P - 16c_1\Omega b_1{}^3 P = 0,$$

$\cos^3(w)$:

$$4c_2c_3v_e{}^2a_0a_1{}^3 + 16Q a_1{}^3 P + 16c_1\Omega a_1{}^3 P - 12c_2c_3v_e{}^2a_0a_1b_1{}^2 + 16\mu^2 a_0 a_1 P^2 - 48Q a_1 b_1{}^2 P - 48c_1\Omega a_1 b_1{}^2 P = 0,$$

$\sin(w)\cos^3(w)$:

$$4c_2c_3v_e{}^2a_1{}^3b_1 + 24\mu^2 a_1 b_1 P^2 - 4c_2c_3v_e{}^2a_1b_1{}^3 = 0,$$

$\cos^4(w)$:

$$c_2c_3v_e{}^2a_1{}^4 + 12\mu^2 a_1{}^2 P^2 - 12\mu^2 b_1{}^2 P^2 - 6c_2c_3v_e{}^2a_1{}^2b_1{}^2 + c_2c_3v_e{}^2 b_1{}^4 = 0.$$

It appears that there are three more equations than the six variables sought for. Such a large number of coupled nonlinear algebraic equations can be obviously very challenging for human manipulation. On the contrary, computer algebra tools such as Maple, Mathematica or MuPAD are very powerful for such tasks, especially as these equations are all polynomial. For the explicit solutions that are of interest to us in this paper, and disregarding the trivial (constant) solutions, we find with the help of Maple that they depend on the combination of the physical parameters as considered below.

Case 1

$$\mu = \pm 2\frac{\sqrt{c_1 P\left(-12 + c_2c_3b_1{}^2\right)Qc_2c_3}b_1}{c_1 P\left(-12 + c_2c_3b_1{}^2\right)}, \quad v_e = \pm 4\frac{\sqrt{3}\sqrt{c_1 P\left(-12 + c_2c_3b_1{}^2\right)Q}}{c_1\left(-12 + c_2c_3b_1{}^2\right)}, \tag{21a}$$

$$a_0 = a_1 = 0, \quad b_1 \geq 0, \quad \Omega = -\frac{Q}{c_1}. \tag{21b}$$

According to Eqs. (4), (17), and Eqs. (21), we obtain the following exact bright solitary wave solutions of Eq. (3)

$$A(x,T) = \left[b_1 \operatorname{sech}\left(2\frac{\sqrt{c_1 P\left(-12 + c_2 c_3 {b_1}^2\right) Q c_2 c_3} b_1}{c_1 P\left(-12 + c_2 c_3 {b_1}^2\right)} (x - v_e T) \right) \right]^{1/2} \exp\left[\mathrm{i}\Theta\right], \tag{22}$$

where $\Theta = (q_e x - \Omega T + \phi(\tau))$ with ϕ and q_e are given in Eqs. (6) and (7). The intensity of these solutions is given by

$$|A(x,T)|^2 = b_1 \operatorname{sech}\left(2\frac{\sqrt{c_1 P\left(-12 + c_2 c_3 {b_1}^2\right) Q c_2 c_3} b_1}{c_1 P\left(-12 + c_2 c_3 {b_1}^2\right)} (x - v_e T) \right), \tag{23}$$

and the graphical representation is given in Fig. 3 for some arbitrary chosen values of parameters.

