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Primary resonance of fractional-order Duffing–van der Pol oscillator by harmonic balance method*

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The dynamical properties of fractional-order Duffing–van der Pol oscillator are studied, and the amplitude–frequency response equation of primary resonance is obtained by the harmonic balance method. The stability condition for steady-state solution is obtained based on Lyapunov theory. The comparison of the approximate analytical results with the numerical results is fulfilled, and the approximations obtained are in good agreement with the numerical solutions. The bifurcations of primary resonance for system parameters are analyzed. The results show that the harmonic balance method is effective and convenient for solving this problem, and it provides a reference for the dynamical analysis of similar nonlinear systems.

Keywords: fractional-order oscillator, harmonic balance method, primary resonance

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1. Introduction

In recent years, the research of fractional calculus has exercised strong attractions.[1–10] In the field of engineering, a lot of problems can be better described by fractional-order equations, for instance, the description of the memory and hereditary properties in various materials and processes,[9,10] the model of rheological medium and viscoelastic fluid.[11–13] Many works had been fulfilled on fractional-order dynamic system, where the dynamic properties and the effects of the fractional-order terms are concerned. Those classical quantitative analysis methods are extended successfully to analyze the dynamical properties of fractional-order systems. Wahi and Chatterjee[14] demonstrated the method of averaging for conservative oscillators which may be strongly nonlinear, under small perturbations including delayed and/or fractional derivative terms. Chen *et al.*[15] derived the averaged Itô equations of amplitude modulation and phase difference via stochastic averaging method. Shen *et al.*[16] investigated the primary resonance of dry-friction oscillator with fractional-order PID controller of velocity feedback by Krylov–Bogolubov–Mitropolsky (KBM) asymptotic method. Xu *et al.*[17] investigated stochastic Duffing oscillator with fractional-order damping term by combining with Lindstedt–Poincaré (LP) method and the multiple-scale approach. Shen *et al.*[18] extended the incremental harmonic balance (IHB) method to analyze the dynamical properties of fractional-order nonlinear oscillator, which is a very effective method to deal with both strongly and weakly nonlinear systems. Xie and Lin[19] investigated the free vibration of van der Pol oscillator with small fractional damping by using the method of two-scale expansion. Rand *et al.*[20] obtained the approximation of fractional Mathieu equation by the harmonic balance method. Yang and Zhu[21] obtained the approximate solutions of the fractional-order linear system with harmonic excitation by the harmonic balance method. Guo *et al.*[22] proposed an analytical technique based on the method of harmonic balance for predicting and generating the steady-state solution of the fractional differential system.

The Duffing–van der Pol equation is essentially a combination of Duffing and van der Pol equations, and it can be used as a mathematical model in many fields,[23] such as aircraft wings at high angles of attack, thin panels in supersonic flows, and the structure vibration caused by the flow, *etc*. Lots of research work has been carried out on the Duffing–van der Pol system. Awrejcewicz and Mrozowski[24] studied chaotic dynamics of a particular non-linear oscillator having Duffing type stiffness, van der Pol damping and dry friction. Kimiaeifar *et al.*[25] solved the Duffing–van der Pol equation analytically using homotopy analysis method. Kyzioland Okniński[26] obtained the periodic steady-state solutions of the Duffing–van der Pol equation by the KBM approach. Recently, Duffing–van der Pol oscillator with fractional-order derivative had attracted more and more attentions. For example, Leung *et al.*[27] investigated a Duffing–van der Pol oscillator having fractional derivatives and time delays by the residue harmonic method. Matouk[28] studied the stability analysis of the fractional-order modified autonomous van der Pol–Duffing circuit by using the fractional Routh–Hurwitz criteria.

The harmonic balance (HB) method is a widely used

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method to deal with both integer order systems and fractional-order systems with strong and weak nonlinearity.[20,21,29–31] Accordingly, the Duffing–van der Pol oscillator with fractional-order damping term is studied by HB method in this paper. The paper is organized as follows. Section 2 presents the amplitude–frequency response equation of primary resonance by using HB method. In Section 3, the stability condition for steady-state solution is presented. In Section 4, the numerical solutions of system are obtained by the power series method. In Section 5, the bifurcations of primary resonance are investigated. The main conclusions are drawn at last.

