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A more general form of lump solution, lumpoff, and instanton/rogue wave solutions of a reduced (3 + 1)-dimensional nonlinear evolution equation*

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In this manuscript, a reduced $(3+1)$-dimensional nonlinear evolution equation is studied. We first construct the bilinear formalism of the equation by using the binary Bell polynomials theory, then explore a lump solution to the special case for $z = x$. Furthermore, a more general form of lump solution of the equation is found which possesses seven arbitrary parameters and four constraint conditions. By cutting the lump by the induced soliton(s), lumpoff and instanton/rogue wave solutions are also constructed by the more general form of lump solution.

Keywords: a reduced $(3+1)$-dimensional nonlinear evolution equation, more general form of lump solution, soliton induced by lump, lumpoff and instanton/rogue wave solutions

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1. Introduction

Rogue waves correspond to large-amplitude waves are extreme events that seldom appear on the ocean surfaces. Such waves can be accompanied by deep troughs (holes), which occur before and/or after the largest crest. In the last decades, a large interest including theories and experiments has spread from oceanography into several other areas of research, such as nonlinear optical systems,[1–3] plasmas,[4–6] fluid dynamics and atmosphere,[7–9] Bose–Einstein condensations (BECs),[10,11] financial system,[12,13] microwave oscillators,[14] capillary flow,[15] *etc*. Various physical models of the rogue wave phenomenon have been intensively developed and many laboratory experiments[16–18] conducted. These investigations makes scientists to understand the physics of the huge wave appearance and its relation to environmental conditions. Though the origin of rogue waves is still a matter of debate,[19] there are two important and common features characterizing rogue wave phenomena which have been known and accepted: 1) its amplitude is more than twice (or 2.5 times) of the average amplitude of the significant wave height; 2) a rogue wave "appears from nowhere and disappears without a trace", in other words, rogue wave is believed to be unpredictable.

Many nonlinear physical models are used to describe rogue waves. For example, the lump waves, known as a kind rational function solutions, is an algebraically localized wave decayed in all space directions and exists in all time which can be used to describe rouge waves in some sence. Recent studies show lump solutions have attracted more and more attention. Many physical systems, such as $(2+1)$-dimensional KPI equation, BKP equation, KdV equation, CDGKS equation, and Ito equation have been proved to possess lump solutions.[20–23] Generally, there are many methods to search for lump solutions, such as the multiple variable separation method,[24] the common expression method,[25] *etc*. Moreover, recent study shows the interactions between lumps and other nonlinear waves such as Refs. [26]– [30]. These progress are very helpful in understanding of the physical mechanisms of the rogue wave phenomenon.

On the other hand, in quantum field theories the instantons studies allow scientists to see the previously hidden logarithmic structure of the states and operators.[31,32] Physicists believe that instantons are the key to explore the interactions principle in the standard model. Studies exhibit that instantons have been shown in integrable systems, such as DS equation[24] via the multiple linear variable separation approach.

In this paper, we consider a $(3+1)$-dimensional nonlinear evolution equation (NEE)

$$3u_{xz} - (2u_t - 2uu_x + 2u_{xxx})_y + 2(u_x\partial_x^{-1}u_y)_x = 0, \qquad (1)$$

which was first introduced in the study of the algebraic geometrical solutions[33] in nonlinear science. Recently, Chen and Li have studied the rouge wave of this equation.[34] The

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authors in Ref. [35] found m-lump solutions and interactive solutions and the Darboux transformations were presented in Ref. [36], which can be used to find positon, negaton, soliton, and complexiton solutions of the NEE. Though the background of this $(3+1)$-dimensional made is not clear in physics or other sciences, it is not hard to find the relationship between the equation and the famous Kortewege–de Vries (KdV) equation[37] which describe shallow water waves. As far as we know, though the lump solutions of the NEE has been studied in Ref. [34], which can be used to describe rogue waves in shallow water waves, the physical mechanisms of the rogue wave have not been given.

This paper is organized as follows: In the next section, we first introduce the known form of the lump solutions for the reduced nonlinear evolution equation, then extend a more general form to obtain lump solutions with maximum numbers of arbitrary parameters. In Section 3, a lumpoff, a lump being cutoff by a soliton with a special dispersive relation is demonstrated with a more general form. The lumpoff possesses a special dispersive relation which shows the soliton induced by the lump. Due to the invariant of the lump, the lump will keep its path until it is cutoff by the induced lump. In Section 4, the instanton/rogue waves are found with the same dispersive relation. Because the lump is cutoff between the two induced solitons, it will become a rogue wave when it reaches a huge wave height, otherwise it will become an instanton. The perspectives in the study of this kind of rogue wave phenomenon are discussed in the last section.

2. Bilinear formalism of the (3 + 1)-dimensional NEE

Now we try to construct the bilinear formalism of the NEE Eq. (1). According to decoupling binary Bell polynomials theories, we assume

$$u = c(t)q_{xx}, \tag{2}$$

where q is a real function with respect to variable x, y, z, and t, and $c(t) \equiv c$ is an undetermined function with respect to variable t.

Substitute Eq. (2) into NEE Eq. (1), then integrate over x twice, the equation reads

$$-2c_t q_y - 2cq_{xxxy} + 2c^2 q_{xx}q_{xy} - 2cq_{yt} + 3cq_{xz} = 0. \tag{3}$$

Because we aim to find the bilinear form of the NEE Eq. (1), which means the first term $-2c_t q_y$ has to be canceled out, c has to be fixed as a constant. Moreover, the terms $-2cq_{xxxy}$ and $2c^2 q_{xx}q_{xy}$ can be related to the known P- polynomial[38–40] by

$$-2cq_{xxxy} + 2c^2 q_{xx}q_{xy} = -6P_{3x,y}(q) = -6(q_{xxxy} + 3q_{xx}q_{xy}),$$

where P is the P-polynomial if $c = -3$. Thus c has to be selected as -3 which leads to Eq. (3) to be transformed to the binary Bell polynomials form

$$-2P_{3x,y}(q) - 2P_{yt}(q) + 3P_{xz}(q) = 0. \tag{4}$$

Due to the relation of binary Bell polynomials and bilinear operators, equation (4) can be written in the bilinear formalism

$$(-2D_tD_y - 2D_x^3D_y + 3D_zD_x)f\cdot f = 0, \tag{5}$$

where the bilinear operators D_t, D_x, D_y, and D_z are defined as[41]

$$\begin{aligned} &D_x^m D_y^n D_z^l D_t^k f\cdot g \\ &= \lim_{x=x',y=y'z=z',t=t'} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^l \\ &\quad\times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k f(x,y,z,t)\cdot g(x',y',z',t'), \end{aligned}$$

and f and g are real functions with respect to variable x, y, z, and t.