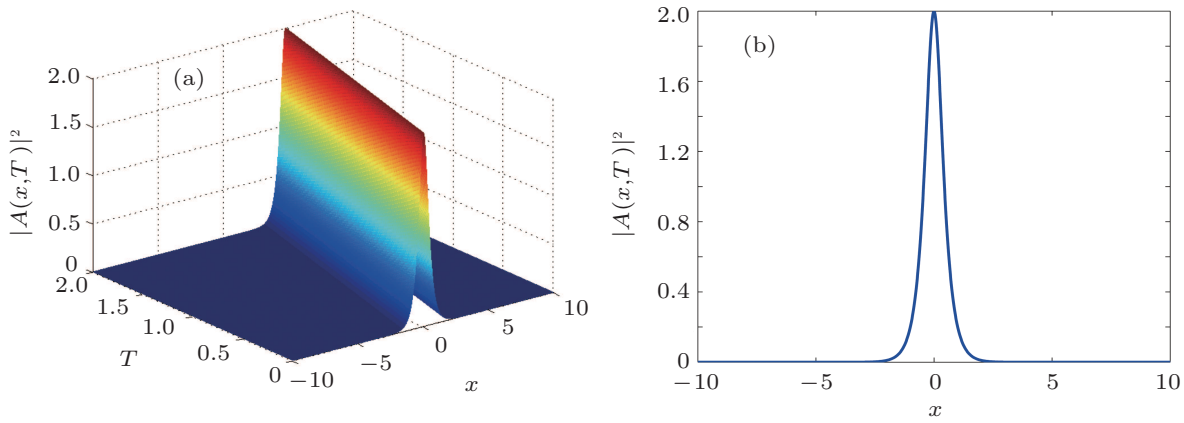


Fig. 3. (color online) The three-dimensional (3D) and two-dimensional (2D) intensities of the bright solitary wave solution Eq. (22) with the parameters $P = Q = 1$, $s_1 = -1.25$, $s_2 = -2.5$, $b_1 = 1$, and $-10 < x < 10$: (a) $0 < T < 2$, (b) $T = 0$.

Case 2

$$\mu = \pm\frac{\sqrt{P(2c_2c_3a_0 + c_2c_3{a_0}^2c_1 + 3c_1) Q c_2 c_3} a_0}{P(2c_2c_3a_0 + c_2c_3{a_0}^2c_1 + 3c_1)}, \quad v_e = \pm 2\frac{\sqrt{-(6c_2c_3a_0 + 3c_2c_3{a_0}^2c_1 + 9c_1)PQ}}{2c_2c_3a_0 + c_2c_3{a_0}^2c_1 + 3c_1}, \tag{24a}$$

$$a_0 \geq 1, \quad a_1 = \pm a_0, \quad b_1 = 0,$$

$$\Omega = -\frac{Q\left(c_2c_3{a_0}^2 + 3\right)}{2c_2c_3a_0 + c_2c_3{a_0}^2c_1 + 3c_1}. \tag{24b}$$

According to Eqs. (4), (17), and Eqs. (24), we obtain the following exact kink and anti-kink solitary wave solutions of Eq. (3)

$$A(x,T) = \left[a_0 \pm a_0 \tanh\left(\pm\frac{\sqrt{P(2c_2c_3a_0 + c_2c_3{a_0}^2c_1 + 3c_1) Q c_2 c_3} a_0}{P(2c_2c_3a_0 + c_2c_3{a_0}^2c_1 + 3c_1)} (x - v_e T) \right) \right]^{1/2} \exp\left[\mathrm{i}\Theta\right], \tag{25}$$

where $\Theta = (q_e x - \Omega T + \phi(\tau))$ with ϕ and q_e being given in Eqs. (6) and (7). The intensity of these solutions is given by

$$|A(x,T)|^2 = a_0 \pm a_0 \tanh\left(\pm\frac{\sqrt{P(2c_2c_3a_0 + c_2c_3{a_0}^2c_1 + 3c_1) Q c_2 c_3} a_0}{P(2c_2c_3a_0 + c_2c_3{a_0}^2c_1 + 3c_1)} (x - v_e T) \right), \tag{26}$$

and the graphical representation is given in Fig. 4 for some arbitrary chosen values of parameters.