2. Approximate periodic solution

The fractional-order Duffing–van der Pol oscillator considered is as follows:

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+\left[c_2x^2(t)-c_1\right]D_t^p\left[x(t)\right]+kx(t)+\alpha_1x^3(t)=F_1\cos(\omega t), \tag{1}$$

where m is the mass, c_1 and c_2 the fractional-order damping coefficients, k is the linear stiffness coefficient, α_1 is the nonlinear stiffness coefficient, F_1 is the amplitude of the external harmonic excitation, ω is the angular frequency of the external harmonic excitation. And p is the order of fractional-order derivative, which is restricted as $0\le p\le 1$. The Caputo's definition of fractional-order derivative is adopted here. The fractional-order term in Eq. (1) is defined as

$$D_t^p[x(t)]=\begin{cases}\dfrac{1}{\Gamma(1-p)}\displaystyle\int_0^t\frac{x'(\tau)}{(t-\tau)^p}\mathrm{d}\tau, & 0\le p<1,\\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}, & p=1,\end{cases} \tag{2}$$

where $\Gamma(z)$ is Gamma function satisfying $\Gamma(z+1)=z\Gamma(z)$.

To study the approximate periodic solutions by HB method, and equation (1) can be transformed into

$$x''+(\mu_2x^2-\mu_1)D_t^p[x]+\omega_{\mathrm{n}}^2x+\alpha x^3=F\cos(\omega t). \tag{3}$$

In Eq. (3) x'' denotes the second derivative with respect to t, $\mu_1=c_1/m$, $\mu_2=c_2/m$, $\omega_{\mathrm{n}}^2=k/m$, $\alpha=\alpha_1/m$, $F=F_1/m$.

The first-order approximate periodic solution contains only one harmonic term, which could be supposed as

$$x=A\cos(\omega t+\theta), \tag{4}$$

where A is the amplitude and θ is the phase to be determined.

By introducing $\varphi=\omega t+\theta$, one could obtain a transformation of Eq. (3)

$$\omega^2\ddot{x}+(\mu_2x^2-\mu_1)\omega^pD_\varphi^p[x]+\omega_{\mathrm{n}}^2x+\alpha x^3=F\cos(\varphi-\theta), \tag{5}$$

where $\ddot{x}$ denotes the second derivative with respect to φ.

Substituting Eq. (4) into the left-hand side of Eq. (5), for the integer-order parts one could obtain

$$\begin{aligned}&\omega^2\ddot{x}+\omega_{\mathrm{n}}^2x+\alpha x^3\\ =\;&-A\omega^2\cos(\varphi)+A\omega_{\mathrm{n}}^2\cos(\varphi)\\ &+\frac{3}{4}\alpha A^3\cos(\varphi)+\frac{1}{4}\alpha A^3\cos(3\varphi).\end{aligned} \tag{6}$$

For the fractional-order part in Eq. (5), if $p=1$ one could have

$$\begin{aligned}&(\mu_2x^2-\mu_1)\omega^pD_\varphi^p[x]\\ =\;&\left(\mu_1\omega A-\frac{1}{4}\mu_2\omega A^3\right)\sin(\varphi)-\frac{1}{4}\mu_2\omega A^3\sin(3\varphi).\end{aligned} \tag{7}$$

When $0\le p<1$, the fractional-order term in Eq. (5) can be taken as a periodic function with the period tends to infinity. Expanding into Fourier series and taking N-th harmonic terms, one could have

$$D_\varphi^p[x]=E_0+\sum_{r=1}^{N}\left[E_{1r}\cos(r\varphi)+E_{2r}\sin(r\varphi)\right], \tag{8a}$$

$$E_0=\lim_{T\to\infty}\frac{1}{T}\int_0^T\left[\frac{1}{\Gamma(1-p)}\int_0^\varphi\frac{-A\sin(\tau)}{(\varphi-\tau)^p}\mathrm{d}\tau\right]\mathrm{d}\varphi, \tag{8b}$$

$$E_{1r}=\lim_{T\to\infty}\frac{2}{T}\int_0^T\left[\frac{1}{\Gamma(1-p)}\int_0^\varphi\frac{-A\sin(\tau)}{(\varphi-\tau)^p}\mathrm{d}\tau\right]\times\cos(r\varphi)\mathrm{d}\varphi, \tag{8c}$$

$$E_{2r}=\lim_{T\to\infty}\frac{2}{T}\int_0^T\left[\frac{1}{\Gamma(1-p)}\int_0^\varphi\frac{-A\sin(\tau)}{(\varphi-\tau)^p}\mathrm{d}\tau\right]\times\sin(r\varphi)\mathrm{d}\varphi. \tag{8d}$$

