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Ting-Ting Liu(刘婷婷), Ya-Yun Hu(胡亚运), Jing Zhao(赵静), Ming Zhong(钟鸣), Pei-Qing Tong(童培庆)

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The entanglement of deterministic aperiodic quantum walks*

Ting-Ting Liu(刘婷婷)[1], Ya-Yun Hu(胡亚运)[2], Jing Zhao(赵静)[3], Ming Zhong(钟鸣)[1,†], and Pei-Qing Tong(童培庆)[1,4,‡]

[1] *Department of Physics and Institute of Theoretical Physics, Nanjing Normal University, Nanjing* 210023, *China*
[2] *Physics Department,* 104 *Davey Laboratory, Pennsylvania State University, University Park, PA* 16802, *USA*
[3] *College of Mathematical and Physical Sciences, Anshun University, Anshun* 561000, *China*
[4] *Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing Normal University, Nanjing* 210023, *China*

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We study the entanglement between the internal (coin) and the external (position) degrees of freedom in the dynamic and the static deterministic aperiodic quantum walks (QWs). For the dynamic (static) aperiodic QWs, the coin depends on the time (position) and takes two coins $C(\alpha)$ and $C(\beta)$ arranged in the two classes of generalized Fibonacci (GF) and the Thue–Morse (TM) sequences. We found that for the dynamic QWs, the entanglement of three kinds of the aperiodic QWs are close to the maximal value, which are all much larger than that of the homogeneous QWs. Further, the first class of GF (1st GF) QWs can achieve the maximum entangled state, which is similar to that of the dynamic disordered QWs. And the entanglement of 1st GF QWs is greater than that of the TM QWs, being followed closely by the entanglement of the second class of GF (2nd GF) QWs. For the static QWs, the entanglement of three kinds of the aperiodic QWs are also close to the maximal value and 1st GF QWs can achieve the maximum entangled state. The entanglement of the TM QWs is between 1st GF QWs and 2nd GF QWs. However, the entanglement of the static disordered QWs is less than that of three kinds of the aperiodic QWs. This is different from those of the dynamic QWs. From these results, we can conclude that the dynamic and static 1st GF QWs can also be considered as maximal entanglement generators.

Keywords: quantum walks, quantum entanglement, aperiodic, entanglement production

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1. Introduction

Recently the quantum walks (QWs)[1–4] have attracted extensive attention. This is because of the diffusion of QWs is faster than that of the classical random walks[1–9] and QWs are widely used in quantum information.[6–9] The another reason is that QWs can be used to simulate and explain dynamical behaviors in variety of the physical systems, such as Landau–Zener tunneling,[10] the Klein paradox,[11] and Bloch oscillations.[12] In addition, many experimental schemes have been suggested and reported to implement QWs, such as optical lattices,[13] trapped ions,[14–16] waveguide lattices,[17,18] light pulses in optical fibers,[19,20] and single photons.[21,22]

On the other hand, the entanglement in quantum state has far reaching application in information fields such as quantum cryptography, quantum computation, quantum teleportation, and quantum algorithms.[23–26] Only quantum system has this fantastic property.[27,28] In recent years, the entanglement of QWs has drawn much attention. Carneiro *et al.*[29] studied the entanglement of the homogeneous QWs between the coin and position degrees of freedom. They used the von Neumann entropy to measure the entanglement and found that the entanglement with Hadamard coin tends to be 0.872. Then, Vieira *et al.*[30] studied the entanglement of the dynamic disordered QWs and pointed out that the entanglement of the dynamic disordered QWs approaches one, the maximal value of entanglement, and the entanglement is independent of the initial condition of the walker. Therefore, the dynamic disorder QW can be considered as a maximal entanglement generator. They also studied the static disordered QWs and showed that the entanglement oscillate around a value smaller than 1. This is different from that of the dynamic disordered QWs.[31,32]

Since the discovery of quasicrystals by Shechtman *et al.*[33] and the studies of many experimental works of quasiperiodic superlattices,[34–36] a great deal of attention has been focused on studies of aperiodic systems. Despite the absence of translational invariance in these structures, they are perfectly ordered. Such systems can be regarded as intermediate between homogeneous and disordered systems. The aperiodic systems exhibit some unusual physical properties. For example, the electronic energy spectra of the Fibonacci lattice is singular continuous and the wave functions are critical, *i.e.*, neither localized nor extended.[37,38] The widely studied aperiodic systems are the two-element Fibonacci,[35–45] generalized Fibonacci (GF),[39,46–50] and Thue–Morse (TM)

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†Corresponding author. E-mail: mzhong@njnu.edu.cn
‡Corresponding author. E-mail: pqtong@njnu.edu.cn
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systems.[39,45,51–54] They can be classified by different mathematical and/or physical properties. Burrows and Sulston defined the degrees of sequence disorder for two-element aperiodic sequences by the Shannon entropy.[55] They found that the second class of GF (2nd GF) sequences are more disordered than that of TM sequences, being followed by the first class of GF (1st GF) sequences. The degrees of sequence disorder for three kinds of the aperiodic sequences are between those of the disordered and homogeneous sequences. On the other hand, based on the wandering number, Luck showed that the quantum phase transition (QPT) of the Fibonacci, 1st GF and TM quantum Ising chain (QIC) belong to the same universality class of homogeneous QIC.[39] The QPT of the 2nd GF QIC is similar to that of the disordered QIC. Recently, the dynamic Fibonacci[40–46] and nonuniform QWs[56–60] have been widely investigated. It is found that the dynamic Fibonacci QWs leads to a sub-ballistic dynamic behavior characterized, *i.e.*, $\sigma(t) \propto t^{\alpha}(0.5 < \alpha < 1)$.[40] This behavior is between that of the homogeneous QWs ($\alpha = 1$) and the dynamic disordered QWs ($\alpha = 1/2$). It is also found that the dynamic behavior of the dynamic GF QWs is the sub-ballistic behavior and the α of the 2nd GF QWs is greater than that of the 1st GF QWs.[46] Therefore, there are two very interesting questions. One question is what is the entanglement of different dynamic (static) aperiodic QWs? Another is what is the relationship between the entanglement of different dynamic (static) aperiodic QWs and their relationship to entanglement of the disordered and homogeneous QWs? To address these questions, we study the entanglement of the dynamic and static aperiodic QWs.