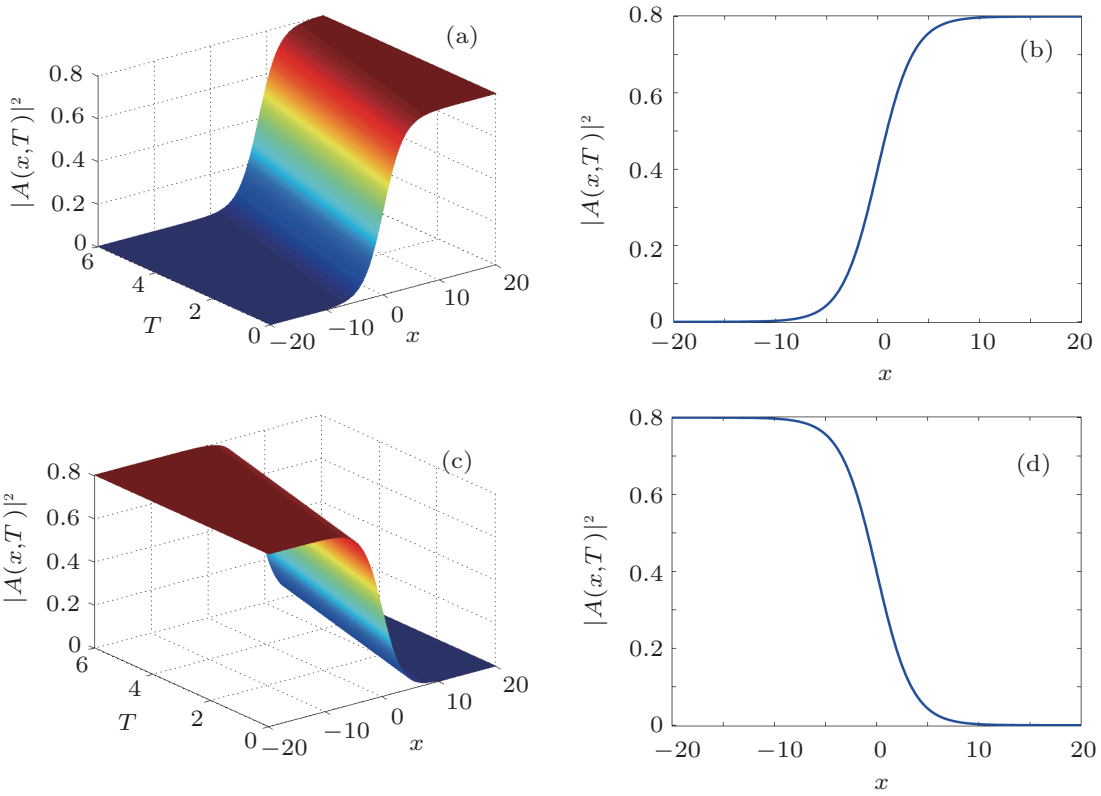


Fig. 4. (color online) The three-dimensional and two-dimensional intensities of the kink ($a_1 = a_0$) and anti-kink ($a_1 = -a_0$) solitary wave solutions of Eq. (25) with the parameters chosen as $P = Q = 1$, $s_1 = -1$, $s_2 = -2$, $a_0 = 0.4$, and $-20 < x < 20$: (a) $0 < T < 6$, (b) $T = 0$.

Case 3

$$\mu = \pm 2\frac{\sqrt{-P\left(8c_2c_3a_0+5c_2c_3a_0^2c_1+12c_1\right)Qc_2c_3}a_0}{P\left(8c_2c_3a_0+5c_2c_3a_0^2c_1+12c_1\right)},$$

$$\Omega = -\frac{Q\left(5c_2c_3a_0^2+12\right)}{8c_2c_3a_0+5c_2c_3a_0^2c_1+12c_1}, \tag{27a}$$

$$a_0 \geq 1, \quad b_1 = \pm a_0, \quad a_1 = 0,$$

$$v_e = \pm 4\frac{\sqrt{-\left(24c_2c_3a_0+15c_2c_3a_0^2c_1+36c_1\right)PQ}}{8c_2c_3a_0+5c_2c_3a_0^2c_1+12c_1}. \tag{27b}$$