In order to calculate the Fourier coefficients, two important results about the definite integrals in Ref. [18] are introduced as

$$\lim_{T\to\infty}\int_0^T\frac{\sin(\varphi)}{\varphi^p}\mathrm{d}\varphi=\Gamma(1-p)\cos\left(\frac{p\pi}{2}\right), \tag{9a}$$

$$\lim_{T\to\infty}\int_0^T\frac{\cos(\varphi)}{\varphi^p}\mathrm{d}\varphi=\Gamma(1-p)\sin\left(\frac{p\pi}{2}\right). \tag{9b}$$

Firstly, E_{1r} is calculated. Substituting $s=\varphi-\tau$, $\mathrm{d}s=-\mathrm{d}\tau$ into Eq. (8c), one could obtain

$$\begin{aligned}E_{1r}&=\lim_{T\to\infty}\frac{2}{T}\int_0^T\left[\frac{1}{\Gamma(1-p)}\int_0^\varphi\frac{-A\sin(\varphi-s)}{s^p}\mathrm{d}s\right]\cos(r\varphi)\mathrm{d}t\\ &=E_{1r1}+E_{1r2},\end{aligned} \tag{10a}$$

$$E_{1r1}=\frac{A}{\Gamma(1-p)}\lim_{T\to\infty}\frac{2}{T}\int_0^T\left[\int_0^\varphi\frac{\sin(s)}{s^p}\mathrm{d}s\right]\cos(\varphi)\times\cos(r\varphi)\mathrm{d}\varphi, \tag{10b}$$

$$E_{1r2}=-\frac{A}{\Gamma(1-p)}\lim_{T\to\infty}\frac{2}{T}\int_0^T\left[\int_0^\varphi\frac{\cos(s)}{s^p}\mathrm{d}s\right]\times\sin(\varphi)\cos(r\varphi)\mathrm{d}\varphi. \tag{10c}$$

Using integration by parts and Eq. (8c), one could obtain

$$\begin{aligned}E_{1r1}=\;&\frac{A}{2\Gamma(1-p)}\lim_{T\to\infty}\left\{\frac{2\varphi+\sin(2\varphi)}{T}\left[\int_0^\varphi\frac{\sin(s)}{s^p}\mathrm{d}s\right]\right\}\Bigg|_0^T\\ &-\frac{A}{2\Gamma(1-p)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\left[\frac{2\varphi+\sin(2\varphi)}{\varphi^p}\sin(\varphi)\right]\mathrm{d}\varphi\end{aligned}$$

$$= A\cos\left(\frac{p\pi}{2}\right), \quad r=1, \tag{11a}$$

$$E_{1r1} = 0, \quad r>1. \tag{11b}$$

$$E_{1r2} = 0. \tag{11c}$$

Substituting Eq. (11) into Eq. (10), one could have

$$E_{1r} = \begin{cases} A\cos\left(\frac{p\pi}{2}\right), & r=1, \\ 0, & r>1. \end{cases} \tag{12}$$

Similarly, one could obtain

$$E_0 = 0, \tag{13a}$$

$$E_{2r} = \begin{cases} -A\sin\left(\frac{p\pi}{2}\right), & r=1, \\ 0, & r>1. \end{cases} \tag{13b}$$

Furthermore, one could obtain

$$\begin{aligned} &(\mu_2 x^2-\mu_1)\omega^p D_\varphi^p[x] \\ &= \left(\frac{3}{4}\mu_2 A^3-\mu_1 A\right)\omega^p\cos\left(\frac{p\pi}{2}\right)\cos(\varphi) \\ &\quad+\frac{1}{4}\mu_2 A^3\omega^p\cos\left(\frac{p\pi}{2}\right)\cos(3\varphi) \\ &\quad+\left(\mu_1 A-\frac{1}{4}\mu_2 A^3\right)\omega^p\sin\left(\frac{p\pi}{2}\right)\sin(\varphi) \\ &\quad-\frac{1}{4}\mu_2 A^3\omega^p\sin\left(\frac{p\pi}{2}\right)\sin(3\varphi). \end{aligned} \tag{14}$$

From Eq. (7) and Eq. (14), it can be concluded that the fractional-order part in Eq. (5) can be expressed as the unified form by Eq. (14) for the fractional order $0 \le p \le 1$.

