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N-soliton solutions for the nonlocal two-wave interaction system via the Riemann–Hilbert method*

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In this paper, a nonlocal two-wave interaction system from the Manakov hierarchy is investigated via the Riemann–Hilbert approach. Based on the spectral analysis of the Lax pair, a Riemann–Hilbert problem for the nonlocal two-wave interaction system is constructed. By discussing the solutions of this Riemann–Hilbert problem in both the regular and non-regular cases, we explicitly present the N-soliton solution formula of the nonlocal two-wave interaction system. Moreover, the dynamical behaviour of the single-soliton solution is shown graphically.

Keywords: nonlocal two-wave interaction system, Riemann–Hilbert problem, N-soliton solutions

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1. Introduction

In recent decades, study of the coupled nonlinear Schrödinger equation of Manakov type

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + 2(|u|^2 + |v|^2)u = 0, \\ &\mathrm{i}v_t + v_{xx} + 2(|u|^2 + |v|^2)v = 0 \end{aligned} \tag{1}$$

has attracted a lot of attention due to its use for modelling equations in diverse areas including deep water waves, nonlinear fibre optics, plasma physics and others (see, *e.g.*[1–3] and references therein). This model was first proposed by Manakov to study intense electromagnetic pulse propagation in birefringent fibre. Subsequently, this system was derived as a key model for light-wave propagation in optical fibres.[4] Much research on the coupled nonlinear Schrödinger equation (1) has been conducted. For example, multi-soliton solutions, collisions, algebro-geometric solutions, asymptotic behaviour of solutions, and other properties have been widely investigated for the coupled nonlinear Schrödinger equation of Manakov type.[5–13]

The aim of this paper is to study the Cauchy problem of the nonlocal two-wave interaction system from the Manakov hierarchy[14–17]

$$\begin{aligned} &u_t - u_x + v\partial^{-1}(uv^*) = 0, \quad (x,t) \in \mathbb{R} \times [0, +\infty), \\ &v_t - 2v_x - u\partial^{-1}(u^*v) = 0, \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{aligned} \tag{2}$$

via the Riemann–Hilbert approach,[18–22] where $u = u(x,t)$ and $v = v(x,t)$ are the wave envelopes, and the star denotes complex conjugation, by which the multi-soliton solutions for this system are constructed. We assume here the rapidly decreasing conditions $u(x,t), v(x,t) \to 0$ as $x \to \pm\infty$, and $(\partial^{-1})f(x) = \int_{-\infty}^{x} f(x')\,\mathrm{d}x'$. System (2) can be used to describe the soliton wave phenomenon in nonlinear waves arising in a nonlinear dispersive medium.[15] Furthermore, a similar prototypical system of evolution equations which models the nonlocal interaction of two waves in quadratic nonlinear media is investigated in Ref. [17]; the solvability of its initial value problem is shown and its conservation laws are explicitly presented. The related three-wave resonant interaction system is studied in Refs. [14] and [16], the bounded rational soliton solutions are obtained in Ref. [14] by means of the Darboux–Dressing method, and the multi-dark–dark–dark soliton solutions for this three-wave resonant interaction system are constructed through the generalized Darboux transformation method.[16] Moreover, various types of other exact solutions have been obtained such as breathers, rogue waves and other solutions. For the latest literature in this area, we refer to Refs. [23]–[29].

The structure of this paper is as follows. In Section 2, we start with the spectral analysis of the Lax pair for the nonlocal two-wave interaction system (2). Then we formulate the corresponding Riemann–Hilbert problem for the nonlocal two-wave interaction system. In Section 3, we study the inverse scattering transformation of the nonlocal two-wave interaction system (2) and solve the Riemann–Hilbert problem. In Section 4, the spatial and temporal evolution of scattering data is discussed. In Section 5, N-soliton solutions of the nonlocal two-wave interaction system (2) are constructed.

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2. Direct scattering transformation

In this section, we shall study the direct scattering problems for the nonlocal two-wave interaction system (2) and construct the corresponding Riemann–Hilbert problem. Let us consider the spectral and auxiliary problems associated with the nonlocal two-wave interaction system (2)

$$Y_x = UY = \begin{pmatrix} -\mathrm{i}\lambda & u & v \\ -u^* & \mathrm{i}\lambda & 0 \\ -v^* & 0 & \mathrm{i}\lambda \end{pmatrix} Y, \tag{3}$$

$$Y_t = VY = \begin{pmatrix} -2\mathrm{i}\lambda & u & 2v \\ -u^* & 0 & -\partial^{-1}(u^*v) \\ -2v^* & \partial^{-1}(uv^*) & 2\mathrm{i}\lambda \end{pmatrix} Y, \tag{4}$$

where λ is a spectral parameter, and $Y(x,t,\lambda)$ is a matrix function. Assume that the potentials u and v decay to zero sufficiently fast as $x \to \pm\infty$. It is convenient for us to introduce a new matrix spectral function