Based on the property of the binary Bell polynomials and the bilinear operator, let $q = 2\ln(f)$, then u is related to f by

$$u = -6(\ln f)_{xx}. \tag{6}$$

This means the $(3+1)$-dimensional NEE Eq. (1) possesses the bilinear formalism Eq. (5) under the transformation Eq. (6). In other words, the bilinear equation (5) is equivalent to NEE Eq. (1) by the transformation $u = -6(\ln f)_{xx}$. Once the solution f to the bilinear equation (5) is found, the solution to the NEE Eq. (1) is also obtained by the transformation Eq. (6).

3. Lump solution to the reduced NEE

To construct the lump solutions of the NEE Eq. (1), we take a special reduction: $z = x$. It is easy to find when $z = x$, the bilinear equation Eq. (5) becomes

$$(-2D_tD_y - 2D_x^3D_y + 3D_x^2)f\cdot f = 0, \tag{7}$$

which is equivalent to the following form

$$\begin{aligned} &-2f_{ty}f + 2f_tf_y - 2f_{xxxy}f + 6f_{xxy}f_x - 6f_{xx}f_{xy} \\ &+ 2f_{xxx}f_y + 3f_{xx}f - 3f_x^2 = 0. \end{aligned} \tag{8}$$

Therefore, if the solution f solves the bilinear equation (7) or Eq. (8), $u = -6\ln(f)_{xx}$ is also the solution to the NEE Eq. (1) for $z = x$.

3.1. Lump solutions

In this section, we seek lump solutions of the equivalent equation (8) in the form of

$$f = \sum_{i\le j=0}^{3} a_{ij}x_ix_j + f_0, \tag{9}$$

with $x_1 = x$, $x_2 = y$, $x_3 = t$, $x_0 = 1$, and a_{ij}, f_0 being constants to be determined. Alternatively, f in expansion form is

$$\begin{aligned} f \ =\ & a_{11}x^2 + 2a_{12}xy + 2a_{13}xt + a_{22}y^2 + 2a_{23}yt + a_{33}t^2 \\ & + 2a_{01}x + 2a_{02}y + 2a_{03}t + a_{00} + f_0, \end{aligned} \tag{10}$$

which contains 11 parameters for a_{ij}, $i \le j$ with $i,\ j = 0,\ 1,\ 2,\ 3$, and f_0.

Substituting Eq. (10) into Eq. (8) and collecting the coefficients of $\{x,\ y,\ t\}$ yields ten equations to determine the parameters of a_{ij}, $i \le j$ with $i,\ j = 0,\ 1,\ 2,\ 3$, and f_0

$$3a_{11}a_{13} - 2a_{12}a_{33} = 0, \tag{11}$$
$$3a_{11}a_{12} - 2a_{13}a_{22} = 0, \tag{12}$$
$$3a_{11}^2 + 2a_{11}a_{23} - 4a_{13}a_{12} = 0, \tag{13}$$
$$-3a_{11}a_{33} + 6a_{13}^2 - 2a_{33}a_{23} = 0, \tag{14}$$
$$-3a_{11}a_{22} + 6a_{12}^2 - 2a_{23}a_{22} = 0, \tag{15}$$
$$-3a_{03}a_{11} + 6a_{01}a_{13} - 2a_{33}a_{02} = 0, \tag{16}$$
$$-2a_{03}a_{22} + 6a_{01}a_{12} - 3a_{11}a_{02} = 0, \tag{17}$$
$$-3a_{11}a_{23} + 6a_{13}a_{12} - 2a_{33}a_{22} = 0, \tag{18}$$
$$-2a_{03}a_{12} + 3a_{01}a_{11} + 2a_{01}a_{23} - 2a_{13}a_{02} = 0, \tag{19}$$
$$\begin{aligned} & -3a_{00}a_{11} + 2a_{00}a_{23} - 4a_{03}a_{02} - 3f_0a_{11} \\ & + 2f_0a_{23} + 6a_{01}^2 + 12a_{11}a_{12} = 0. \end{aligned} \tag{20}$$

One can easily prove the following five constraints of a_{i3} $(i = 0,\ 1,\ 2,\ 3)$, and f_0

$$a_{03} = \frac{6a_{01}a_{12} - 2a_{11}a_{02}}{2a_{22}}, \tag{21}$$
$$a_{13} = \frac{3}{2}\frac{a_{12}a_{11}}{a_{22}}, \tag{22}$$
$$a_{23} = -\frac{3}{2}\frac{a_{11}a_{22} - 2a_{12}^2}{a_{22}}, \tag{23}$$
$$a_{33} = \frac{9}{4}\frac{a_{11}^2}{a_{22}}, \tag{24}$$
$$f_0 = -a_{00} + \frac{a_{01}^2a_{22} - 2a_{01}a_{12}a_{02} + 2a_{11}a_{12}a_{22} + a_{11}a_{02}^2}{a_{11}a_{22} - a_{12}^2}, \tag{25}$$

solve the ten equations (11)–(20) with a_{01}, a_{02}, a_{11}, a_{12}, a_{22} being arbitrary constants (a_{00} is canceled out according to Eq. (25)).

It is necessary to point out that in order to solve the ten determining equations (11)–(20), we use three non-zero conditions for $a_{11} \neq 0$, $a_{22} \neq 0$, and $a_{11}a_{22} - a_{12}^2 \neq 0$. If the non-zero conditions are not satisfied, we cannot find the lump solutions to the NEE Eq. (1).