According to Eqs. (4), (17), and Eqs. (27a) and (27b), we obtain the following exact bright and dark solitary wave solutions of Eq. (3):

$$A(x,T) = \left[a_0 \pm a_0 \operatorname{sech}\left(2\frac{\sqrt{-P\left(8c_2c_3a_0+5c_2c_3a_0^2c_1+12c_1\right)Qc_2c_3}a_0}{P\left(8c_2c_3a_0+5c_2c_3a_0^2c_1+12c_1\right)}(x-v_eT)\right)\right]^{1/2}\exp\left[\mathrm{i}\Theta\right], \tag{28}$$

where $\Theta = (q_e x - \Omega T + \phi(\tau))$ with ϕ and q_e being given in Eqs. (6) and (7). The intensity of these solutions is given by the following expression

$$|A(x,T)|^2 = a_0 \pm a_0 \operatorname{sech}\left(2\frac{\sqrt{-P\left(8c_2c_3a_0+5c_2c_3a_0^2c_1+12c_1\right)Qc_2c_3}a_0}{P\left(8c_2c_3a_0+5c_2c_3a_0^2c_1+12c_1\right)}(x-v_eT)\right), \tag{29}$$

and the graphical representation is given in Fig. 5 for some arbitrary chosen values of parameters.

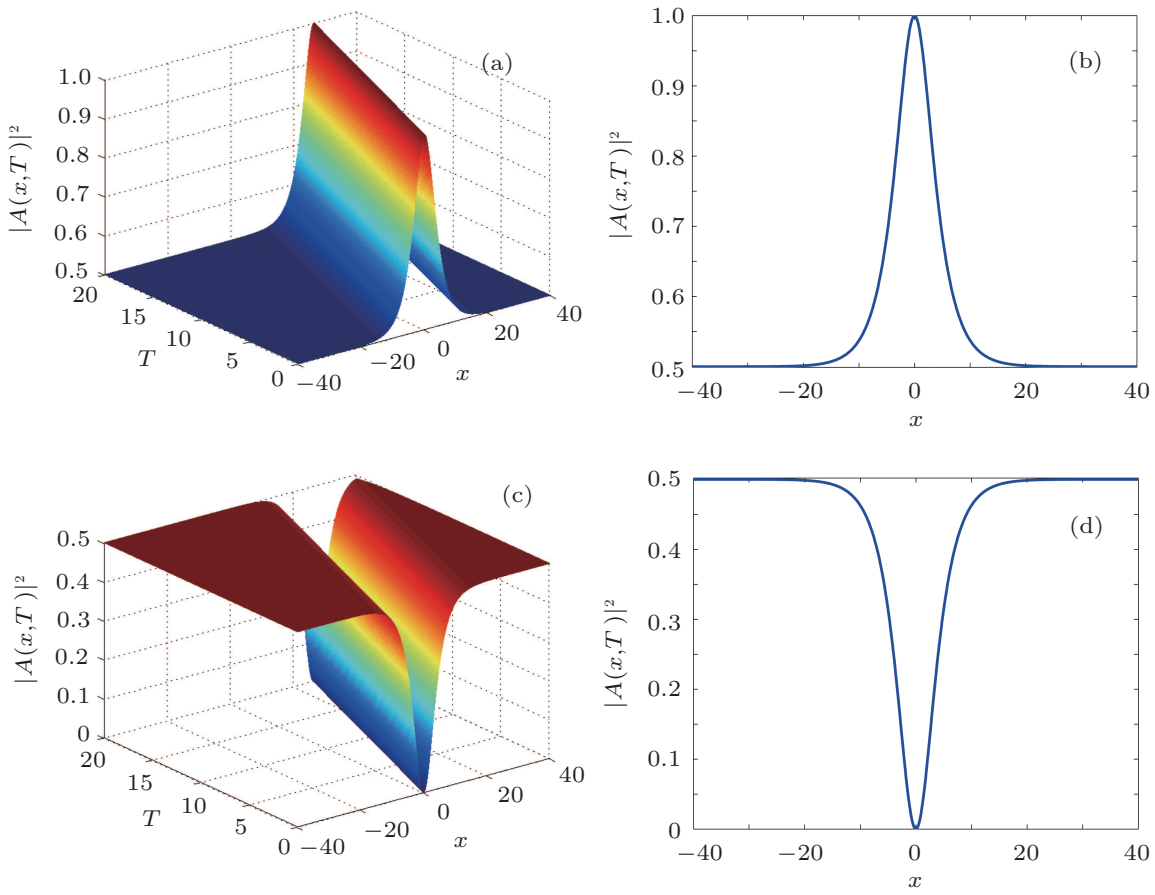


Fig. 5. (color online) The 3D and 2D intensities of the bright ($b_1 = a_0$) and dark ($b_1 = -a_0$) solitary wave solutions Eq. (28) with the parameters $P = -Q = 1$, $s_1 = -1$, $s_2 = -2$, $a_0 = 0.5$, and $-40 < x < 40$: (a) $0 < T < 20$, (b) $T = 0$.

Case 4