$$J(x,t,\lambda) = Y(x,t,\lambda)E_1^{-1}, \tag{5}$$

where

$$E_1 = \mathrm{e}^{\mathrm{i}\lambda\sigma_0 x - 2\mathrm{i}\lambda\sigma_1 t},$$
$$\sigma_0 = \mathrm{diag}\,(-1,1,1), \quad \sigma_1 = \mathrm{diag}\,(1,0,-1), \tag{6}$$

and E_1 is a solution of the above spectral equations at $x \to \pm\infty$. Therefore, the spectral problem equations (3) and (4) can be rewritten as

$$J_x = \mathrm{i}\lambda[\sigma_0, J] + QJ, \tag{7}$$

$$J_t = -2\mathrm{i}\lambda[\sigma_1, J] + Q_1 J, \tag{8}$$

where $[\sigma_i, J] = \sigma_i J - J\sigma_i$, $i = 0, 1$,

$$Q = \begin{pmatrix} 0 & u & v \\ -u^* & 0 & 0 \\ -v^* & 0 & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & u & 2v \\ -u^* & 0 & -\partial^{-1}(u^*v) \\ -2v^* & \partial^{-1}(uv^*) & 0 \end{pmatrix}. \tag{9}$$

It is easy to see that both the matrices Q and Q_1 possess the "symmetry" conditions, *i.e.*

$$Q^\dagger = -Q, \quad \sigma_0 Q \sigma_0 = -Q, \tag{10}$$

$$Q_1^\dagger = -Q_1, \tag{11}$$

where the superscript † denotes the Hermitian of a matrix.

In the scattering process, the time t is fixed and can be treated as a dummy variable, hence it will be suppressed in our notation. We assume that the matrix Jost solutions $J_\pm(x,\lambda)$ of the spectral problem (7) fulfil the asymptotic condition

$$J_+(x,\lambda) \to I, \quad x \to +\infty,$$
$$J_-(x,\lambda) \to I, \quad x \to -\infty, \tag{12}$$

where I denotes the 3×3 unit matrix. Noticing the identity

$$(\det J)_x = \det J \cdot \mathrm{tr}\,(J_x J^{-1}), \tag{13}$$

and $\mathrm{tr}\,(Q) = 0$, we arrive at

$$\det J_\pm = 1 \tag{14}$$

for all (x,λ), where $\det J$ denotes the determinant of matrix J and $\mathrm{tr}\,(\cdot)$ represents the trace of a matrix. Besides, it is easy to verify that the matrix Jost solutions $J_\pm$ solve the Volterra integral equations:

$$J_+(x,\lambda) = I - \int_x^{+\infty} \mathrm{e}^{\mathrm{i}\lambda\sigma_0(x-\zeta)} Q J_+(\zeta,\lambda) \mathrm{e}^{-\mathrm{i}\lambda\sigma_0(x-\zeta)}\,\mathrm{d}\zeta,$$
$$J_-(x,\lambda) = I + \int_{-\infty}^{x} \mathrm{e}^{\mathrm{i}\lambda\sigma_0(x-\zeta)} Q J_-(\zeta,\lambda) \mathrm{e}^{-\mathrm{i}\lambda\sigma_0(x-\zeta)}\,\mathrm{d}\zeta. \tag{15}$$

Hence $J_\pm(x,\lambda)$ admit analytical continuations off the real axis $\lambda \in \mathbb{R}$ if the above Volterra integrals converge. Let us split $J_\pm$ into column vectors, *i.e.*, $J_\pm = ([J_\pm]_1, [J_\pm]_2, [J_\pm]_3)$. Then the column vectors $[J_-]_1, [J_+]_2, [J_+]_3$ can be analytically continued to the upper half plane $\lambda \in \mathbb{C}_+$, while the column vectors $[J_+]_1, [J_-]_2, [J_-]_3$ can be analytically continued to the lower half plane $\lambda \in \mathbb{C}_-$. Furthermore, if we denote $E = \mathrm{e}^{\mathrm{i}\lambda\sigma_0 x}$, it follows easily that both J_+E and J_-E are fundamental matrix solutions of the spectral problem (3), which indicates that they are not independent and are linearly related by a scattering matrix $S(\lambda)$

$$J_-E = J_+ES(\lambda), \quad \lambda \in \mathbb{R}, \tag{16}$$

with $S(\lambda) = (s_{ij}(\lambda))_{3\times 3}$. In view of Eqs. (14) and (16), we have

$$\det S(\lambda) = 1. \tag{17}$$

In what follows, we shall construct the Riemann–Hilbert problem. To this end, we introduce the following matrix function

$$\begin{aligned} P^+ &= J_-H_1 + J_+H_2 + J_+H_3 \\ &= J_+(ES(\lambda)E^{-1}H_1 + H_2 + H_3) \\ &= J_+ \begin{pmatrix} s_{11} & 0 & 0 \\ \mathrm{e}^{2\mathrm{i}\lambda x}s_{21} & 1 & 0 \\ \mathrm{e}^{2\mathrm{i}\lambda x}s_{31} & 0 & 1 \end{pmatrix}, \end{aligned} \tag{18}$$

where

$$H_1 = \mathrm{diag}\,(1,0,0),\ \ H_2 = \mathrm{diag}\,(0,1,0),$$
$$H_3 = \mathrm{diag}\,(0,0,1). \tag{19}$$