Applying the results into Eq. (10), we find that

$$\begin{aligned} f_1 \ =\ & a_{11}x^2 + \frac{3a_{12}a_{11}}{a_{22}}xt + 2a_{12}xy + \frac{9a_{11}^2}{4a_{22}}t^2 \\ & - \frac{3(a_{11}a_{22} - 2a_{12}^2)yt}{a_{22}} + a_{22}y^2 \\ & + \frac{3(2a_{01}a_{12} - a_{11}a_{02})}{a_{22}}t + 2a_{01}x + 2a_{02}y \\ & + \frac{a_{01}^2a_{22} - 2a_{01}a_{12}a_{02} + 2a_{11}a_{12}a_{22} + a_{11}a_{02}^2}{a_{11}a_{22} - a_{12}^2}, \end{aligned} \tag{26}$$

then the lump solution u to the NEE Eq. (1) with $z = x$ is obtained

$$u = -\frac{12a_{11}}{f_1} + \frac{6(2a_{11}x + \frac{3a_{12}a_{11}}{a_{22}}t + 2a_{12}y + 2a_{01})^2}{f_1^2}, \tag{27}$$

with a_{01}, a_{02}, a_{11}, a_{12}, a_{22} being arbitrary constants and three non-zero conditions for $a_{11} \neq 0$, $a_{22} \neq 0$, and $a_{11}a_{22} - a_{12}^2 \neq 0$.

We can conclude that the solutions (26) and (27), containing ten parameters with five constraints, and three non-zero conditions are general lump solutions to NEE Eq. (1) for $z = x$. The known lump solutions in Ref. [42], which contain nine parameters with three constraints and two non-zero conditions can be considered as a special case of our results.

The results show that in order to solve the ten determining equations (11)–(20), we need only five constraints conditions and two non-zero conditions for $a_{22} \neq 0$ and $a_{11}a_{22} - a_{12}^2 \neq 0$. Thus it motivates us to finding similar solutions which may contain more parameters with less constraint conditions.

3.2. Lump solutions with more freedom

To contain more parameters in the solution, we take a special ansatz

$$f = \sum_{i \le j = 0}^{3} a_{ij}x_ix_j + f_0, \tag{28}$$

with $x_1 = x$, $x_2 = y$, $x_3 = t$, and $x_0 = 1$, and a_{ij}, $i \le j$, $i,\ j = 0,\ 1,\ 2,\ 3$ is defined as

$$a_{ij} = \langle A_i | A_j \rangle = \sum_{m=1}^{n} A_{im}A_{jm}, \tag{29}$$

where

$$|A_1\rangle = \boldsymbol{k},\ \ |A_2\rangle = \boldsymbol{p},\ \ |A_3\rangle = \boldsymbol{\omega},\ \ |A_4\rangle = \boldsymbol{\alpha}, \tag{30}$$

are vectors and k_m, p_m, ω_m, and α_m are constants to be determined.

Substituting Eqs. (28)–(30) into Eq. (8) and eliminating all the coefficients of $\{x,\ y,\ t\}$, we find the same ten determining equations (11)–(20) with the five solutions of Eqs. (21)-(25). But further calculations show that the following two constraints

$$\omega_m = \frac{3}{2}\frac{-a_{11}p_m + 2a_{12}k_m}{a_{22}}, \tag{31}$$
$$f_0 = -a_{00} + \frac{a_{01}^2a_{22} - 2a_{01}a_{12}a_{02} + 2a_{11}a_{12}a_{22} + a_{11}a_{02}^2}{a_{11}a_{22} - a_{12}^2}, \tag{32}$$

solve the five solutions of Eqs. (21)–(25) with the new definition of a_{ij}!

It is plausible that our work is similar to others, but using two requirements of ω_m and f_0, we actually decrease the constraint conditions. And we will check that the solutions have a maximum number of free parameters, in other words, the arbitrary parameters are meaningless when n is no limiting.

First we consider $n=2$. It can be seen that there are nine parameters totally, including k_1, k_2, p_1, p_2, ω_1, ω_2, α_1, α_2, and f_0. But according to Eq. (31), ω_1 and ω_2 are determined by

$$\omega_1 = \frac{3(k_1^2 p_1 + 2k_1 k_2 p_2 - k_2^2 p_1)}{2(p_1^2+p_2^2)}, \tag{33}$$

$$\omega_2 = \frac{3(k_2^2 p_2 + 2k_1 k_2 p_1 - k_1^2 p_2)}{2(p_1^2+p_2^2)}, \tag{34}$$

and f_0 is

$$f_0 = \frac{2(k_1^2+k_2^2)(k_1 p_1 + k_2 p_2)(p_1^2+p_2^2)}{(k_1 p_2 - k_2 p_1)^2}. \tag{35}$$

That means there are six arbitrary independent constants and three constraints of Eqs. (33)–(35) for $n=2$.

Then we take $n=3$, the solution contains 13 parameters for $\{k_m,\ p_m,\ \omega_m,\ \alpha_m\}$, $m=1,\ 2,\ 3$, and f_0 but only 11 parameters are independent according to Eq. (28). The constraint conditions of ω_i and f_0 are provided as Eqs. (31) and (32), respectively

$$\omega_1 = \frac{3[p_1(+k_1^2-k_2^2-k_3^2)+2k_1(k_2p_2+k_3p_3)]}{2(p_1^2+p_2^2+p_3^2)}, \tag{36}$$

$$\omega_2 = \frac{3[p_2(-k_1^2+k_2^2-k_3^2)+2k_2(k_1p_1+k_3p_3)]}{2(p_1^2+p_2^2+p_3^2)}, \tag{37}$$

$$\omega_3 = \frac{3[p_3(-k_1^2-k_2^2+k_3^2)+2k_3(k_1p_1+k_2p_2)]}{2(p_1^2+p_2^2+p_3^2)}, \tag{38}$$

$$\begin{aligned} f_0 =& -[(k_1p_2-k_2p_1)^2+(k_2p_3-k_3p_2)^2 \\ &+(k_1p_3-k_3p_1)^2]^{-1}\{[\alpha_1(k_2p_3-k_3p_2) \\ &+\alpha_2(k_3p_1-k_1p_3)+\alpha_3(k_1p_2-k_2p_1)]^2 \\ &-2(k_1^2+k_2^2+k_3^2)(k_1p_1 \\ &+k_2p_2+k_3p_3)(p_1^2+p_2^2+p_3^2)\}. \end{aligned} \tag{39}$$

It means that we have seven independent free parameters with four constraints in this case.