$$\mu = \pm 2\frac{\sqrt{P\left(2c_2c_3a_0 + c_2c_3a_0{}^2c_1 + 3c_1\right)Qc_2c_3a_0}}{P\left(2c_2c_3a_0 + c_2c_3a_0{}^2c_1 + 3c_1\right)}, \quad \Omega = -\frac{Q\left(c_2c_3a_0{}^2 + 3\right)}{2c_2c_3a_0 + c_2c_3a_0{}^2c_1 + 3c_1}, \tag{30a}$$

$$a_0 \geq 1, \quad a_1 = \pm a_0, \quad b_1 = \pm \mathrm{i}a_0,$$

$$v_{\mathrm{e}} = \pm 2\frac{\sqrt{-\left(6c_2c_3a_0 + 3c_2c_3a_0{}^2c_1 + 9c_1\right)PQ}}{2c_2c_3a_0 + c_2c_3a_0{}^2c_1 + 3c_1}. \tag{30b}$$

According to Eqs. (4), (17), and Eqs. (30a) and (30b), we obtain the following exact kink and anti-kink solitary wave solutions of Eq. (3):

$$A(x,T) = \left[a_0\left(1 \pm \tanh\left(\mu\left(x - v_{\mathrm{e}}T\right)\right) \pm \mathrm{i}\,\mathrm{sech}\left(\mu\left(x - v_{\mathrm{e}}T\right)\right)\right)\right]^{1/2} \exp\left(\mathrm{i}\Theta\right), \tag{31}$$

where $\Theta = \left(q_{\mathrm{e}}x - \Omega T + \phi(\tau)\right)$ with ϕ and q_{e} being given in Eqs. (6) and (7). The intensity of these solutions is given by the following expression

$$|\,A(x,T)\,|^2 = a_0\left[\left(1 \pm \tanh\left(\mu\left(x - v_{\mathrm{e}}T\right)\right)\right)^2 + \mathrm{sech}\left(\mu\left(x - v_{\mathrm{e}}T\right)\right)^2\right]^{1/2}, \tag{32}$$

and the graphical representation is given in Fig. 6 for some arbitrary chosen values of parameters.