The paper is organized as follows. The models of the dynamic and the static deterministic aperiodic QWs and the entanglement are introduced in Section 2 and Section 3, respectively. In Section 4, the numerical results of the entanglement of the aperiodic QWs are given and discussed. Section 5 is a short summary.

2. Models

Consider a one-dimensional discrete QWs. The system can be described by a Hilbert space $H_{\rm p} \otimes H_{\rm c}$, where $H_{\rm p}$ denotes the position space Z spanned by $\{|\, i\rangle, i \in Z\}$, and $H_{\rm c}$ denotes the inner coin space spanned by $\{|\uparrow\rangle, |\downarrow\rangle\}$. A general state for a quantum walker can be written as[31]

$$|\,\Psi(t)\rangle = \sum_{-\infty}^{+\infty}[R(i,t)\,|\uparrow\rangle \otimes |\,i\rangle + L(i,t)\,|\downarrow\rangle \otimes |\,i\rangle], \tag{1}$$

with normalization condition demanding that $\sum_{-\infty}^{+\infty}[|\,R(i,t)\,|^2 + |\,L(i,t)\,|^2] = 1$. At t step the state of QW is

$$|\,\Psi(t)\rangle = U(t)\,|\,\Psi(t-1)\rangle, \tag{2}$$

where,

$$U(t) = \mathcal{S}\mathcal{C}(t), \tag{3}$$

and

$$\mathcal{S} = \sum_i |\uparrow\rangle\langle\uparrow| \otimes |\,i+1\rangle\langle i\,| + \sum_i |\downarrow\rangle\langle\downarrow| \otimes |\,i-1\rangle\langle i\,|, \tag{4}$$

$$\mathcal{C}(t) = \sum_{i=-\infty}^{\infty} C(i,t) \otimes |\,i\rangle\langle i\,|\,. \tag{5}$$

Here, $\mathcal{S}$ is the conditional displacement operator, moves the walker at site j to the site $j+1$ if its coin state is $|\uparrow\rangle$ and to the site $j-1$ if its coin state is $|\downarrow\rangle$. $C(i,t)$ is the quantum coin acting on the internal degree of freedom at site i and time t. In our studies, we select the coin operator $C(i,t)$ as

$$C(i,t) = \begin{pmatrix} \cos\theta(i,t) & \sin\theta(i,t) \\ \sin\theta(i,t) & -\cos\theta(i,t) \end{pmatrix}, \quad \theta \in [0, \pi/2]. \tag{6}$$

When $\theta(i,t)$ is independent of the time and position, *i.e.*, $\theta(i,t) = \theta$, the model becomes to the ordinary homogeneous QWs. In this paper, we consider two kinds of the aperiodic QWs: the dynamic and the static aperiodic QWs. For the dynamic aperiodic QWs, $\theta(i,t) = \theta(t)$, *i.e.*, the quantum coin is only dependent on the time. The time evolution of the system is obtained by applying $U(t)$ to an arbitrary state at time $t-1$, then the recurrence relations can be written as[31]

$$\begin{aligned} R(i,t) &= \cos\theta(t)R(i-1,t-1) + \sin\theta(t)L(i-1,t-1), \\ L(i,t) &= \sin\theta(t)R(i+1,t-1) - \cos\theta(t)L(i+1,t-1). \end{aligned} \tag{7}$$

For the static aperiodic QWs, $\theta(i,t) = \theta(i)$, *i.e.*, the quantum coin is only dependent on the position. The recurrence relations become

$$\begin{aligned} R(i,t) &= \cos\theta(i-1)R(i-1,t-1) \\ &\quad + \sin\theta(i-1)L(i-1,t-1), \\ L(i,t) &= \sin\theta(i+1)R(i+1,t-1) \\ &\quad - \cos\theta(i+1)L(i+1,t-1). \end{aligned} \tag{8}$$

In order to study the deterministic aperiodic QWs, the $\theta(t)$ $(\theta(i))$ takes two values α and β arranged in the generalized Fibonacci (GF) and the Thue–Morse (TM) sequences at the t-th step (i-th position) of the walker. The GF sequences for coin operators are generated from a seed (*e.g.*, $C(\alpha)$) by the following rule[39]

$$\begin{cases} C(\alpha) \to C(\alpha)^n C(\beta)^m, \\ C(\beta) \to C(\alpha), \end{cases} \tag{9}$$

with n and m being positive integers and $C(\alpha)^n$ representing a string of n $C(\alpha)$'s. The length F_l of the l-th generation is given by the recursion relation $F_l = nF_{l-1} + mF_{l-2}$ with $F_0 = F_1 = 1$.

We can classify the GF sequences by a wandering index ω. The wandering index of the GF sequences is

$$\omega = \frac{\ln\left|\dfrac{n-\sqrt{n^2+4m}}{2}\right|}{\ln\dfrac{n+\sqrt{n^2+4m}}{2}}. \tag{10}$$

The cases with $\omega < 0$ ($n+1 > m$) correspond to the first class of (1st) GF sequences and those with $\omega > 0$ ($n+1 < m$) correspond to the second class of (2nd) GF sequences.