For $n\geq 4$ case, because the numbers of constraint conditions for ω_k and f_0 is more than or equal to 5, the total numbers of independent parameters are less than or equal to 6. Compared with $n=3$ case, the numbers of independent arbitrary parameters decreases. Thus it is impossible for us to find more arbitrary free parameters in the lump solutions for $n\geq 4$.

So we conclude that the solution of Eqs. (28)–(30) for $n=3$ with constraint conditions (36)–(39) is a more general lump solution which includes seven arbitrary independent parameters. Then the solution f to Eq. (8) for $n=3$ is

$$\begin{aligned} f_{\rm lump} \equiv& a_{11}x^2+2a_{12}xy+2a_{13}xt+a_{22}y^2+2a_{23}yt \\ &+a_{33}t^2+2a_{01}x+2a_{02}y+2a_{03}t \\ &+\frac{a_{01}^2a_{22}-2a_{01}a_{12}a_{02}+2a_{11}a_{12}a_{22}+a_{11}a_{02}^2}{a_{11}a_{22}-a_{12}^2}, \end{aligned} \tag{40}$$

where a_{ij}, $i\leq j$, are determined by Eq. (29). Then the corresponding lump solution is written as

$$u_{\rm lump} = -\frac{12a_{11}}{f_{lump}} + \frac{6(2a_{11}x+\frac{3a_{12}a_{11}}{a_{22}}t+2a_{12}y+2a_{01})^2}{f_{lump}^2}. \tag{41}$$

According to Eq. (6), we have to require $f>0$ which leads to requirement of $f_0>0$ to insure u analyticity. From the special expressions of f_0 given by Eq. (35) for $n=2$ and Eq. (39) for $n=3$, we know the simplest condition is $a_{12}>0$.

Based on Eq. (41), we can find the physical properties of a lump much more concretely. It is easy to check that the position of the lump where the maximum value of a lump is

$$\begin{aligned} x_0 =& -\frac{a_{13}a_{22}t-a_{12}a_{23}t+a_{01}a_{22}-a_{12}a_{02}}{a_{11}a_{22}-a_{12}^2}, \\ y_0 =& \frac{-a_{11}a_{23}t+a_{12}a_{13}t+a_{01}a_{12}-a_{11}a_{02}}{a_{11}a_{22}-a_{12}^2}, \end{aligned} \tag{42}$$

by some simple differential calculations of $\{u_x=0,\ u_y=0\}$. That means the lump moves along the straight line

$$y = \frac{(a_{11}a_{23}-a_{13}a_{12})x+a_{01}a_{23}-a_{13}a_{02}}{a_{13}a_{22}-a_{12}a_{23}}, \tag{43}$$

with a constant velocity of

$$\begin{aligned} \boldsymbol{v} =& \frac{1}{a_{11}a_{22}-a_{12}^2}[(-a_{13}a_{22}+a_{12}a_{23})\boldsymbol{i} \\ &+(-a_{11}a_{23}+a_{12}a_{13})\boldsymbol{j}]. \end{aligned} \tag{44}$$

And it is not difficult to verify that the amplitude of a lump is

$$A = \left|-\frac{6(a_{11}a_{22}-a_{12}^2)}{a_{22}a_{12}}\right|, \tag{45}$$

which reveals that the amplitude is also a constant.

A figure of a lump is shown in Fig. 1 with the arbitrary parameters chosen as

$$\begin{aligned} &\alpha_1=1,\ \alpha_2=0,\ \alpha_3=1,\ k_1=1,\ k_2=2,\ k_3=1, \\ &p_1=2,\ p_2=-1,\ p_3=1, \end{aligned} \tag{46}$$

panel (a) shows the structure at $t=0$ and panel (b) is the corresponding projective density plot of u. It is clear that the corresponding highest point is located at

$$x=-\frac{t}{2}-\frac{9}{35},\quad y=\frac{3t}{2}-\frac{16}{35}. \tag{47}$$

So the solution is moving with a constant speed of

$$\boldsymbol{v}=-\frac{1}{2}\boldsymbol{i}+\frac{3}{2}\boldsymbol{j}, \tag{48}$$

along the straight line

$$y=-3x-\frac{43}{35}. \tag{49}$$

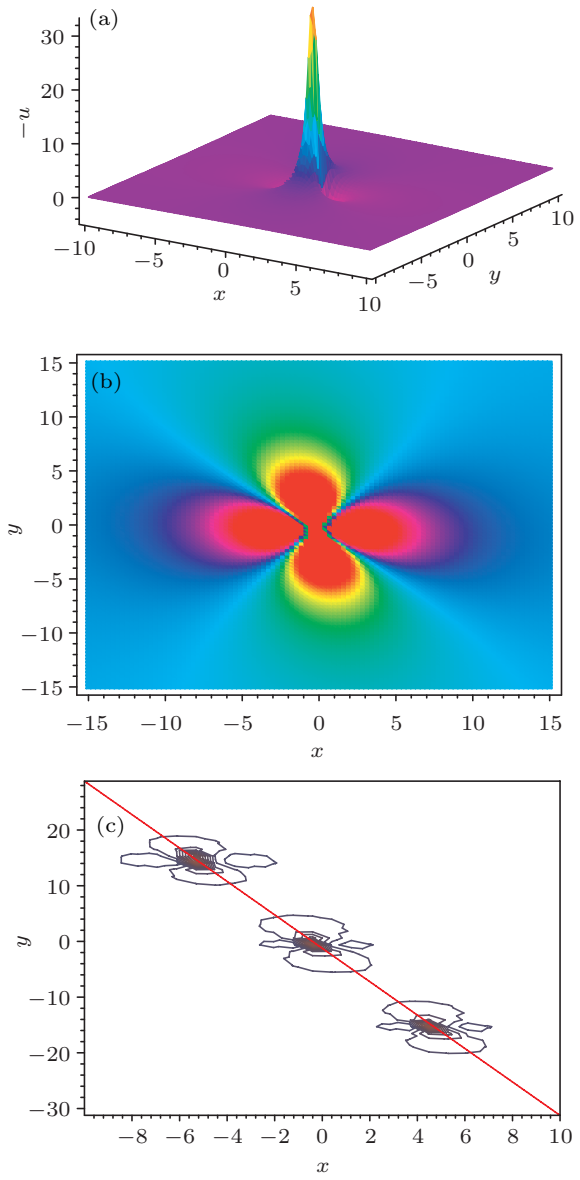


Fig. 1. (color online) The exhibition of the lump solution determined by Eq. (41) with the parameters chosen in Eq. (46). Panel (a) shows the lump structure at $t = 0$. Panel (b) is the projective density plot of u at $t = 0$. Panel (c) is the path of the lump at $t = -10, 0, 10$ with the red line being Eq. (49).