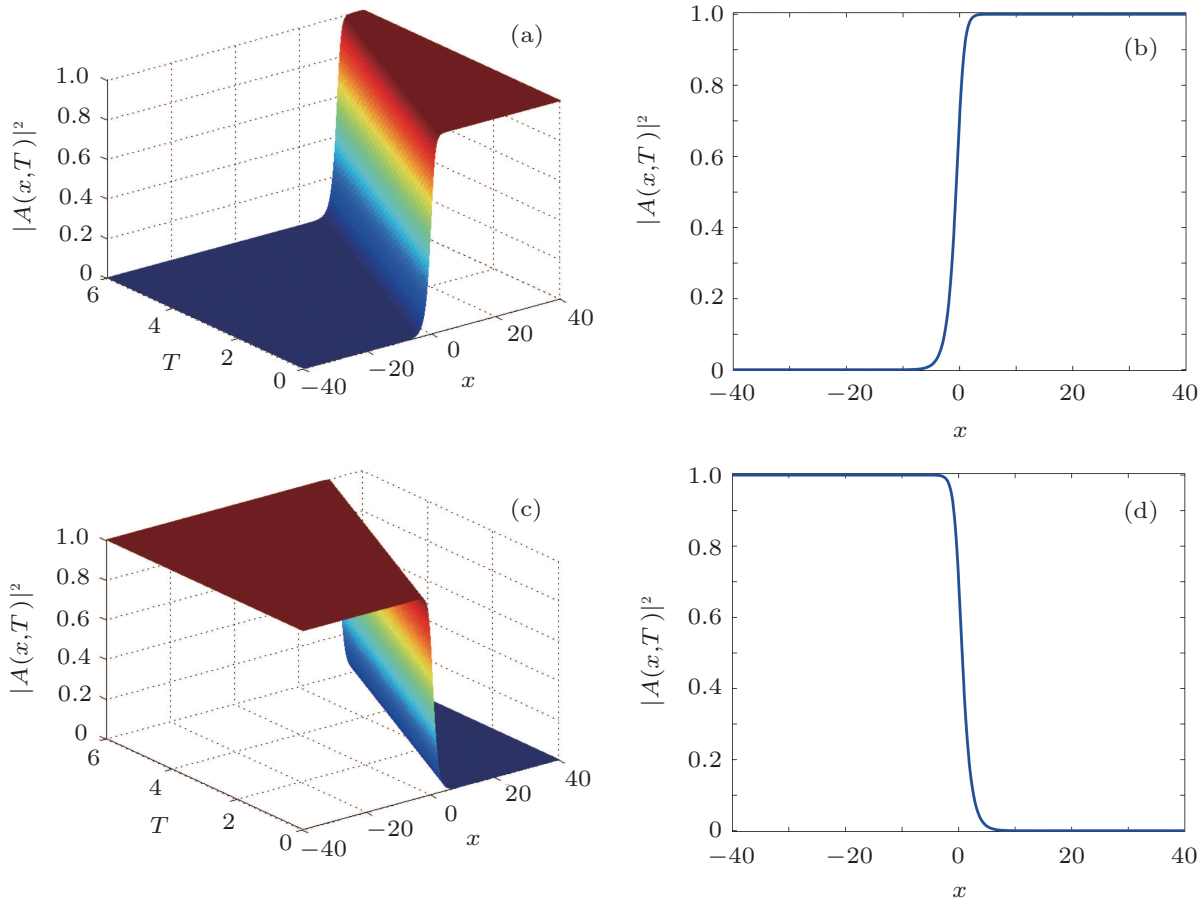


Fig. 6. (color online) The 3D and 2D intensities of the kink ($a_1 = a_0$) and anti-kink ($a_1 = -a_0$) solitary wave solutions Eq. (31) with the parameters $P = -Q = 1$, $s_1 = -0.5$, $s_2 = -0.75$, $a_0 = 0.5$, and $-40 < x < 40$: (a) $0 < T < 6$, (b) $T = 0$.