Substituting Eq. (6) and Eq. (14) into Eq. (5) and remaining first-order harmonic terms, one could have

$$\begin{aligned} &A\omega_{\rm n}^2\cos(\varphi)-A\omega^2\cos(\varphi)+\frac{3}{4}\alpha A^3\cos(\varphi) \\ &+\left(\frac{3}{4}\mu_2 A^3-\mu_1 A\right)\omega^p\cos\left(\frac{p\pi}{2}\right)\cos(\varphi) \\ &+\left(\mu_1 A-\frac{1}{4}\mu_2 A^3\right)\omega^p\sin\left(\frac{p\pi}{2}\right)\sin(\varphi) \\ &= F\cos(\theta)\cos(\varphi)+F\sin(\theta)\sin(\varphi). \end{aligned} \tag{15}$$

According to harmonic balance principle, one could obtain

$$\begin{aligned} &A\omega_{\rm n}^2-A\omega^2+\frac{3}{4}\alpha A^3 \\ &+\left(\frac{3}{4}\mu_2 A^3-\mu_1 A\right)\omega^p\cos\left(\frac{p\pi}{2}\right)=F\cos(\theta), \end{aligned} \tag{16a}$$

$$\left(\mu_1 A-\frac{1}{4}\mu_2 A^3\right)\omega^p\sin\left(\frac{p\pi}{2}\right)=F\sin(\theta). \tag{16b}$$

Eliminating the parameter θ and substituting the original parameters of system, the amplitude–frequency response equation can be obtained as

$$A^2\left[K-m\omega^2\right]^2+A^2\left[C(p)\omega\right]^2=F_1^2, \tag{17}$$

where

$$\begin{aligned} K &= k+\frac{3}{4}\alpha_1 A^2+K(p), \\ K(p) &= \left(\frac{3}{4}c_2 A^2-c_1\right)\omega^p\cos\left(\frac{p\pi}{2}\right), \\ C(p) &= \left(\frac{1}{4}c_2 A^2-c_1\right)\omega^{p-1}\sin\left(\frac{p\pi}{2}\right). \end{aligned}$$

And the phase-frequency response equation can be expressed as

$$\theta = \arctan\left[\frac{-C(p)\omega}{K-m\omega^2}\right]. \tag{18}$$

From Eq. (17), the peak value of the primary resonance can be obtained as

$$A_{\max} = \frac{F_1}{|C(p)\omega|}, \tag{19}$$

when $K = m\omega^2$.

In Eq. (17) and Eq. (18), the parameter K denotes the equivalent stiffness coefficient of the system. The parameter $K(p)$ is defined as the equivalent stiffness coefficient, and $C(p)$ is defined as the equivalent damping coefficient, which are both induced by the fractional-order term. Letting $\omega_r = \sqrt{K/m}$, and ω_r can be defined as the equivalent natural frequency of system. When $p=1$, one could get $K(p)=0$ and $C(p) = (c_2A^2/4)-c_1$, it is obviously consistent with the integer–order case. It can be found that the viscoelastic damping has equivalent stiffness and damping effect at the same time.

3. Stability condition for steady-state solution

The stability of the steady-state solution is studied in this section. The steady-state solution can be expressed as

$$x_{\rm p} = \bar{A}\cos(\omega t+\bar{\theta}). \tag{20}$$

And $x_{\rm p}$ satisfies Eq. (16). The small perturbations near the periodic motion $x_{\rm p}$ can be assumed as ΔA and $\Delta\theta$. Let $A = \bar{A}+\Delta A$ and $\theta = \bar{\theta}+\Delta\theta$. Substituting them into Eq. (16) and ignoring higher-order terms, it yields

$$\left(\mu_1-\frac{3}{4}\mu_2\bar{A}^2\right)\omega^p\sin\left(\frac{p\pi}{2}\right)\Delta A-F\cos(\bar{\theta})\Delta\theta=0, \tag{21a}$$

$$\begin{aligned} &\left[\omega_{\rm n}^2-\omega^2+\frac{9}{4}\alpha\bar{A}^2+\left(\frac{9}{4}\mu_2\bar{A}^2-\mu_1\right)\omega^p\cos\left(\frac{p\pi}{2}\right)\right]\Delta A \\ &+F\sin(\bar{\theta})\Delta\theta=0. \end{aligned} \tag{21b}$$