It should be pointed out that when $z = y$, we can also find the lump solution of the NEE. One can easily find that the lump solution in this case contains five arbitrary constants which is much more simple than the case $z = x$. Here we do not show it in detail.

4. Lumpoff solution to NEE

A lumpoff is a cutoff lump, which is a lump cut by something (say, a soliton) before or after a special time (off-time) that means the lump exists only before or after the off-time. In this section, we focus on the lumpoff solution to the reduced NEE.

To find the lumpoff solution, we take the ansatz

$$f_{\text{lumpoff}} \equiv f_{\text{lump}} + a_0 \exp(k_0 x + p_0 y + \omega_0 t + x_0), \tag{50}$$

where a_0, k_0, p_0, ω_0, and x_0 are constants to be determined. It can be seen that the lump will only emerge at a special area $k_0 x + p_0 y + \omega_0 t + x_0 < 0$ because the exponential part is dominant for the area $k_0 x + p_0 y + \omega_0 t + x_0 > 0$. That means when $k_0 x + p_0 y + \omega_0 t + x_0 < 0$, the lump will be visible, otherwise, the lump will be invisible for $k_0 x + p_0 y + \omega_0 t + x_0 > 0$. Thus the lump is cutoff or blocked by the exponential part (actually, a soliton) before or after a special time which becomes a lumpoff.

With a direct calculation by substituting the ansatz Eq. (50) into the bilinear form Eq. (8), we find that k_0, p_0, and ω_0 satisfy the following equations

$$\omega_0 = -k_0^3 + \frac{3k_0^2}{2p_0}, \tag{51}$$

$$p_0 = \frac{(a_{22}k_0^2 + a_{12})k_0}{a_{11}}, \tag{52}$$

$$k_0^4 = \frac{a_{11}a_{22} - a_{12}^2 - 2a_{12}a_{22}k_0^2}{a_{22}^2}, \tag{53}$$

where a_{11}, a_{12}, and a_{22} are defined in Eq. (29), with a_0 and x_0 being arbitrary constants.

It is interesting that the results demonstrate a special soliton induced by lump. Equation (51) is a special dispersive relation which indicates ω_0 being related to k_0 and p_0, while k_0 and p_0 are completely determined by the lump according to Eqs. (52) and (53). Thus the soliton (the exponential part) is induced by the lump. The existence of the soliton is based on the existence of the lump. If the lump does not exist, the soliton will also disappear. Once the soliton is induced, due to the domination of the exponential part, the lump will be invisible. The lump is thus cutoff by the soliton induced by itself.

According to Eq. (53), we have two cases: $k_0 > 0$ and $k_0 < 0$, which show two different ways for the lump cutoff. One is the lump is cut by the induced soliton so that the lump will be invisible after a special time (off-time), the other is that the lump is blocked by the induced soliton before the off-time and the lump will be visible eventually.

Based on the transformation $u = -6(\ln f)_{xx}$, we obtain the lumpoff solution of the NEE for $z = x$

$$\begin{aligned} u_{\text{lumpoff}} &= \frac{6[2a_{11}x + 2a_{13}t + 2a_{12}y + 2a_{01} + a_0k_0\exp(\xi)]^2}{f_{\text{lumpoff}}^2} \\ &\quad - \frac{6[2a_{11} + a_0k_0^2\exp(\xi)]}{f_{\text{lumpoff}}}, \end{aligned} \tag{54}$$

where $\xi \equiv k_0 x + p_0 y + \omega_0 t + x_0$, and k_0, p_0, and ω_0 are shown in Eqs. (51) and (52).

For instance, if we select the arbitrary parameters as

$$\begin{aligned} &\alpha_1 = 1, \ \alpha_2 = 0, \ \alpha_3 = 1, \ k_1 = 1, \ k_2 = 2, \ k_3 = 1, \\ &p_1 = 2, \ p_2 = -1, \ p_3 = 1, \ a_0 = 2, \ x_0 = 0, \end{aligned} \tag{55}$$

then k_0, p_0, and ω_0 provided by Eqs. (52) and (53) are

$$k_0 = \frac{30}{6}, \quad p_0 = \frac{30}{6}, \quad \omega_0 = \frac{30}{9}, \tag{56}$$

and

$$k_0 = -\frac{30}{6}, \quad p_0 = -\frac{30}{6}, \quad \omega_0 = -\frac{30}{9}, \tag{57}$$

then we obtain two kinds of lumopff solutions. Equations (55), (56), and (52) describe a special lump being cutoff by the induced soliton. Figure 2 is the evolution plot of the lumopff at (a) $t = -15$, (b) $t = 0$, (c) $t = 5$, and (d) $t = 25$, respectively. The lump is cutoff by the induced soliton and eventually is invisible.