The typical example of the 1st GF sequence is a sequences with $m = n = 1$, which is the well-known Fibonacci sequence. It is generated by the following rule:

$$\begin{cases} C(\alpha) \to C(\alpha)C(\beta), \\ C(\beta) \to C(\alpha). \end{cases} \tag{11}$$

The infinite Fibonacci sequence is given by

$$C(\alpha)C(\beta)C(\alpha)C(\alpha)C(\beta)C(\alpha)C(\beta)C(\alpha)C(\alpha)C(\beta)....$$

Another well-studied aperiodic system is the TM system. The generating rule of the TM sequence for the coin operators is

$$\begin{cases} C(\alpha) \to C(\alpha)C(\beta), \\ C(\beta) \to C(\beta)C(\alpha). \end{cases} \tag{12}$$

The length F_l of the l-th generation is $F_l = 2^l$.

3. Entanglement

Similarly, we also use the von Neumann entropy S of the reduced density matrix in the coin space to quantify the entanglement.[29–32] That is

$$\begin{aligned} S(t) &= -\mathrm{Tr}[\rho_C(t)\log_2\rho_C(t)] \\ &= -\lambda_+\log_2\lambda_+ - \lambda_-\log_2\lambda_-, \end{aligned} \tag{13}$$

where,

$$\begin{aligned} \rho_C(t) &= \mathrm{Tr}_P[\rho(t)] \equiv \mathrm{Tr}_P[|\Psi\rangle\langle\Psi|] \\ &= \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{12}*(t) & \rho_{22}(t) \end{pmatrix}, \end{aligned} \tag{14}$$

and $\mathrm{Tr}_P[\cdot]$ is the trace over the position degree of freedom. The

$$\begin{aligned} &\rho_{11}(t) = \sum_i |R(i,t)|^2, \quad \rho_{22}(t) = \sum_i |L(i,t)|^2, \\ &\rho_{12}(t) = \sum_i R(i,t) \times L^*(i,t), \end{aligned} \tag{15}$$

and L^* is the complex conjugate of L. $\lambda_\pm$ are the eigenvalues of $\rho_C(t)$ and

$$\lambda_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \rho_{11}(t)[1-\rho_{11}(t)] + |\rho_{12}(t)|^2}. \tag{16}$$

$S = 1$ is the maximal entanglement between the coin and position degrees of freedom in QWs.

To gain further insights into the asymptotic limit of $S(t)$, we studied $S(t)$ evolution over time by

$$\Delta_S(t) = S(t) - S(t-1). \tag{17}$$

If $\Delta_S(t)$ is close to zero in the long time limit, then the $S(t)$ can achieve stationary value. In order to determine whether to approach the maximal value of entanglement, we further studied the differences of the entanglement and the maximal value ($S = 1$) evolution over time, *i.e.*

$$\Delta_E(t) = 1 - S(t). \tag{18}$$

If $\Delta_E(t)$ reduces to zero in the long time limit, then the entanglement can approach the maximal entanglement.

4. Results and discussions

In the following, we study the entanglement of the aperiodic QWs. Due to the nonuniformity of the aperiodic sequences, the numerical results need to be averaged as done in the random case.[31] In our numerical calculations, we generate a very long but finite the GF or the TM sequence of size L according to the rule (9) or rule (12). We select continuous sequence of size N from the sequence of size L ($L \gg N$) randomly. N is the size of time steps (position spaces) for the dynamic (static) aperiodic QWs. Then we obtain the values of $S(t)$. Finally, we perform the averaging over 100 realizations, and results averaging over more different selections are similar to those averaging over 100 different selections. In short, we use $\langle S(t)\rangle$ to denote the averaged entanglement. In our numerical calculations, we chose the initial state as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle \otimes |0\rangle + \frac{\mathrm{i}}{\sqrt{2}}|\downarrow\rangle \otimes |0\rangle. \tag{19}$$

For different initial states, the entanglement of the aperiodic QWs are similar.

4.1. Dynamic QWs

In Fig. 1, we give a typical result of $\langle S(t)\rangle$ as a function of time for the dynamic Fibonacci QWs with $\alpha = \pi/3, \beta = \pi/6$. We can see that the averaged entanglement grows rapidly from 0 in the first few steps, then grows slowly over time and tends to be a stationary value finally. In the inset of Fig. 1, we also give the $\Delta_S(t)$ [Fig. 1(a)] and the $\Delta_E(t)$ [Fig. 1(b)] as functions of time, respectively. It can be seen that the $\Delta_S(t)$ oscillates around 0 and the amplitude of oscillation decreases over time. It means that $\langle S(t)\rangle$ tends to a stationary value. And the $\Delta_E(t)$ decreases over time and follows a power law approximatively, $\Delta_E(t) \sim t^{-\gamma}$ with $\gamma \approx 0.25$. Therefore, the averaged entanglement approaches the maximal value $S = 1$, it means that the dynamic Fibonacci QWs can achieve the maximum entangled state. This behavior is similar to that of the dynamic disordered QWs.

In order to study the entanglement of different sequences of the dynamic 1st GF QWs, we study the entanglement of the QWs with $n = 2, m = 1$ and $n = 3, m = 1$, respectively. The numerical results are shown in Fig. 2. We can see that $\langle S(t)\rangle$ as functions of time are all similar to that of the Fibonacci case. This indicates that the entanglement of the different sequences

of the dynamic 1st GF QWs all approach the maximal value $S = 1$, which is similar to that of the dynamic disordered QWs.

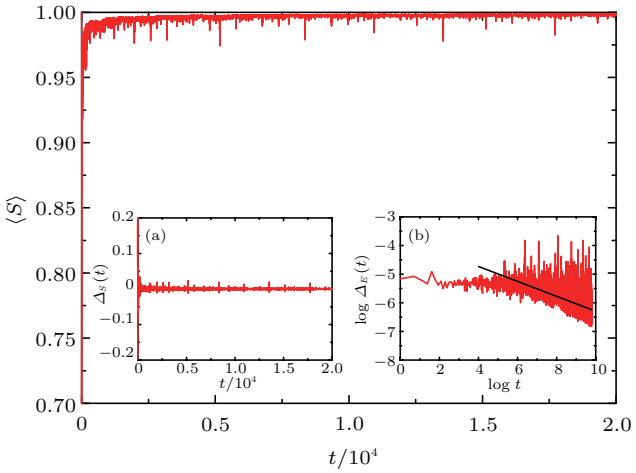


Fig. 1. (color online) Averaged entanglement $\langle S(t)\rangle$ for the dynamic Fibonacci QWs ($m = 1, n = 1$) with $\alpha = \pi/3$, $\beta = \pi/6$. The insets (a) and (b) are $\Delta_S(t)$ and $\Delta_E(t)$, respectively. The solid curve was obtained by fitting of the simulated data, where we have dropped the first 100 points to better highlight the asymptotic behavior. The slope of this solid is (-0.25 ± 0.002).