Case 5

$$\mu = \pm 2\frac{\sqrt{Pc_1\left(27{c_1}^2 - 5c_2c_3\right)Qc_2c_3}}{Pc_1\left(27{c_1}^2 - 5c_2c_3\right)}, \quad \Omega = \frac{Q\left(27{c_1}^2 + c_2c_3\right)}{c_1\left(-27{c_1}^2 + 5c_2c_3\right)}, \tag{33a}$$

$$a_0 = -\frac{1}{3c_1}, \quad a_1 = \pm a_0, \quad b_1 = \pm \mathrm{i}a_0,$$

$$v_{\mathrm{e}} = \pm 6\frac{\sqrt{-\left(81{c_1}^2 - 15c_2c_3\right)QPc_1}}{27{c_1}^2 - 5c_2c_3}. \tag{33b}$$

According to Eqs. (4), (17), and Eqs. (33a) and (33b), we obtain the following exact kink and anti-kink solitary wave solutions of Eq. (3):

$$A(x,T) = \left[-\frac{1}{3c_1}\left(1 \pm \tanh\left(\mu\left(x - v_{\mathrm{e}}T\right)\right) \pm \mathrm{i}\,\mathrm{sech}\left(\mu\left(x - v_{\mathrm{e}}T\right)\right)\right)\right]^{1/2} \exp\left(\mathrm{i}\Theta\right), \tag{34}$$

where $\Theta = \left(q_{\mathrm{e}}x - \Omega T + \phi(\tau)\right)$ with ϕ and q_{e} being given in Eqs. (6) and (7). The intensity of these solutions is given by the following equation

$$|A(x,T)|^2 = \frac{1}{3c_1}\left[\left(1 \pm \tanh\left(\mu\left(x - v_{\mathrm{e}}T\right)\right)\right)^2 + \mathrm{sech}\left(\mu\left(x - v_{\mathrm{e}}T\right)\right)^2\right]^{1/2}, \tag{35}$$

and the graphical representation is given in Fig. 7 for some arbitrary chosen values of parameters.