From the analytic property of $J_\pm$, we get that P^+ is analytic in $\lambda \in \mathbb{C}_+$. In view of Eq. (15), it follows that

$$P^+(x,\lambda) \to I, \quad \lambda \in \mathbb{C}_+ \to \infty. \tag{20}$$

In order to obtain the analytic counterpart of P^+ in $\mathbb{C}_-$, we start with the adjoint spectral equation of Eq. (7):

$$K_x = \mathrm{i}\lambda[\sigma_0, K] - KQ. \tag{21}$$

It is not difficult to verify that $J_\pm^{-1}$ satisfies Eq. (21), where $J_\pm^{-1}$ is partitioned into rows in the following form

$$J_+^{-1} = \begin{pmatrix} [J_+^{-1}]^1 \\ [J_+^{-1}]^2 \\ [J_+^{-1}]^3 \end{pmatrix}, \quad J_-^{-1} = \begin{pmatrix} [J_-^{-1}]^1 \\ [J_-^{-1}]^2 \\ [J_-^{-1}]^3 \end{pmatrix}. \tag{22}$$

In view of Eq. (16), we have

$$E^{-1}J_-^{-1} = R(\lambda)E^{-1}J_+^{-1}, \quad \lambda \in \mathbb{R}, \tag{23}$$

where $R(\lambda) = S^{-1}(\lambda) = (r_{kj})_{3\times 3}$. Similar discussions lead to the conclusion that the row vectors $[J_+^{-1}]^1$, $[J_-^{-1}]^2$, $[J_-^{-1}]^3$ are analytic in $\lambda \in \mathbb{C}_+$, while other row vectors $[J_-^{-1}]^1$, $[J_+^{-1}]^2$, $[J_+^{-1}]^3$ are analytic for $\lambda \in \mathbb{C}_-$. Noticing the analytic property of J_- and J_+^{-1}, it follows that s_{11} admits analytic continuation to $\mathbb{C}_+$, and s_{22}, s_{33}, s_{23} and s_{32} can be analytically continued to $\mathbb{C}_-$, while s_{13}, s_{31}, s_{12} and s_{21} are only defined for $\lambda \in \mathbb{R}$. Similarly, r_{22}, r_{23}, r_{32} and r_{33} can be analytically continued to $\mathbb{C}_+$, r_{11} allows analytical continuation to $\mathbb{C}_-$, while r_{12}, r_{21}, r_{13} and r_{31} are defined only for $\lambda \in \mathbb{R}$. Therefore, we obtain that the adjoint Jost solutions

$$\begin{aligned} P^- &= \begin{pmatrix} [J_-^{-1}]^1 \\ [J_+^{-1}]^2 \\ [J_+^{-1}]^3 \end{pmatrix} \\ &= H_1[J_-^{-1}] + H_2[J_+^{-1}] + H_3[J_+^{-1}] \\ &= [H_1ER(\lambda)E^{-1} + H_2 + H_3]J_+^{-1} \\ &= \begin{pmatrix} r_{11} & r_{12}\,\mathrm{e}^{-2\mathrm{i}\lambda x} & r_{13}\,\mathrm{e}^{-2\mathrm{i}\lambda x} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} J_+^{-1} \end{aligned} \tag{24}$$

are analytic for $\lambda \in \mathbb{C}_-$. In addition, we have that

$$P^-(x,\lambda) \to \mathrm{I}, \quad \lambda \in \mathbb{C}_- \to \infty. \tag{25}$$

By now, we have obtained two matrix functions P^+ and P^-, which are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. On the real line, they are related by

$$P^-(x,\lambda)P^+(x,\lambda) = G(x,\lambda), \quad \lambda \in \mathbb{R}, \tag{26}$$

where

$$\begin{aligned} G(x,\lambda) &= (H_1ERE^{-1} + H_2 + H_3)J_+^{-1}J_+(ESE^{-1}H_1 + H_2 + H_3) \\ &= \begin{pmatrix} r_{11}s_{11} + r_{12}s_{21} + r_{13}s_{31} & r_{12}\,\mathrm{e}^{-2\mathrm{i}\lambda x} & r_{13}\,\mathrm{e}^{-2\mathrm{i}\lambda x} \\ s_{21}\,\mathrm{e}^{2\mathrm{i}\lambda x} & 1 & 0 \\ s_{31}\,\mathrm{e}^{2\mathrm{i}\lambda x} & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & r_{12}\,\mathrm{e}^{-2\mathrm{i}\lambda x} & r_{13}\,\mathrm{e}^{-2\mathrm{i}\lambda x} \\ s_{21}\,\mathrm{e}^{2\mathrm{i}\lambda x} & 1 & 0 \\ s_{31}e^{2\mathrm{i}\lambda x} & 0 & 1 \end{pmatrix}, \end{aligned} \tag{27}$$

where $r_{11}s_{11} + r_{12}s_{21} + r_{13}s_{31} = 1$ is followed by the fact that $R(\lambda)S(\lambda) = \mathrm{I}$. Equation (26) is just the matrix Riemann–Hilbert problem we needed, and the associated canonical condition is as follows:

$$P^\pm(x,\lambda) \to I, \quad \lambda \to \infty. \tag{28}$$

3. Solutions to the Riemann–Hilbert problem

In this section, we shall solve the Riemann–Hilbert problem (26) in both regular and non-reguluar cases.[21] Before this, from Eqs. (14) and (18), it is easy to see that

$$\det P^+(\lambda) = s_{11}(\lambda), \tag{29}$$
$$\det P^-(\lambda) = r_{11}(\lambda). \tag{30}$$