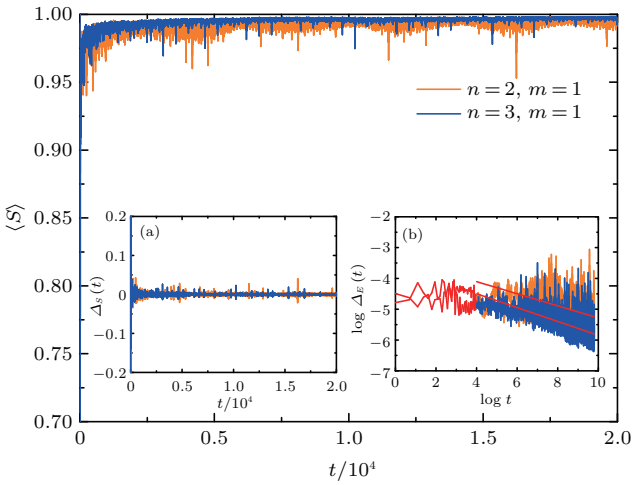


Fig. 2. (color online) Averaged entanglement $\langle S(t)\rangle$ for the dynamic 1st GF QWs with $\alpha = \pi/3, \beta = \pi/6$. The orange and blue curves correspond to $n = 2, m = 1$ and $n = 3, m = 1$, respectively. The insets (a) and (b) are $\Delta_S(t)$ and $\Delta_E(t)$, respectively. The solid curves were obtained by fitting of the simulated data, where we have dropped the first 100 points to better highlight the asymptotic behavior. The slopes of these solids are (-0.20 ± 0.003) and (-0.22 ± 0.003), respectively.

To compare the entanglement of the other aperiodic QWs, we give the numerical results of the dynamic 2nd GF QWs and the dynamic TM QWs in Figs. 3 and 4, respectively. In Fig. 3, we show the averaged entanglement of two sequences with $n = 1, m = 2$ and $n = 1, m = 3$, respectively. We can see that $\langle S(t)\rangle$ of these two sequences all tend to the stationary value. This illustrates that the entanglement of the different sequences of the dynamic 2nd GF QWs have same behavior. It also can be see that the entanglement of the dynamic TM QWs tend to the stationary value (see Fig. 4). However, their entanglement are different from that of the dynamic 1st GF QWs and can not reach the maximal entanglement value.

In order to compare the entanglement of different aperiodic QWs with that of the homogeneous QWs and the disordered QWs, we give the results of $\langle S(t)\rangle$ as functions of time for those QWs in Fig. 5. It is found that the entanglement of those QWs with $\alpha = \pi/3, \beta = \pi/6$ follow the order:

$$S_{\text{disordered}} \approx S_{\text{1st GF}} > S_{\text{TM}} > S_{\text{2nd GF}} > S_{\text{homogeneous}}. \quad (20)$$

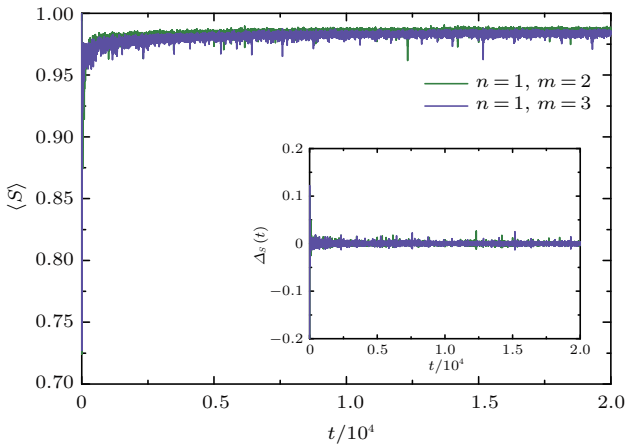


Fig. 3. (color online) Averaged entanglement $\langle S(t)\rangle$ for the dynamic 2nd GF QWs with $\alpha = \pi/3, \beta = \pi/6$. The green and violet curves correspond to $n = 1, m = 2$ and $n = 1, m = 3$, respectively. The inset is $\Delta_S(t)$.

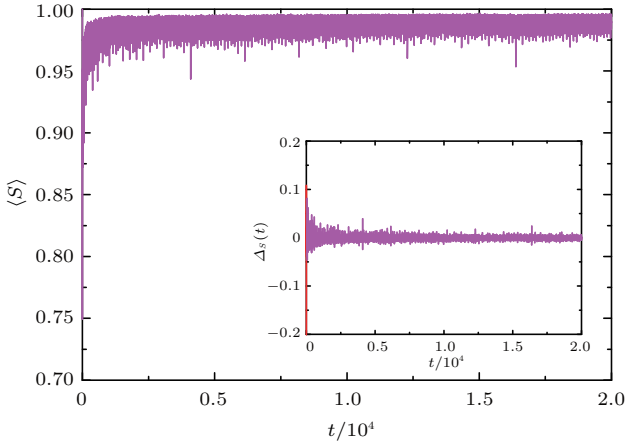


Fig. 4. (color online) Averaged entanglement $\langle S(t)\rangle$ for the dynamic TM QWs with $\alpha = \pi/3, \beta = \pi/6$. The inset is $\Delta_S(t)$.