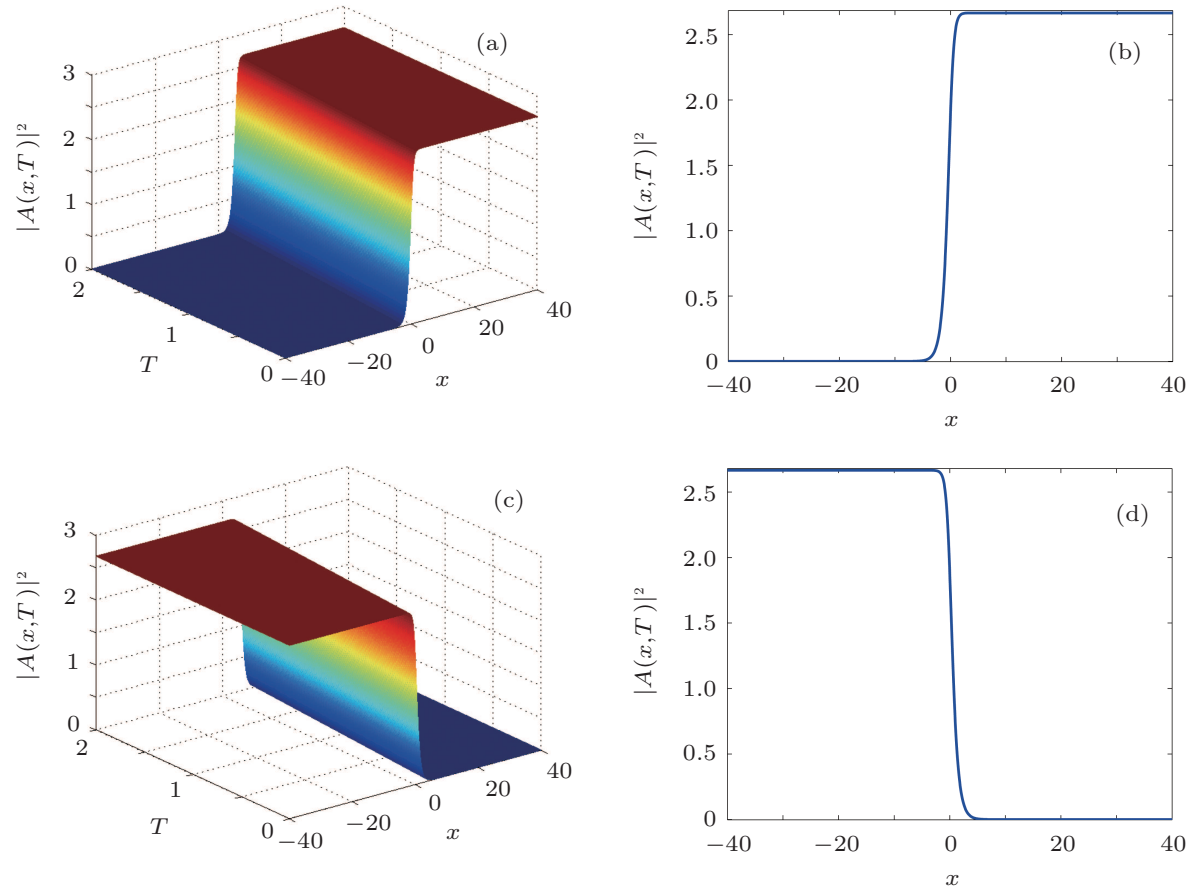


Fig. 7. (color online) The 3D and 2D intensities of the kink ($a_1 = a_0$) and anti-kink ($a_1 = -a_0$) solitary wave solutions Eq. (34) with the parameters $P = -Q = 1$, $s_1 = -0.5$, $s_2 = -0.75$, and $-40 < x < 40$: (a) $0 < T < 2$, (b) $T = 0$.

3.2. The standard Duffing equation

When either of the conditions $v_e = 0$ or $s_1 = -s_2$ or $s_1 = 36s_2/5$ is realized, equation (8) degenerates to the standard Duffing equation. This is one of the most studied equations and its solutions (periodical and non-periodic solutions) are well examined and expressed by using the elliptic functions, hyperbolic functions, and trigonometric functions.[52,71,72] In the present article, we will restrict ourselves here to only point out that the new terms can be used to induce solutions of some desired nature in parameter regions where such solutions would not be obtained in the standard NLS equation. For example, it is well known that the solution of the cubic Duffing equation corresponds to bright soliton only if the coefficient of the linear term is negative while that of the cubic term is positive. From Eq. (8), this last condition cannot be met for the classical NLS equation when $PQ < 0$. However, for either of $s_1 = -s_2$ or $s_1 = 3s_2/5$ with $s_1 \neq 0$, it is possible to first fix the value of the free parameter Ω so as to make the cubic coefficient positive and then also fix v_e so that the linear coefficient is negative, thereby realizing the conditions of existence of bright solitons.

4. Conclusion

In this paper, we have introduced a novel model of the nonlinear Schrödinger equation containing nonlinear time-derivative terms which make the model to be equivalent to a generalized NLS equation with saturable nonlinearities. This new type of NLS equation has been obtained herein as the one governing the dynamics of modulated gap waves in a basic discrete electrical transmission line.

Seeking analytical solutions of our model, we have shown that the new terms induce an additional quintic term in the Duffing oscillator when the classical travelling wave transformation is used to reduce the NLS equation to an ODE. The sine-Gordon expansion-method has been employed to obtain varied exact analytical solutions of the model. Four types of solutions, namely the kink soliton, the antikink soliton, the dark soliton, and the bright soliton have been obtained. Such solutions give rise to particle-like structures, such as magnetic monopoles, and extended structures, like, domain walls and cosmic strings, that have implications in cosmology of the early universe.[73] As such, the results reported here can be of significant importance for such problems as the transmission of information in nonlinear waveguide, in nonlinear electri-

cal transmission lines and many other domains. Likewise, the sine-Gordon expansion-method used here is powerful and can be also applied to other nonlinear equations in mathematical physics.

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