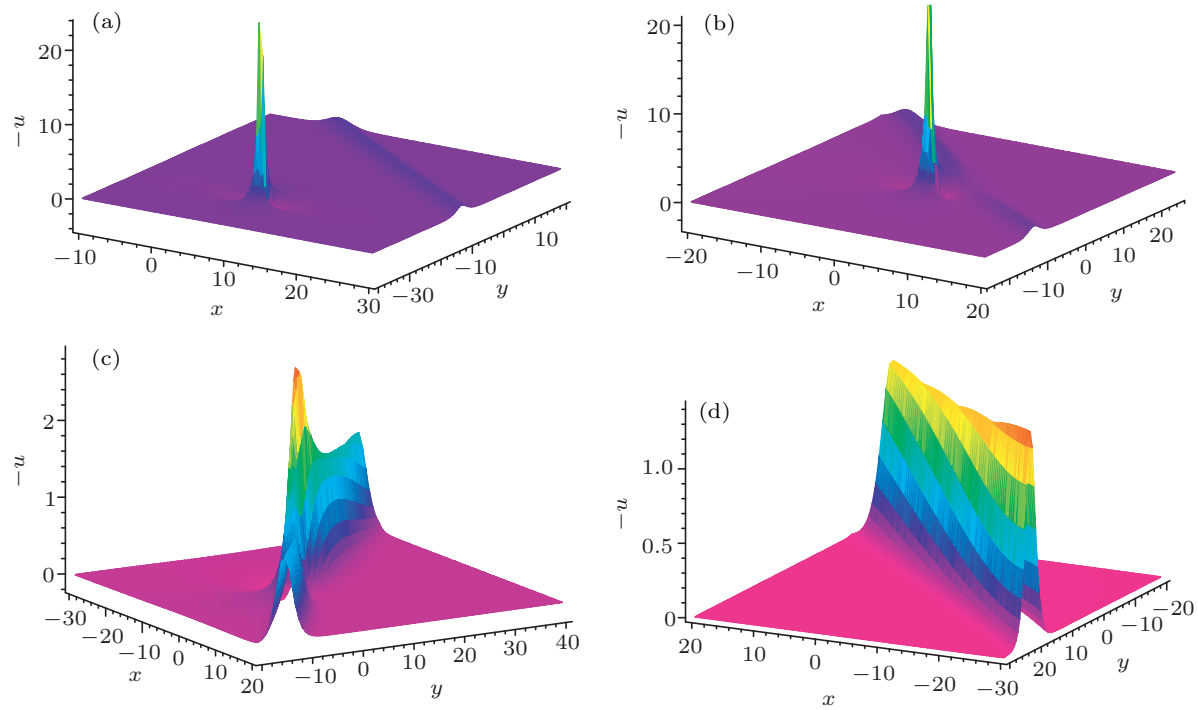


Fig. 2. (color online) Evolution plots of lumpoff solution by choosing Eqs. (55) and (56) at time (a) $t = -15$, (b) $t = 0$, (c) $t = 5$, and (d) $t = 25$, respectively which shows that the lump is cutoff by the induced soliton and eventually is invisible.

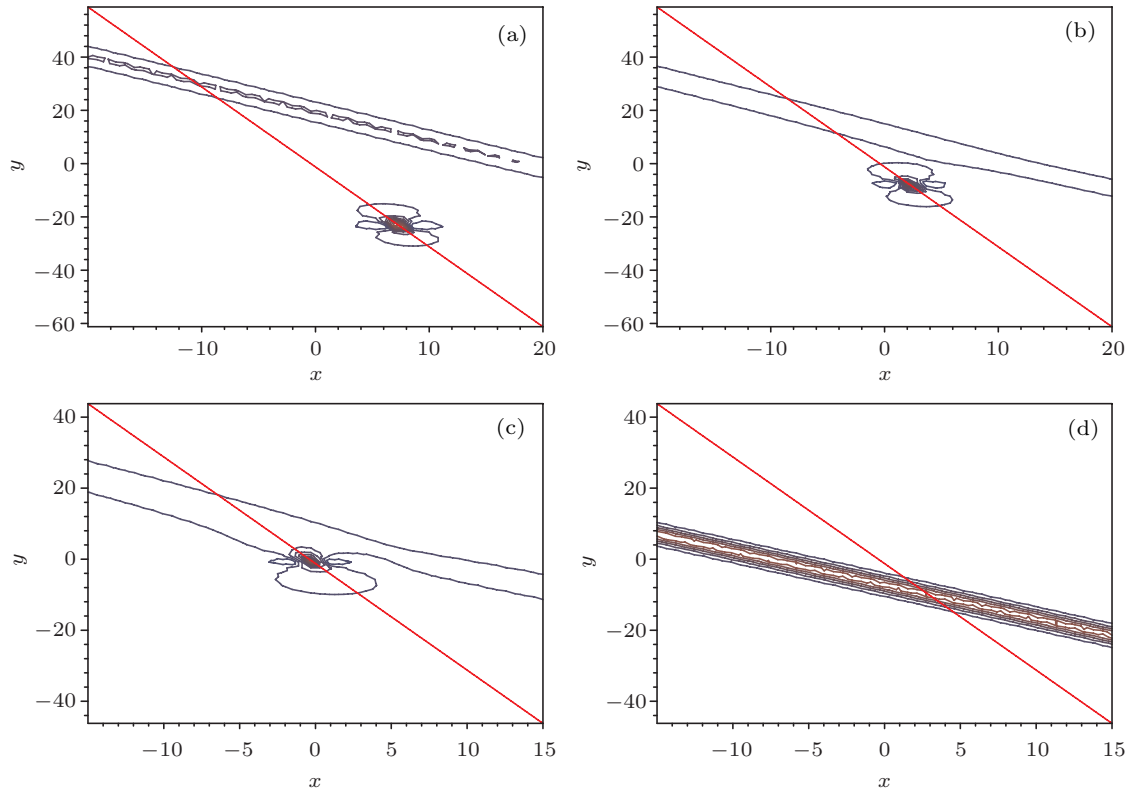


Fig. 3. (color online) The path and the positions of the lump at (a) $t = -15$, (b) $t = -5$, (c) $t = 0$, and (d) $t = 25$.

Because the lump part remains unchanged, the corresponding positions of the lump are Eq. (49). Figure 3 reveals the positions of the lump at (a) $t=-15$, (b) $t=-5$, (c) $t=0$, and (d) $t=25$, respectively with the red being Eq. (49).

Figure 4 is the density plots of the lumpoff provided by Eqs. (55), (57), and (52) which describe a special lump separates out from the induced soliton. The induced soliton blocks the lump which makes the lump be invisible at (a) $t=-15$. After the off-time, the lump is visible at (b) $t=-2$, (c) $t=2$, and (d) $t=10$, respectively. The lump is invisible at first and then separates out from the induced soliton.