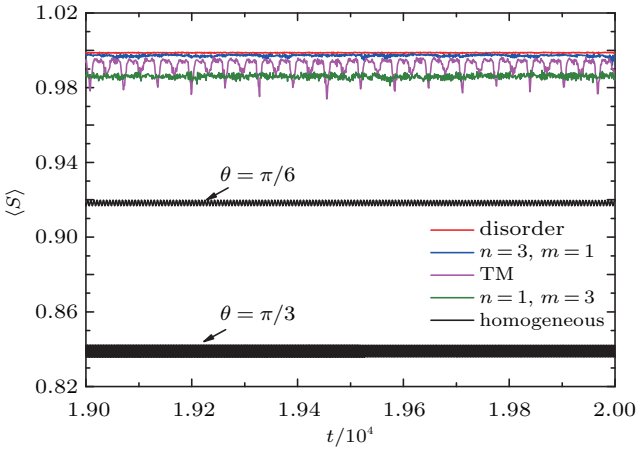


Fig. 5. (color online) Averaged entanglement $\langle S(t)\rangle$ for the dynamic disordered QWs (red), the dynamic 1st GF QWs (blue), the dynamic TM QWs (pink), the dynamic 2nd GF QWs (green) with $\alpha = \pi/3, \beta = \pi/6$, and the homogeneous QWs (black) with $\theta = \pi/3$ and $\theta = \pi/6$.

For further understanding the entanglement of the dynamic QWs, we study the distributions $P(\rho_{12})$ of the nondiagonal elements ρ_{12} of density matrix ρ. In Fig. 6, we give the distributions of the dynamic disordered, 1st GF, 2nd GF, TM, and homogeneous QWs in different time periods, respectively. We can see that the $P(\rho_{12})$ of the disordered, 1st GF and TM QWs are all Poisson distribution (see Figs. 6(a)–6(c)). For the disordered and 1st GF QWs, the $P(\rho_{12})$ are centered around the origin $\rho_{12}=0$ and the peaks of $P(\rho_{12})$ at $\rho_{12}=0$ get higher as time increasing. It means $\rho_{12}\to 0$ and entanglement $S\to 1$ in the limit $t\to\infty$. Therefore, the entanglement of the 1st GF QWs is similar to that of disordered QWs. However, for the TM QWs, the peaks of $P(\rho_{12})$ keep unchanged as time evolution, which means ρ_{12} cannot achieve 0 and the entanglement cannot become the maximum entanglement in the limit $t\to\infty$. In contrast, the $P(\rho_{12})$ of the 2nd GF and homogeneous QWs are Gaussian distribution (see Figs. 6(d) and 6(e)). The ρ_{12} corresponding the peaks of $P(\rho_{12})$ are nonzero value, which means $\rho_{12}\neq 0$ and the entanglement will be $S=(1/2+\rho_{12})\log(1/2+\rho_{12})+(1/2-\rho_{12})\log(1/2-\rho_{12})\neq 1$. However, the ρ_{12} corresponding the peaks of $P(\rho_{12})$ of the 2nd GF QWs are close to 0. Therefore, the entanglement of the 2nd GF QWs is close to the maximal value.

Finally, we analyze the entanglement of the dynamic aperiodic QWs with the different angles. In Fig. 7, we plot the averaged entanglement as functions of β $(0\leq\beta\leq\pi/2)$ for $\alpha=\pi/4$. We can see that the entanglement of three kinds of the dynamic aperiodic QWs follows the order of expression (20) in most of angles. When β is close to α, the entanglement of three kinds of the dynamic aperiodic QWs tend to be equal and are close to 0.872 (the entanglement of the homogeneous QWs for $\theta=\pi/4$). This is because that when $\alpha=\beta$, these dynamic aperiodic QWs are equal to the homogeneous QWs. Also we found that when the β approaches $\pi/2$, the entanglements of the dynamic 2nd QWs are different for odd m and even m. The entanglement is close to that of the homogeneous QWs for even m. The reason is described below. When $\beta=\pi/2$, the coin becomes $C(\pi/2)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Then,

$$\begin{aligned}R(i,t+2)&=L(i-1,t+1)=R(i,t),\\L(i,t+2)&=R(i+1,t+1)=L(i,t).\end{aligned}\tag{21}$$

It means that the state of the walker at $t+2$ step is same as that at t step. Therefore, the dynamic 2nd QWs with even m for $\beta=\pi/2$ are equivalent of the homogeneous QWs.