Substituting the original parameters of system, one could have

$$\left(c_1-\frac{3}{4}c_2\bar{A}^2\right)\omega^p\sin\left(\frac{p\pi}{2}\right)\Delta A-F_1\cos(\bar{\theta})\Delta\theta=0, \tag{22a}$$

$$\left[k - m\omega^2 + \frac{9}{4}\alpha_1\bar{A}^2 + \left(\frac{9}{4}c_2\bar{A}^2 - c_1\right)\omega^p \cos\left(\frac{p\pi}{2}\right)\right]\Delta A + F_1 \sin(\bar{\theta})\Delta\theta = 0. \quad (22b)$$

Eliminating $\bar{\theta}$ from Eq. (21) based on Eq. (16), and the characteristic determinant can be obtained as

$$\det\begin{bmatrix} A_1 - \lambda & -\rho_1 \\ \rho_2 & A_2 - \lambda \end{bmatrix} = 0, \quad (23)$$

where $A_1 = -C(p)\omega$, $A_2 = \bar{A}A_1$, $\rho_1 = \bar{A}K$,

$$\rho_2 = \left[k - m\omega^2 + \frac{9}{4}\alpha_1\bar{A}^2 + \left(\frac{9}{4}c_2\bar{A}^2 - c_1\right)\omega^p \cos\left(\frac{p\pi}{2}\right)\right].$$

Expanding the determinant, the characteristic equation can be obtained as

$$\lambda^2 - (A_1 + A_2)\lambda + A_1A_2 + \rho_1\rho_2 = 0. \quad (24)$$

From Eq. (23), the stability condition for the steady-state solution can be expressed as

$$\begin{cases} A_1 + A_2 < 0, \\ A_1A_2 + \rho_1\rho_2 > 0. \end{cases} \quad (25)$$

It can be concluded that the solution in the middle is unstable, when the amplitude–frequency response curve has multiple solutions.

4. Comparison with the numerical results

In order to verify the precision of the approximate analytical solution, the numerical results are also provided. The relationship about the explicit numerical approximation of the power series method in Ref. [2] is

$$D^p_{t_n}[y(t_n)] \approx h^{-p}\sum_{j=0}^{n} C^p_j y(t_{n-j}), \quad (26)$$

where $t_n = nh$ is the sample points, h is the sample step, and C^p_j is the fractional binomial coefficient with the iterative relationship as

$$C^p_0 = 1, \quad C^p_j = \left(1 - \frac{1+p}{j}\right)C^p_{j-1}. \quad (27)$$

According to Eq. (26) and Eq. (27), the numerical scheme for Eq. (1) can be expressed as

$$x(t_n) = y(t_{n-1})h - \sum_{j=1}^{n} C^1_j x(t_{n-j}), \quad (28a)$$

$$y(t_n) = \left\{F_1\cos(\omega t_n) - kx(t_n) - \alpha_1 x^3(t_n) - [c_2x^2(t_n) - c_1]z(t_{n-1})\right\}\frac{h}{m} - \sum_{j=1}^{n} C^1_j y(t_{n-j}), \quad (28b)$$

$$z(t_n) = [y(t_n)]h^{1-p} - \sum_{j=1}^{n} C^{1-p}_j z(t_{n-j}). \quad (28c)$$

The sample step is set as $h = 0.004$, and the total computation time is 200 s. After omitting the temporary response in frontal 120 s, the peak value of the posterior response is taken as the steady-state amplitude by the numerical results.

In order to investigate the dynamic response characteristics of the system, the demonstration system parameters are defined as system parameters are defined as $m = 10$, $k = 40$, $c_1 = 0.2$, $c_2 = 0.5$, $\alpha_1 = 0.5$, $p = 0.5$, $F_1 = 2$. Based on Eq. (17), one could analytically obtain the amplitude–frequency response curve, which is shown in Fig. 1 by the solid points. According to Eq. (28), one could obtain the amplitude–frequency response by numerical integration, which are also shown in Fig. 1 and denoted by the circles. From the observation of Fig. 1, it could be found that the approximate analytical solution agrees well with the numerical results.