We first consider the regular Riemann–Hilbert problem (26), *i.e.*, $\det P^+ = s_{11} \neq 0$ and $\det P^- = r_{11} \neq 0$ in their analytic domains. Equation (26) can be rewritten as

$$(P^+)^{-1}(\lambda) - P^-(\lambda) = \hat{G}(\lambda)(P^+)^{-1}(\lambda),\ \lambda \in \mathbb{R}, \tag{31}$$

where $\hat{G} = I - G$. By Plemelj's formula, we obtain that the solution to the regular Riemann–Hilbert problem (26) under condition (28) takes the form

$$(P^+)^{-1}(\lambda) = I + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty} \frac{\hat{G}(\xi)(\hat{P}^+)^{-1}(\xi)}{\xi - \lambda}\,\mathrm{d}\xi,$$
$$\lambda \in \mathbb{C}_+. \tag{32}$$

Next, we consider the non-regular Riemann–Hilbert problem (26), that is, $\det P^+(\lambda)$ and $\det P^-(\lambda)$ possess simple zeros, which are denoted by $\lambda_k \in \mathbb{C}_+$ and $\hat{\lambda}_k \in \mathbb{C}_-$, $1 \leq k \leq N$, respectively, where N is the number of these zeros. Therefore λ_k and $\hat{\lambda}_k$ are also zeros of the scattering coefficients $s_{11}(\lambda)$ and $r_{11}(\lambda)$, where

$$r_{11}(\lambda) = s_{22}(\lambda)s_{33}(\lambda) - s_{23}(\lambda)s_{32}(\lambda). \tag{33}$$

Now we discuss the symmetry properties of the scattering matrix $S(\lambda)$ and the Jost functions. Taking the Hermitian on both sides of the spectral problem (7), in view of Eq. (10), we have

$$(J_\pm^\dagger)_x = \mathrm{i}\lambda^*[\sigma_0, J_\pm^\dagger] - J_\pm^\dagger Q. \tag{34}$$

Since $J_\pm^{-1}$ fulfills Eq. (21), it follows that

$$J_\pm^\dagger(\lambda^*) = J_\pm^{-1}(\lambda), \tag{35}$$

from which, together with Eqs. (18) and (24), $P^{\pm}$ satisfies the property

$$(P^{+})^{\dagger}(\lambda^{*}) = P^{-}(\lambda). \tag{36}$$

In view of Eq. (16), we have

$$S(\lambda) = E^{-1}J_{+}^{-1}J_{-}E. \tag{37}$$

Then we obtain by using Eq. (23) that S satisfies the property

$$S^{\dagger}(\lambda^{*}) = S^{-1}(\lambda), \tag{38}$$

from which we have the following symmetric relation

$$\hat{\lambda}_k = \lambda_k^{*}. \tag{39}$$

Since all the zeros are simple ones, then both $\ker P^{+}(\lambda_k)$ and $\ker P^{-}(\hat{\lambda}_k)$ are one-dimensional and spanned by a single column vector μ_k and row vector $\hat{\mu}_k$, respectively. Hence we have

$$P^{+}(\lambda_k)\mu_k = 0, \quad 1 \le k \le N, \tag{40}$$

$$\hat{\mu}_k P^{-}(\hat{\lambda}_k) = 0, \quad 1 \le k \le N. \tag{41}$$

Taking the Hermitian of the first equation in Eq. (40) and utilizing the property (36), we obtain

$$\mu_k^{\dagger}P^{-}(\hat{\lambda}_k) = 0. \tag{42}$$

Then, comparing it with the second equation in Eq. (40), we see the eigenvectors μ_k and $\hat{\mu}_k$ satisfy the property

$$\hat{\mu}_k = \mu_k^{\dagger}. \tag{43}$$

The solution to this non-regular Riemann–Hilbert problem (26) under the canonical normalization condition (28) is unique and is given by the following theorem.