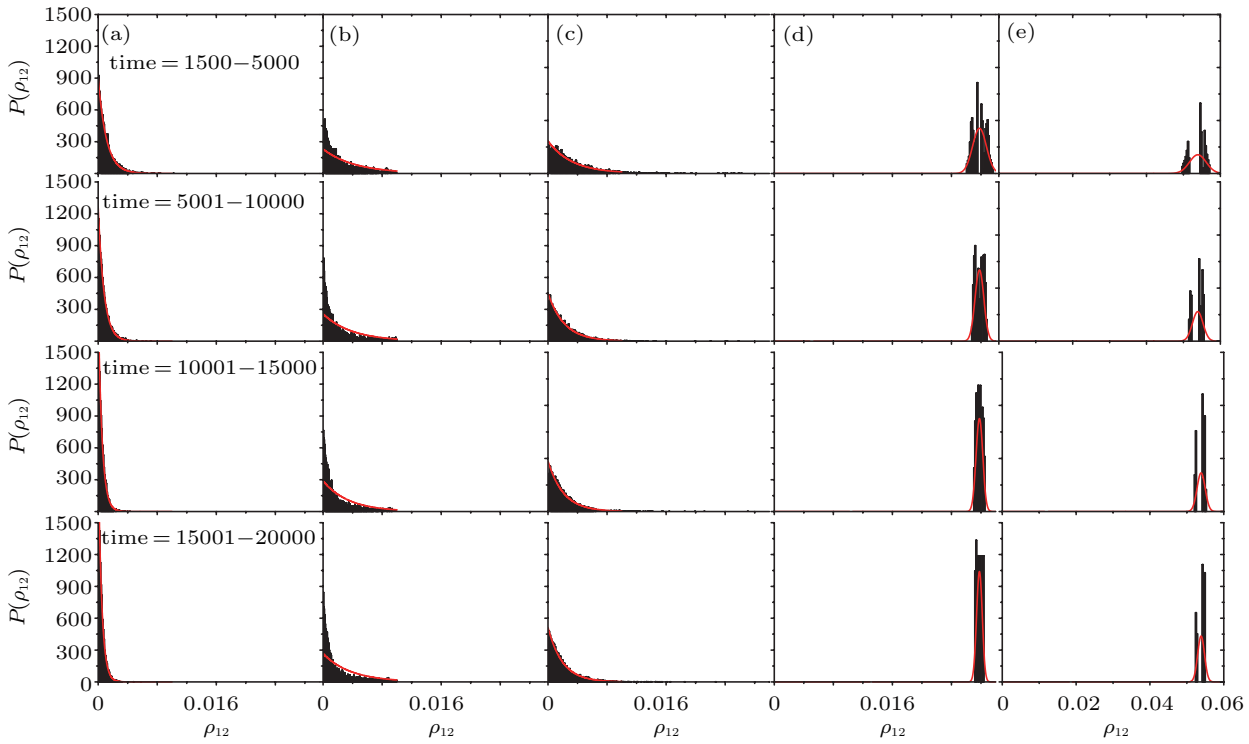


Fig. 6. (color online) The distributions $P(\rho_{12})$ of the nondiagonal elements ρ_{12} of density matrix ρ. The inset (a) is the $P(\rho_{12})$ of the dynamic disordered QWs; inset (b) is the $P(\rho_{12})$ of the dynamic 1st GF QWs; inset (c) is the $P(\rho_{12})$ of the dynamic TM QWs; inset (d) is the $P(\rho_{12})$ of the dynamic 2nd GF QWs; and inset (e) is the $P(\rho_{12})$ of the homogeneous QWs.

4.2. Static QWs

In Figs. 8–10, we give the numerical results for the entanglement of the static 1st GF QWs, the static 2nd GF QWs and the static TM QWs, respectively. In Fig. 8, we show the results of $\langle S(t)\rangle$ as functions of time for three sequences of the static 1st GF QWs. We can see that the entanglement of the different sequences of the static 1st GF QWs all tend to the stationary value finally. The $\Delta_E(t)$ follows a power law approximatively,

$\Delta_E(t) \sim t^{-\gamma}$ with $\gamma \approx 0.33$ (see the inset (b) of Fig. 8). This illustrates that the static 1st GF QWs can achieve the maximum entangled state.

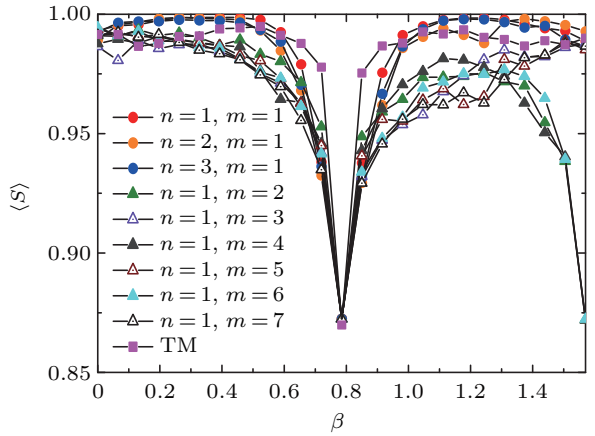


Fig. 7. (color online) Averaged entanglement of three kinds of the aperiodic QWs for $\alpha = \pi/4, 0 \le \beta \le \pi/2$.

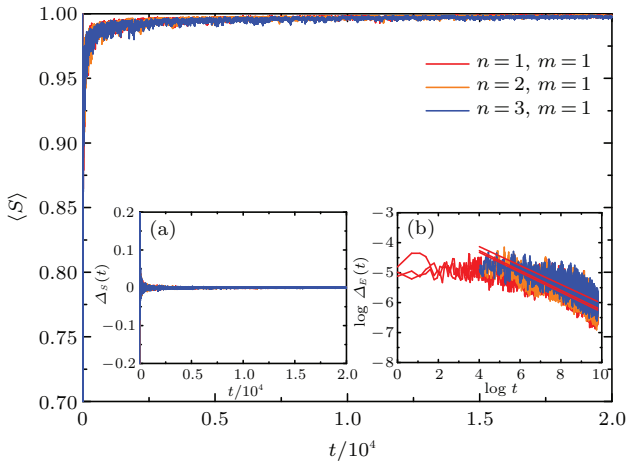


Fig. 8. (color online) Averaged entanglement $\langle S(t)\rangle$ for the static 1st GF QWs with $\alpha = \pi/3, \beta = \pi/6$. The red, orange, and blue curves correspond to $n = 1, m = 1$; $n = 2, m = 1$, and $n = 3, m = 1$, respectively. The insets (a) and (b) are $\Delta_S(t)$ and $\Delta_E(t)$, respectively. The solid curves were obtained by fitting of the simulated data, where we have dropped the first 100 points to better highlight the asymptotic behavior. The slopes of these solids are (-0.33 ± 0.002).