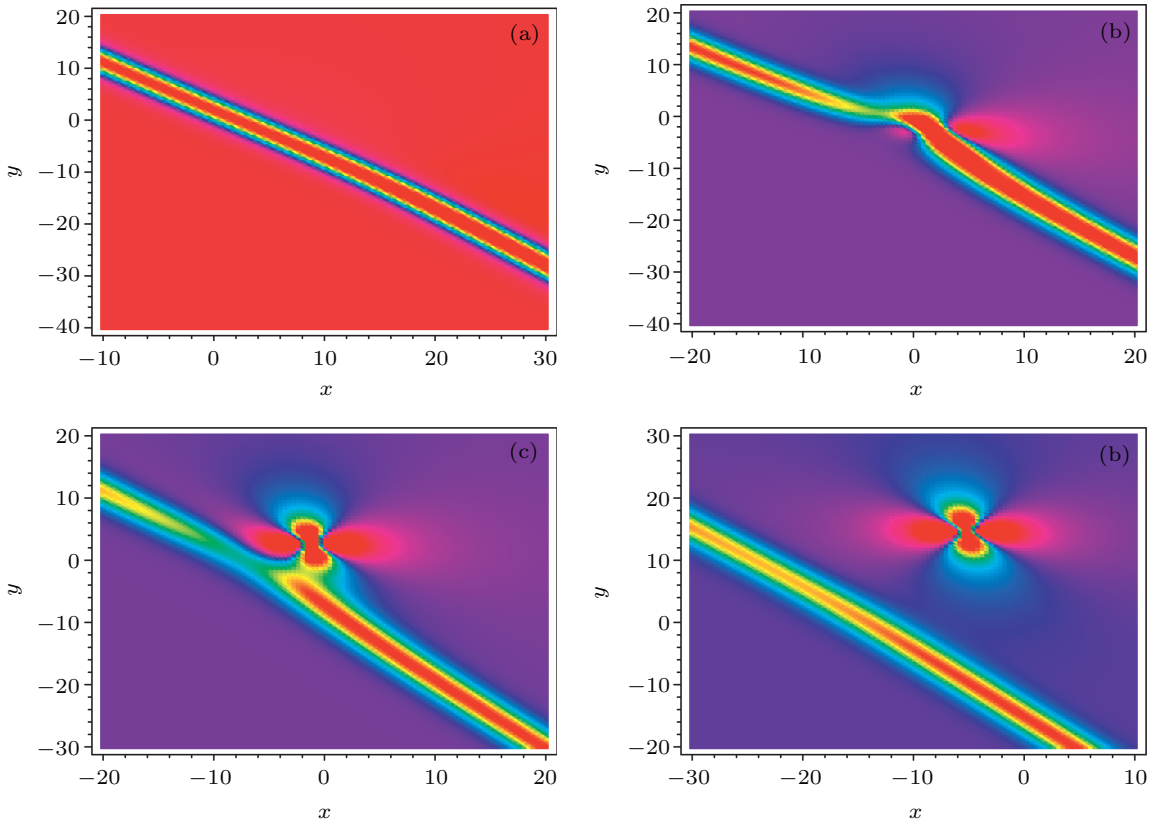


Fig. 4. (color online) The density plots of lumpoff solution with parameters selected in Eqs. (55) and (57) at (a) $t=-15$, (b) $t=-2$, (c) $t=2$, and (d) $t=10$, respectively. After the off-time, the lump is visible all the time.

5. Instanton/rogue wave solutions

The lumpoff solution is a lump cutoff by a induced line-soliton. A lump can also be confined between two off-times, forward-off-time (tf) and backward-off-time (tb). In other words, the lump appears at tf and disappears at tb $\geq$ tf. As known to all, an instanton/rogue wave is a localized wave decayed in all space and time directions. Thus, an algebraic rogue wave/instanton can also be produced by cutting a lump between two off-times, forward-off-time (tf) and backward-off-time (tb).

To obtain the instanton/rouge wave solutions to the NEE, we assume f is in the form of

$$f_{\text{instanton}} \equiv f_{\text{lump}} + a_1 \cosh(k_0 x + p_0 y + \omega_0 t + x_0) + g_0, \quad (58)$$

where a_1 and g_0 are two arbitrary constants to be determined, and $\{k_0,\ p_0,\ \omega_0\}$ are provided by Eqs. (51)–(53). Substituting Eq. (58) and the known results into the bilinear equation Eq. (8), we find the equation is reduced to

$$2a_{11}a_{22}(a_{11}a_{22}-a_{12}^2)g_0 + a_1^2(a_{11}a_{12}a_{22}-a_{12}^3-a_{11}a_{22}^2k_0^2-a_{12}a_{22}k_0^2)=0. \quad (59)$$

Thus the only one possible solution is

$$g_0 = \frac{a_1^2(a_{11}a_{22}+a_{12}^2)k_0^2}{2a_{11}(a_{11}a_{22}-a_{12}^2)} - \frac{a_1^2 a_{12}}{2a_{11}a_{22}}, \quad (60)$$

with a_1 being an arbitrary constant. Then the instanton/rouge wave solution to NEE with $z=x$ is found by the transformation $u=-6(\ln f)_{xx}$,

$$\begin{aligned} u_{\text{instanton}} &= 6[2a_{11}x+2a_{13}t+2a_{12}y+2a_{01} \\ &\quad + a_1k_0\sinh(k_0x+p_0y+\omega_0t+x_0)]^2/f_{\text{instanton}}^2 \\ &\quad - \frac{6[2a_{11}+a_1k_0^2\cosh(k_0x+p_0y+\omega_0t+x_0)]}{f_{\text{instanton}}}. \end{aligned} \quad (61)$$

We obtain the instanton/rouge wave solution Eq. (61) by cutting the lump between two off-times. Due to the existence of the lump, two line solitons are induced according to the special dispersion relation Eq. (51) which is visible all the time because of the domination of the cosh part. The visible solitons lead to the invisible of the lump, thus the lump is visible

only when it moves to the line $k_0x+p_0y+\omega_0t+x_0\sim 0$. Once the lump reaches a large amplitude, it will become a rogue wave, or be an instanton for general amplitudes.

Moreover, the maximum value of wave hight of the instanton/rogue wave is

$$A_{instanton}=\left|-\frac{12\sqrt{a_{11}a_{22}}(a_{11}a_{22}-a_{12}^2)}{a_{22}[a_1(\sqrt{a_{11}a_{22}}-a_{12})+2a_{12}\sqrt{a_{11}a_{22}}]}\right|, \quad (62)$$

which indicates that the amplitude is related to the soliton's parameter a_1 and the lump part of a_{11}, a_{12}, a_{22} by calculating the value of u when the lump arrives at the center of the twin-soliton.

Our results show novel generating and prediction mechanism for this kind of rogue waves. Because the twin-soliton includes enough information (k_0, p_0, and ω_0) of the invisible lump (algebraic) part (a_{11}, a_{22}, and a_{12}), the wave height of the rogue wave may be known.