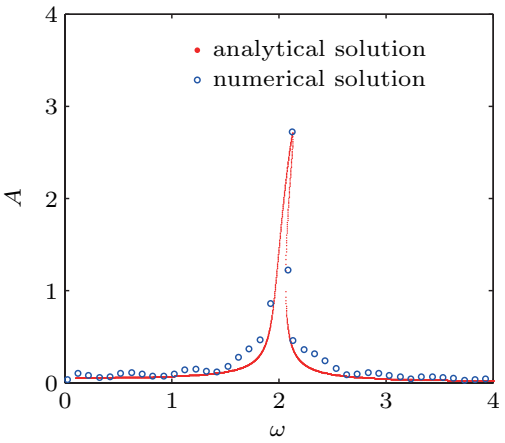


Fig. 1. (color online) Comparison of amplitude–frequency response.

5. Bifurcations of primary resonance

In this section, the bifurcations of primary resonance for the given parameters and $\omega = 2.1$ are presented. When the parameters of the system are changed respectively, the behaviors of resonant amplitudes can be obtained by analytical solution based on Eq. (17), and the results are shown in Figs. 2–6 by the solid points. From these figures, it can be seen that there exist multi-solutions and jump phenomenon all of them, and the intermediate solution is an unstable solution. For the parameters F_1, α_1, p, and c_2, the resonant amplitude maybe has only single solution by choosing appropriate parameter values.

From Eq. (19), it can be concluded that the peak value of the primary resonance is related with the amplitude of the external harmonic excitation F_1, the equivalent natural frequency ω_r, the order of fractional-order derivative p. The peak value of the primary resonance trends to increasing along with the F_1 increased and $\left|(c_2A^2/4 - c_1)\omega_r^p \sin(p\pi/2)\right|$ decreased. When

the order of fractional-order derivative is set as $p = 1$, equation (1) becomes a traditional integer–order system. In order to make the comparison more obvious, the amplitude of the external harmonic excitation is set as $F_1 = 10$. The amplitude–frequency response curves for fractional-order and integer–order systems can be obtained according to Eq. (17), which are shown in Fig. 7. It can be seen that, the peak resonant amplitude of the integer–order system is smaller than that of the fractional-order system.

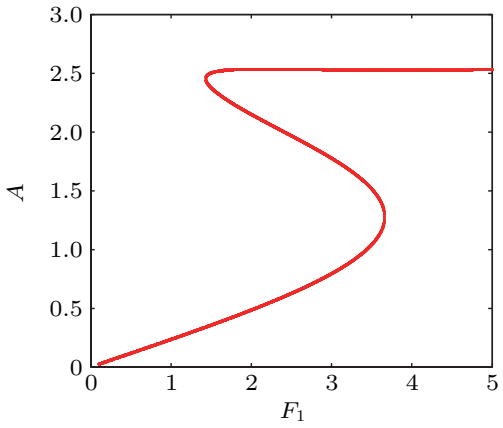


Fig. 2. (color online) Resonant amplitude with F_1 for $m = 10$, $k = 40$, $c_1 = 0.2$, $c_2 = 0.5$, $\alpha_1 = 0.5$, $p = 0.5$, $\omega = 2.1$.

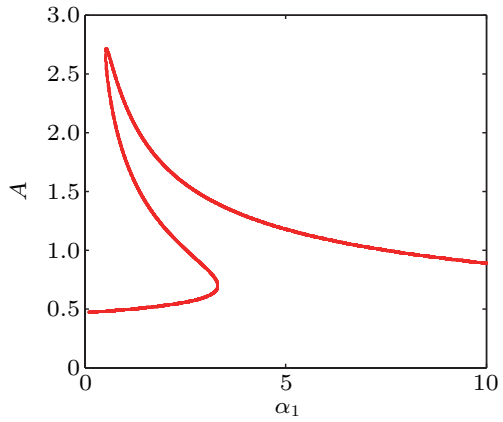


Fig. 3. (color online) Resonant amplitude with α_1 for $m = 10$, $k = 40$, $c_1 = 0.2$, $c_2 = 0.5$, $F_1 = 2$, $p = 0.5$, $\omega = 2.1$.