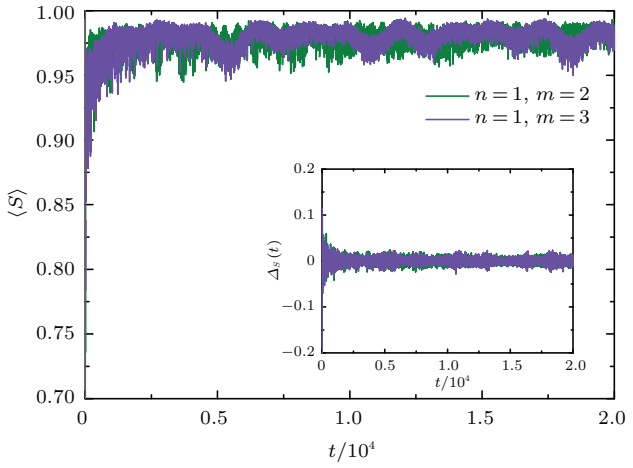


Fig. 9. (color online) Averaged entanglement $\langle S(t)\rangle$ for the static 2nd GF QWs with $\alpha = \pi/3, \beta = \pi/6$. The green and violet curves correspond to $n = 1, m = 2$ and $n = 1, m = 3$, respectively. The inset is $\Delta_S(t)$.

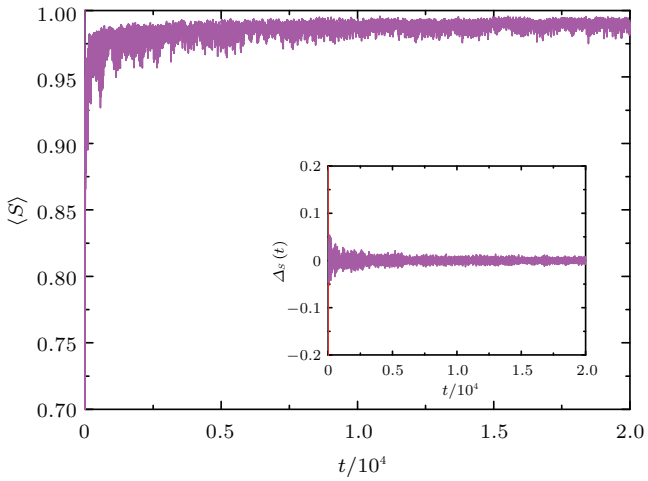


Fig. 10. (color online) Averaged entanglement $\langle S(t)\rangle$ for the static TM QWs with $\alpha = \pi/3, \beta = \pi/6$. The inset is $\Delta_S(t)$.

Similarly, from Figs. 9 and 10, we can also see that the entanglement of the static 2nd GF QWs and the static TM QWs are similar to that of the dynamic cases.

In Fig. 11, we compare the entanglement of the static disordered QWs, three kinds of the static aperiodic QWs and the homogeneous QWs. We found that the entanglement of three kinds of the static aperiodic QWs are all greater than that of the static disordered QWs and the homogeneous QWs. They follow the order:

$$S_{\text{1st GF}} > S_{\text{TM}} > S_{\text{2nd GF}} > S_{\text{disordered}}. \tag{22}$$

This is quite different from that in the dynamic cases.

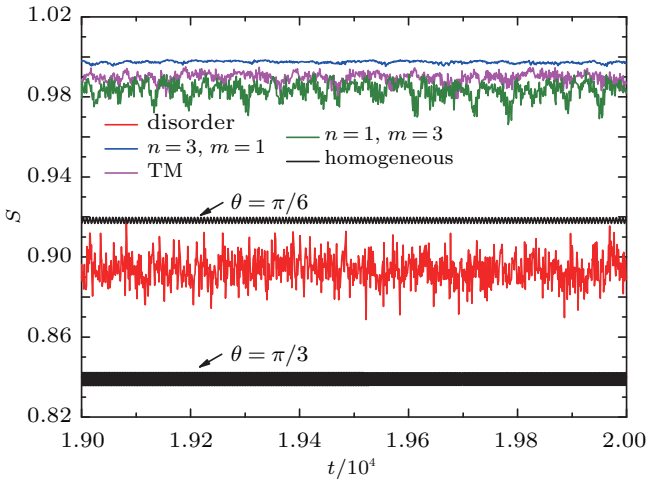


Fig. 11. (color online) Averaged entanglement $\langle S(t)\rangle$ for the static disordered QWs (red), the static 1st GF QWs (blue), the static TM QWs (pink), the static 2nd GF QWs (green) with $\alpha = \pi/3, \beta = \pi/6$, and the homogeneous QWs (black) with $\theta = \pi/3$ and $\theta = \pi/6$.

Similar to that in the dynamic aperiodic QWs, we also analyzing the distributions $P(\rho_{12})$ of the nondiagonal element ρ_{12} of density matrix ρ for the static QWs (see Fig. 12). We found that the $P(\rho_{12})$ of the static disordered, 1st GF, TM and 2nd GF QWs are all Poisson distribution. For the 1st GF QWs,

the distributions are similar to that of the dynamic disordered QWs (see Fig. 12(b)). Then the entanglement of the 1st GF QWs can create the maximal value. However, for the disordered, TM and 2nd GF QWs, the $P(\rho_{12})$ are extensions and the peaks of $P(\rho_{12})$ keep unchanged as time evolution, which means ρ_{12} cannot achieve 0 and the entanglement cannot become the maximal value in the limit $t \to \infty$ (see Figs. 12(c) and 12(d)).