Theorem 1 The solution to the non-regular Riemann–Hilbert problem (26) with zeros determined by Eq. (40) under the canonical normalization condition (28) is

$$P^{+}(\lambda) = \hat{P}^{+}(\lambda)\Upsilon(\lambda), \tag{44}$$

$$P^{-}(\lambda) = \Upsilon^{-1}(\lambda)\hat{P}^{-}(\lambda), \tag{45}$$

where

$$\Upsilon(\lambda) = I + \sum_{j,k=1}^{N} \frac{\mu_j (M^{-1})_{jk}\hat{\mu}_k}{\lambda - \hat{\lambda}_k}, \tag{46}$$

$$\Upsilon^{-1}(\lambda) = I - \sum_{j,k=1}^{N} \frac{\mu_j (M^{-1})_{jk}\hat{\mu}_k}{\lambda - \lambda_j}, \tag{47}$$

and M is an $N \times N$ matrix with its (j,k)th element given by

$$M_{jk} = \frac{\hat{\mu}_j \mu_k}{\hat{\lambda}_j - \lambda_k}, \quad 1 \le j,k \le N, \tag{48}$$

$$\det \Upsilon(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k}{\lambda - \hat{\lambda}_k}, \tag{49}$$

and $\hat{P}^{\pm}(\lambda)$ is the unique solution to the following regular Riemann–Hilbert problem:

$$\hat{P}^{-}(\lambda)\hat{P}^{+}(\lambda) = \Upsilon(\lambda)G(\lambda)\Upsilon^{-1}(\lambda), \quad \lambda \in \mathbb{R}, \tag{50}$$

where $\hat{P}^{\pm}(\lambda)$ are analytic in $\mathbb{C}_{\pm}$, respectively, and $\hat{P}^{\pm}(\lambda) \to I$ as $\lambda \to \infty$.

Proof For a constructive proof of this theorem, we refer to Ref. [21]. □

4. Evolution of scattering data and potential reconstruction

4.1. Spatial and temporal evolution

In this section, we shall establish the spatial and temporal evolution of scattering data, and present the reconstruction of potentials u and v. From the Theorem 1, we know the minimal scattering data needed to work out the non-regular matrix Riemann–Hilbert problem and reconstruct the potentials. We take the scattering data

$$\{s_{21}(\lambda), s_{31}(\lambda), r_{12}(\lambda), r_{13}(\lambda), \lambda \in \mathbb{R}; \lambda_k, \hat{\lambda}_k, \mu_k, \hat{\mu}_k, 1 \le k \le N\}, \tag{51}$$

where μ_k and $\hat{\mu}_k$ are dependent on x, and the others are not. Let us take the x-derivative of Eq. (40), together with the fact that P^{+} satisfies the spectral equation (7), *i.e.*

$$P_x^{+}(\lambda_k, x) = \mathrm{i}\lambda_k[\sigma_0, P^{+}(\lambda_k, x)] + QP^{+}(\lambda_k, x), \tag{52}$$

which implies

$$P_x^{+}(\lambda_k, x)\left(\frac{\mathrm{d}\mu_k}{\mathrm{d}x} - \mathrm{i}\lambda_k\sigma_0\mu_k\right) = 0. \tag{53}$$

Therefore, we have

$$\frac{\mathrm{d}\mu_k}{\mathrm{d}x} - \mathrm{i}\lambda_k\sigma_0\mu_k = 0. \tag{54}$$

The temporal evolution of vector μ_k is derived in a similar manner. Together with these results, we arrive at

$$\mu_k(x,t) = \mathrm{e}^{\mathrm{i}\lambda_k\sigma_0 x - 2\mathrm{i}\lambda_k\sigma_1 t}\mu_k^0, \tag{55}$$

$$\hat{\mu}_k(x,t) = \mu_k^1 \mathrm{e}^{-\mathrm{i}\lambda_k^{*}\sigma_0 x + 2\mathrm{i}\lambda_k^{*}\sigma_1 t}, \tag{56}$$

where $\mu_k^0 = \mu_k|_{x=t=0}$ represents a constant vector and μ_k^1 denotes the Hermitian conjugation of μ_k^0.

Next, we shall present the temporal evolution of scattering data $s_{21}(\lambda)$, $s_{31}(\lambda)$, $r_{12}(\lambda)$, and $r_{13}(\lambda)$. Since $J_{\pm}$ satisfies Eq. (8), we obtain

$$(J_{\pm})_t = -2\mathrm{i}\lambda[\sigma_1, J_{\pm}] + Q_1 J_{\pm}, \tag{57}$$

and from the fact that $J_+\cdot J_+^{-1}=I$ one easily has

$$(J_+^{-1})_t=-J_+^{-1}(J_+)_tJ_+^{-1}. \tag{58}$$

Inserting Eq. (57) into Eq. (58), one obtains

$$(J_+^{-1})_t=-J_+^{-1}\big(-2\mathrm{i}\lambda[\sigma_1,J_+]+Q_1J_+\big)J_+^{-1}$$
$$=2\mathrm{i}\lambda J_+^{-1}\sigma_1-2i\lambda\sigma_1J_+^{-1}-J_+^{-1}Q_1. \tag{59}$$