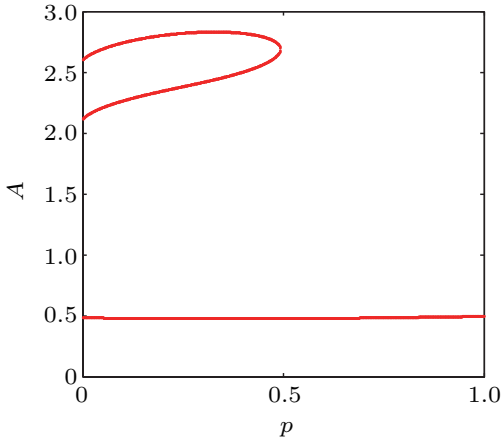


Fig. 4. (color online) Resonant amplitude with p for $m = 10$, $k = 40$, $c_1 = 0.2$, $c_2 = 0.5$, $\alpha_1 = 0.5$, $F_1 = 2$, $\omega = 2.1$.

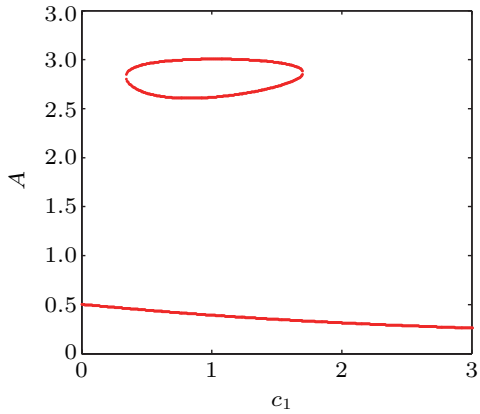


Fig. 5. (color online) Resonant amplitude with c_1 for $m = 10$, $k = 40$, $c_2 = 0.5$, $\alpha_1 = 0.5$, $F_1 = 2$, $p = 0.5$,$\omega = 2.1$.

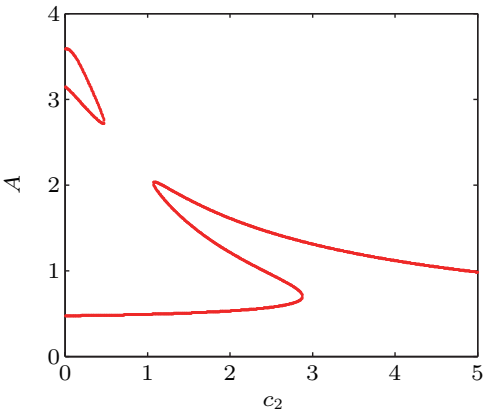


Fig. 6. (color online) Resonant amplitude with c_2 for $m = 10$, $k = 40$, $c_1 = 0.2$,$\alpha_1 = 0.5$, $F_1 = 2$, $p = 0.5$,$\omega = 2.1$.

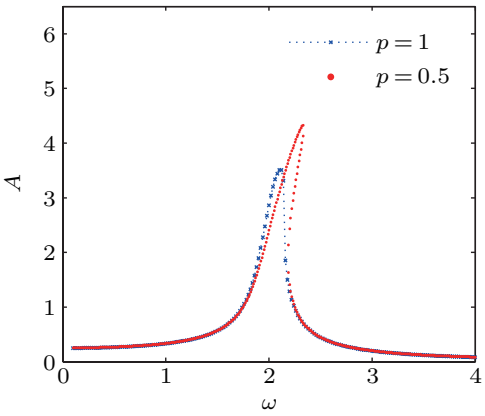


Fig. 7. (color online) Amplitude–frequency response curves for $m = 10$, $k = 40$, $c_1 = 0.2$, $c_2 = 0.5$, $\alpha_1 = 0.5$, $F_1 = 10$.

6. Conclusion and perspectives

The primary resonance of fractional-order of Duffing–van der Pol oscillator is studied in this paper. The approximate analytical solution is obtained by HB method, and the stability condition for steady-state solution is presented based on Lyapunov theory. By comparing with the numerical solutions, the accuracy of the analytical results is demonstrated. Based on amplitude–frequency response equation, the bifurcations of

primary resonance are analyzed, and the effect on amplitude–frequency characteristic for the order of fractional-order term is pointed out. It may be helpful to analyze the forced vibration of other similar fractional-order oscillators.

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