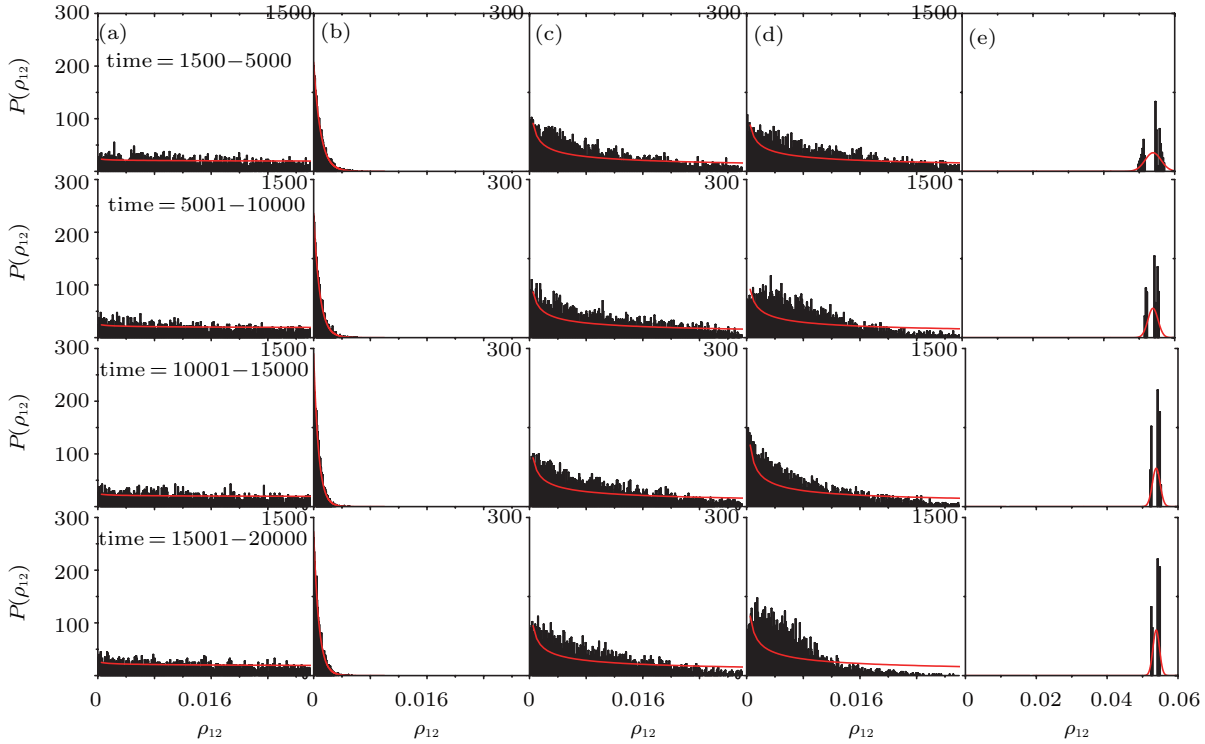


Fig. 12. (color online) The distributions $P(\rho_{12})$ of the nondiagonal elements ρ_{12} of density matrix ρ. The inset (a) is the $P(\rho_{12})$ of the static disordered QWs; inset (b) is the $P(\rho_{12})$ of the static 1st GF QWs; inset (c) is the $P(\rho_{12})$ of static TM QWs; inset (d) is the $P(\rho_{12})$ of the static 2nd GF QWs; and inset (e) is the $P(\rho_{12})$ of the homogeneous QWs.

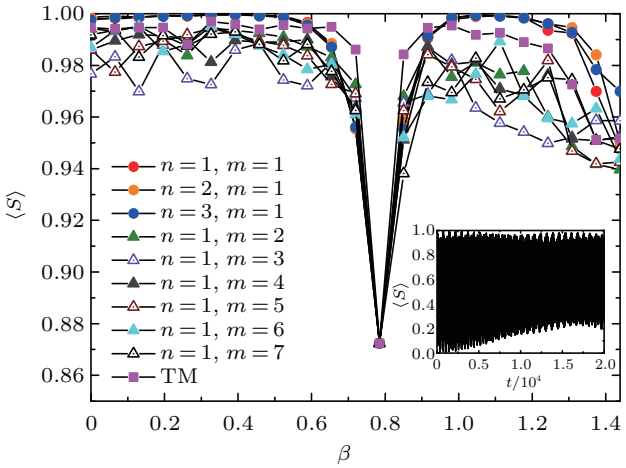


Fig. 13. (color online) Averaged entanglement of three kinds of the aperiodic QWs for $\alpha = \pi/4, 0 \le \beta \le \pi/2$. The inset is the averaged entanglement $\langle S(t) \rangle$ of the static aperiodic QWs for $n = 2, m = 1$ with $\alpha = \pi/4, \beta = \pi/2$.

Similarly, we also study the entanglement of the static aperiodic QWs with different angles in Fig. 13. It can be found that the results of the static aperiodic QWs are similar to that of the dynamic aperiodic QWs. However, when the β approach $\pi/2$, the entanglement of three kinds of the static aperiodic QWs are all away from the maximal entanglement $S = 1$. This is because that when the position with coin $C(\pi/2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the coin state at this position was overturned. It leads the averaged entanglement of the static aperiodic QWs becomes oscillating function and cannot tend to a stationary value (as an example, see the inset of Fig. 13 for $\langle S(t) \rangle$ of the static aperiodic QWs with $n = 2, m = 1$).

5. Conclusions

In this paper, we have discussed the entanglement in the dynamic and static deterministic aperiodic QWs. For the dynamic aperiodic QWs, the entanglement of the 1st GF QWs can reach the maximal entanglement value, which is similar to that of the dynamic disordered QWs. The entanglements of the 2nd GF and TM QWs are all close to maximal value. For the static aperiodic QWs, the entanglement of 1st GF QWs also can reach the maximal entanglement value, which is different from that of the static disordered QWs. The entanglements of the 2nd GF and TM QWs are similar to those of these kinds of the dynamic QWs. From our results, we can conclude that both the dynamic and static 1st GF QWs are the deterministic maximal entanglement generators. At the same time, we found that whether dynamic or static QWs, the entanglement

of the 1st GF QWs is greater than that of the TM QWs, being followed closely by the entanglement of 2nd GF QWs. The entanglements of three kinds of the dynamic aperiodic QWs are between those of the disordered and homogeneous QWs, however, the entanglements of three kinds of the static aperiodic QWs are all greater than those of the disordered and homogeneous QWs. Moreover, the classification of the aperiodic systems according to the entanglement of QWs is different from that by the quantum phase transition of the aperiodic quantum Ising chain[39] and that by the degrees of sequence disorder of those sequences.[55]

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