From Eq. (16), we know that

$$S(\lambda)=E^{-1}J_+^{-1}J_-E, \tag{60}$$

where $E=\mathrm{e}^{\mathrm{i}\lambda\sigma_0x}$. Hence by Eqs. (57)–(60) we have

$$\begin{aligned}S_t&=E^{-1}\big((J_+^{-1})_tJ_-+J_+^{-1}(J_-)_t\big)E\\&=E^{-1}\big(2\mathrm{i}\lambda J_+^{-1}\sigma_1J_--2i\lambda\sigma_1J_+^{-1}J_--J_+^{-1}Q_1J_-+(-2\mathrm{i}\lambda)J_+^{-1}\sigma_1J_-+2\mathrm{i}\lambda J_+^{-1}J_-\sigma_1+J_+^{-1}Q_1J_-\big)E\\&=-2\mathrm{i}\lambda E^{-1}\sigma_1J_+^{-1}J_-E+2\mathrm{i}\lambda E^{-1}J_+^{-1}J_-\sigma_1E=-2\mathrm{i}\lambda\sigma_1E^{-1}J_+^{-1}J_-E+2\mathrm{i}\lambda E^{-1}J_+^{-1}J_-E\sigma_1\\&=-2\mathrm{i}\lambda\sigma_1S+2\mathrm{i}\lambda S\sigma_1=-2\mathrm{i}\lambda[\sigma_1,S],\end{aligned} \tag{61}$$

where we utilize the two facts that

$$E^{-1}\sigma_1=\sigma_1E^{-1}=\mathrm{diag}\,(\mathrm{e}^{\mathrm{i}\lambda x},0,-\mathrm{e}^{-\mathrm{i}\lambda x}),$$
$$E\sigma_1=\sigma_1E=\mathrm{diag}\,(\mathrm{e}^{-\mathrm{i}\lambda x},0,-\mathrm{e}^{\mathrm{i}\lambda x}).$$

By a similar consideration, we also have

$$R_t=-2\mathrm{i}\lambda[\sigma_1,R], \tag{62}$$

which, together with Eq. (61), implies

$$\begin{aligned}&\frac{\partial s_{21}}{\partial t}=2\mathrm{i}\lambda s_{21},\quad \frac{\partial s_{31}}{\partial t}=4\mathrm{i}\lambda s_{31},\\&\frac{\partial r_{12}}{\partial t}=2\mathrm{i}\lambda r_{12},\quad \frac{\partial r_{13}}{\partial t}=-4\mathrm{i}\lambda r_{13},\\&\frac{\partial s_{11}}{\partial t}=\frac{\partial r_{11}}{\partial t}=0.\end{aligned} \tag{63}$$

Thus we obtain

$$\begin{aligned}&s_{21}(t;\lambda)=s_{21}(0;\lambda)\mathrm{e}^{2\mathrm{i}\lambda t},\ s_{31}(t;\lambda)=s_{31}(0;\lambda)\mathrm{e}^{4\mathrm{i}\lambda t},\\&r_{12}(t;\lambda)=r_{12}(0;\lambda)\mathrm{e}^{-2\mathrm{i}\lambda t},\ r_{13}(t;\lambda)=r_{13}(0;\lambda)\mathrm{e}^{-4\mathrm{i}\lambda t}.\end{aligned} \tag{64}$$

In view of Eqs. (15), (23), and (37), we note that the initial scattering data $s_{21}(0,\lambda)$, $s_{31}(0,\lambda)$, $r_{12}(0,\lambda)$, and $r_{13}(0,\lambda)$ are determined by the initial values $u_0(x)$ and $v_0(x)$.

4.2. Potential reconstruction

Noting that P^+ solves the spectral problem (7) and is expanded at large λ as

$$P^+(x,\lambda)=I+\frac{1}{\lambda}P_1^+(x)+O(\lambda^{-2}),\quad \lambda\to\infty, \tag{65}$$

we obtain by inserting this expansion into Eq. (7) that

$$\begin{aligned}Q&=-\mathrm{i}[\sigma_0,P_1^+]\\&=\begin{pmatrix}0&2\mathrm{i}(P_1^+)_{12}&2\mathrm{i}(P_1^+)_{13}\\-2\mathrm{i}(P_1^+)_{21}&0&0\\-2\mathrm{i}(P_1^+)_{31}&0&0\end{pmatrix}.\end{aligned} \tag{66}$$

Hence the potentials u and v could be reconstructed as

$$u=2\mathrm{i}(P_1^+)_{12},\quad v=2\mathrm{i}(P_1^+)_{13}. \tag{67}$$

In view of Theorem 1, the unique solution to the non-regular Riemann–Hilbert problem (26) under the canonical normalization condition (28) is

$$(\hat{P}^+)^{-1}(\lambda)=I+\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{\Upsilon(\xi)\hat{G}(\xi)\Upsilon^{-1}(\xi)(\hat{P}^+)^{-1}(\xi)}{\xi-\lambda}\mathrm{d}\xi,\quad \lambda\in\mathbb{C}_+, \tag{68}$$

where

$$\hat{G}(\lambda;x,t)=I-G=\begin{pmatrix}0&-r_{12}(0;\lambda)\mathrm{e}^{-2\mathrm{i}\lambda(x+t)}&-r_{13}(0;\lambda)\mathrm{e}^{-2\mathrm{i}\lambda(x+2t)}\\-s_{21}(0;\lambda)\mathrm{e}^{2\mathrm{i}\lambda(x+t)}&0&0\\-s_{31}(0;\lambda)\mathrm{e}^{2\mathrm{i}\lambda(x+2t)}&0&\end{pmatrix}. \tag{69}$$

As $\lambda\to\infty$, we arrive at

$$(\hat{P}^+)^{-1}(\lambda)\to I-\frac{1}{2\pi\mathrm{i}\lambda}\int_{-\infty}^{+\infty}\Upsilon(\xi)\hat{G}(\xi)\Upsilon^{-1}(\xi)(\hat{P}^+)^{-1}(\xi)\mathrm{d}\xi. \tag{70}$$

Therefore, we have

$$(\hat{P}^+)(\lambda)\to I+\frac{1}{2\pi\mathrm{i}\lambda}\int_{-\infty}^{+\infty}\Upsilon(\xi)\hat{G}(\xi)\Upsilon^{-1}(\xi)(\hat{P}^+)^{-1}(\xi)\mathrm{d}\xi. \tag{71}$$

For Eq. (46), when $\lambda \to \infty$, we easily obtain

$$\Upsilon(\lambda) \to I + \frac{1}{\lambda}\sum_{j,k=1}^{N} \mu_j (M^{-1})_{jk}\hat{\mu}_k. \tag{72}$$

From Eqs. (44), (71) and (72), we have

$$P_1^+(x,t) = \sum_{j=1}^{N}\sum_{k=1}^{N} \mu_j (M^{-1})_{jk}\hat{\mu}_k + \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{+\infty} \Upsilon(\xi)\hat{G}(\xi)\Upsilon^{-1}(\xi)(\hat{P}^+)^{-1}(\xi)\mathrm{d}\xi. \tag{73}$$

5. *N*-soliton solutions

In this section, we shall give N-soliton solutions of the nonlocal two-wave interaction system, which corresponds to the case $G = I$. Therefore the integral term in Eq. (73) vanishes, that is,

$$P_1^+(x,t) = \sum_{j,k=1}^{N} \mu_j (M^{-1})_{jk}\hat{\mu}_k, \tag{74}$$

and

$$u(x,t) = 2\mathrm{i}\Big(\sum_{j,k=1}^{N} \mu_j (M^{-1})_{jk}\hat{\mu}_k\Big)_{12},$$
$$v(x,t) = 2\mathrm{i}\Big(\sum_{j,k=1}^{N} \mu_j (M^{-1})_{jk}\hat{\mu}_k\Big)_{13}, \tag{75}$$

where $\hat{\mu}_k = \mu_k^\dagger$, vectors μ_j are defined by Eq. (40), and M is given by Eq. (48).

Next, we shall study the properties of these N-soliton solutions. We first consider the simple case where $N = 1$, namely, single-soliton solutions. For simplicity, we set

$$\mu_1^0 = (\alpha_1, \beta_1, \gamma_1)^{\mathrm{T}}, \quad \lambda_1 = \theta_1 + \mathrm{i}\phi_1. \tag{76}$$

Then the single-soliton solutions (75) read

$$u(x,t) = \frac{4\phi_1\alpha_1\beta_1^* \mathrm{e}^{2t\phi_1 - 2\mathrm{i}\theta_1(x+t)}}{|\alpha_1|^2\mathrm{e}^{2\phi_1(x+2t)} + |\beta_1|^2\mathrm{e}^{-2\phi_1 x} + |\gamma_1|^2\mathrm{e}^{-2\phi_1(x+2t)}}, \quad v(x,t) = \frac{4\phi_1\alpha_1\gamma_1^* \mathrm{e}^{-2\mathrm{i}\theta_1(x+2t)}}{|\alpha_1|^2\mathrm{e}^{2\phi_1(x+2t)} + |\beta_1|^2\mathrm{e}^{-2\phi_1 x} + |\gamma_1|^2\mathrm{e}^{-2\phi_1(x+2t)}}. \tag{77}$$

Setting $\alpha_1 = \beta_1 = \gamma_1 = \mathrm{i}$ and $\theta_1 = \phi_1 = 1$, we plot the graphs of the single-soliton solutions for the nonlocal two-wave interaction system (2) as shown in Fig. 1.

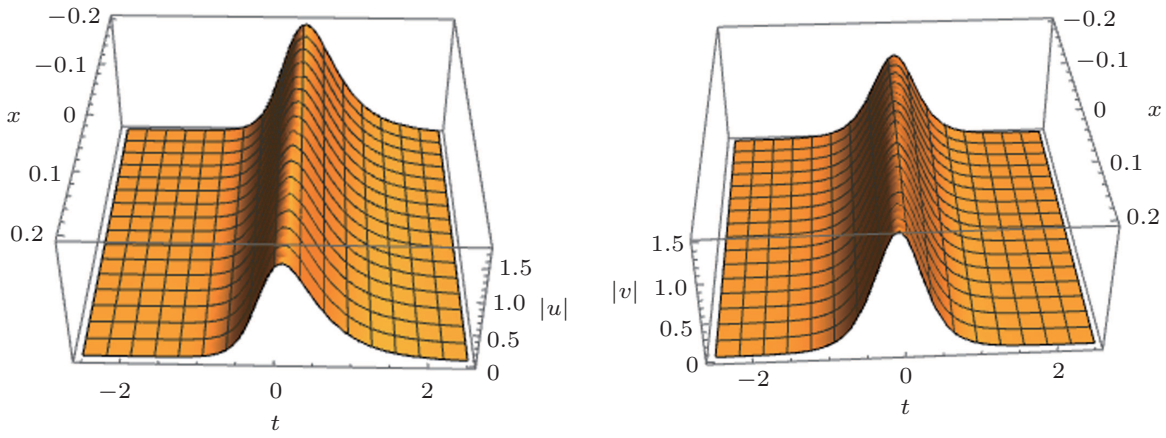


Fig. 1. (color online) Single-soliton solutions for the nonlocal two-wave interaction system (2).

For $N \geq 2$, we introduce the following notations

$$\Theta_{j,1} = -\mathrm{i}\lambda_j(x+2t), \quad \Theta_{j,2} = \mathrm{i}\lambda_j x,$$
$$\mu_j^0 = (\alpha_j, \beta_j, \gamma_j)^{\mathrm{T}}, \quad \mu_k^1 = (v_k^0)^\dagger = (\alpha_k^*, \beta_k^*, \gamma_k^*), \quad 1 \leq j,k \leq N. \tag{78}$$

In view of Eqs. (55) and (56), we have

$$\mu_j = (\alpha_j \mathrm{e}^{\Theta_{j,1}}, \beta_j \mathrm{e}^{\Theta_{j,2}}, \gamma_j \mathrm{e}^{-\Theta_{j,1}})^{\mathrm{T}},$$
$$\hat{\mu}_k = (\alpha_k^* \mathrm{e}^{\Theta_{k,1}^*}, \beta_k^* \mathrm{e}^{\Theta_{k,2}^*}, \gamma_k^* \mathrm{e}^{-\Theta_{k,1}^*}), \tag{79}$$

Then the N-soliton solutions (75) read as

$$u(x,t) = 2\mathrm{i}\sum_{j,k=1}^{N} \alpha_j\beta_k^* \mathrm{e}^{\Theta_{j,1}+\Theta_{k,2}^*}(M^{-1})_{jk},$$

$$v(x,t) = 2\mathrm{i}\sum_{j,k=1}^{N} \alpha_j\gamma_k^* \mathrm{e}^{\Theta_{j,1}-\Theta_{k,1}^*}(M^{-1})_{jk}, \tag{80}$$

where the element of the matrix $M = (M_{jk})_{N\times N}$ is defined by

$$M_{jk} = \frac{1}{\lambda_j^* - \lambda_k}\Big[\alpha_j^*\alpha_k \mathrm{e}^{\Theta_{j,1}^*+\Theta_{k,1}} + \beta_j^*\beta_k \mathrm{e}^{\Theta_{j,2}^*+\Theta_{k,2}} + \gamma_j^*\gamma_k \mathrm{e}^{-\Theta_{j,1}^*-\Theta_{k,1}}\Big]. \tag{81}$$

Moreover, N-soliton solutions (75) can be written as

$$u(x,t) = -2\mathrm{i}\frac{\det F_1}{\det M},$$
$$v(x,t) = -2\mathrm{i}\frac{\det F_2}{\det M}, \tag{82}$$

where two $(N+1)\times(N+1)$ matrices F_1 and F_2 are given by

$$F_1 = \begin{pmatrix} 0 & \beta_1^* \mathrm{e}^{\Theta_{1,2}^*} & \cdots & \beta_N^* \mathrm{e}^{\Theta_{N,2}^*} \\ \alpha_1 \mathrm{e}^{\Theta_{1,1}} & M_{11} & \cdots & M_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N \mathrm{e}^{\Theta_{N,1}} & M_{N1} & \cdots & M_{NN} \end{pmatrix}, \tag{83}$$

$$F_2 = \begin{pmatrix} 0 & \gamma_1^* \mathrm{e}^{-\Theta_{1,1}^*} & \cdots & \gamma_N^* \mathrm{e}^{-\Theta_{N,1}^*} \\ \alpha_1 \mathrm{e}^{\Theta_{1,1}} & M_{11} & \cdots & M_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N \mathrm{e}^{\Theta_{N,1}} & M_{N1} & \cdots & M_{NN} \end{pmatrix}. \tag{84}$$

6. Conclusion

In this paper, we discuss the N-soliton solutions for the nonlocal two-wave interaction system arisen from the Manakov hierarchy by means of the Riemann–Hilbert method. Our presentation is based on the Riemann–Hilbert formulation, which can be summarized as follows: we firstly construct a Riemann–Hilbert problem for the nonlocal two-wave interaction system (2) through the spectral analysis of the corresponding Lax pair. Secondly, we solve the Riemann–Hilbert problem for the nonlocal two-wave interaction system in both the regular and non-regular cases. Thirdly, by reconstructing the potentials and discussing the spatial and temporal evolution of the scattering data, we explicitly give expressions for the N-soliton solutions of the nonlocal two-wave interaction system (2). Finally, the dynamical behaviour of the single-soliton solution is displayed graphically.

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