Because the lump part remains unchanged, once the lump emerges out from the soliton, it will keep its path and velocity until it meets the other soliton.

For example, we take the correlative data as

$$\alpha_1=1,\ \alpha_2=0,\ \alpha_3=\frac{1}{3},\ k_1=1,\ k_2=\frac{1}{2},\ k_3=-1,$$
$$p_1=\frac{1}{2},\ p_2=-1,\ p_3=\frac{1}{2},\ a_1=2,\ x_0=1, \quad (63)$$

a special rogue wave solution of the NEE is obtained with the maximum value of wave hight being about 16.19 according to Eq. (62). Figure 5 exhibits the special rogue wave at times (a) $t=-25$, (b) $t=-5$, (c) $t=0$, (d) $t=5$, and (e) $t=25$, and panel (f) is the wave height in $y=0$ for $t=-25$ in blue, $t=0$ in red, and $t=25$ in green which shows the huge wave height of the rogue wave. Figure 6 exhibits the path of the rogue wave at (a) $t=-25$, (b) $t=-5$, (c) $t=0$, (d) $t=5$, and (e) $t=25$, with (f) being the density plot at $t=0$.

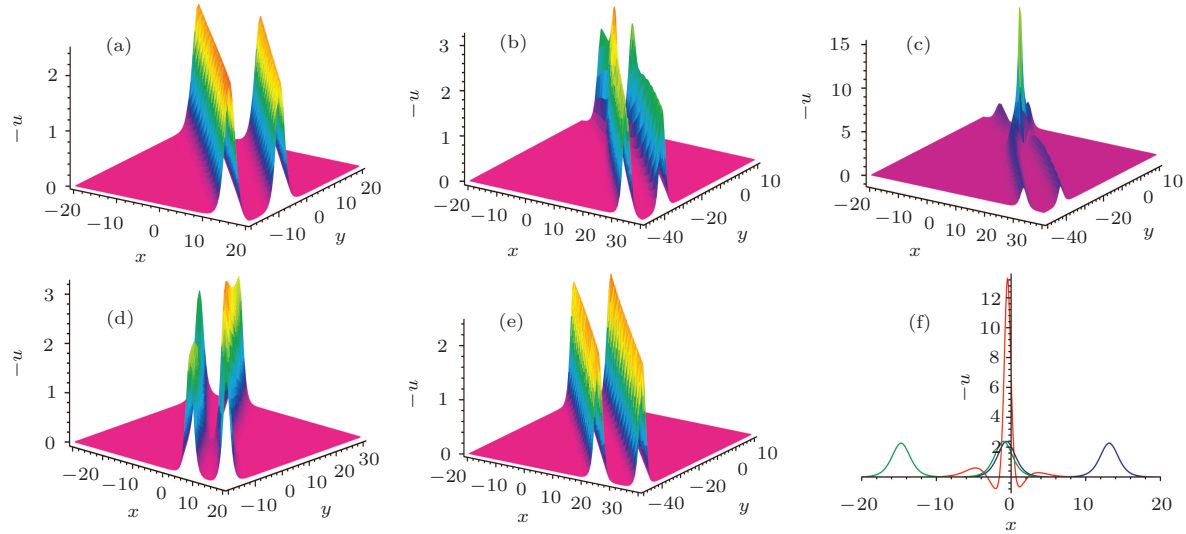


Fig. 5. (color online) Evolution plots of rogue wave solution by choosing Eq. (63) at times (a) $t=-25$, (b) $t=-5$, (c) $t=0$, (d) $t=5$, and (e) $t=25$, respectively. Panel (f) shows the wave height in $y=0$ for $t=-25$ in blue, $t=0$ in red, and $t=25$ in green.

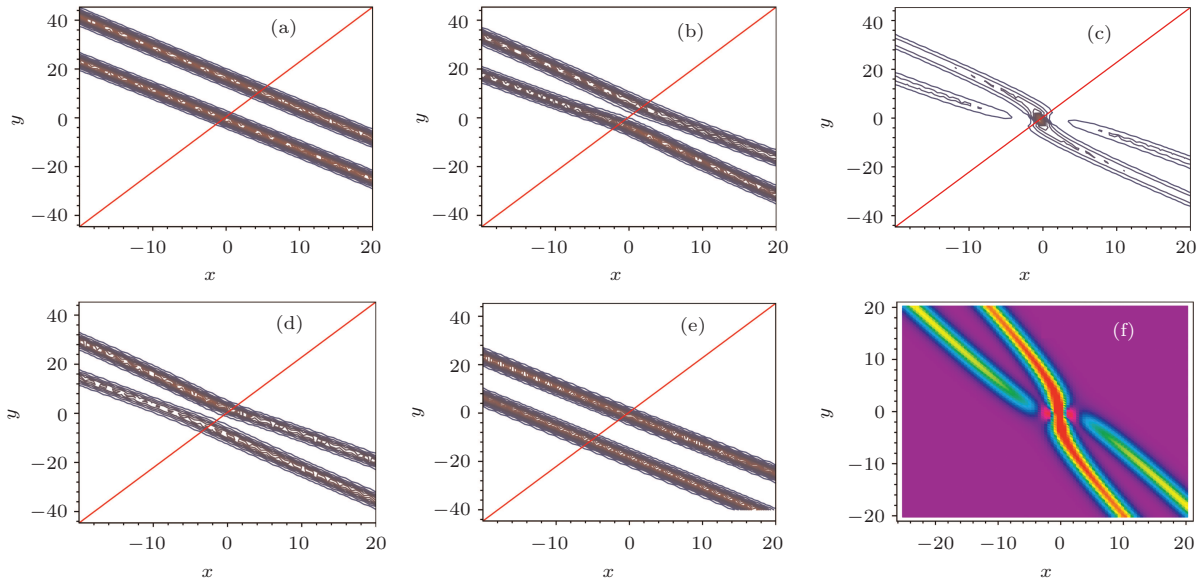


Fig. 6. (color online) The path of the lump is shown by choosing Eq. (63) at times (a) $t=-25$, (b) $t=-5$, (c) $t=0$, (d) $t=5$, and (e) $t=25$, respectively. Panel (f) is the density plot of the rogue wave at $t=0$.

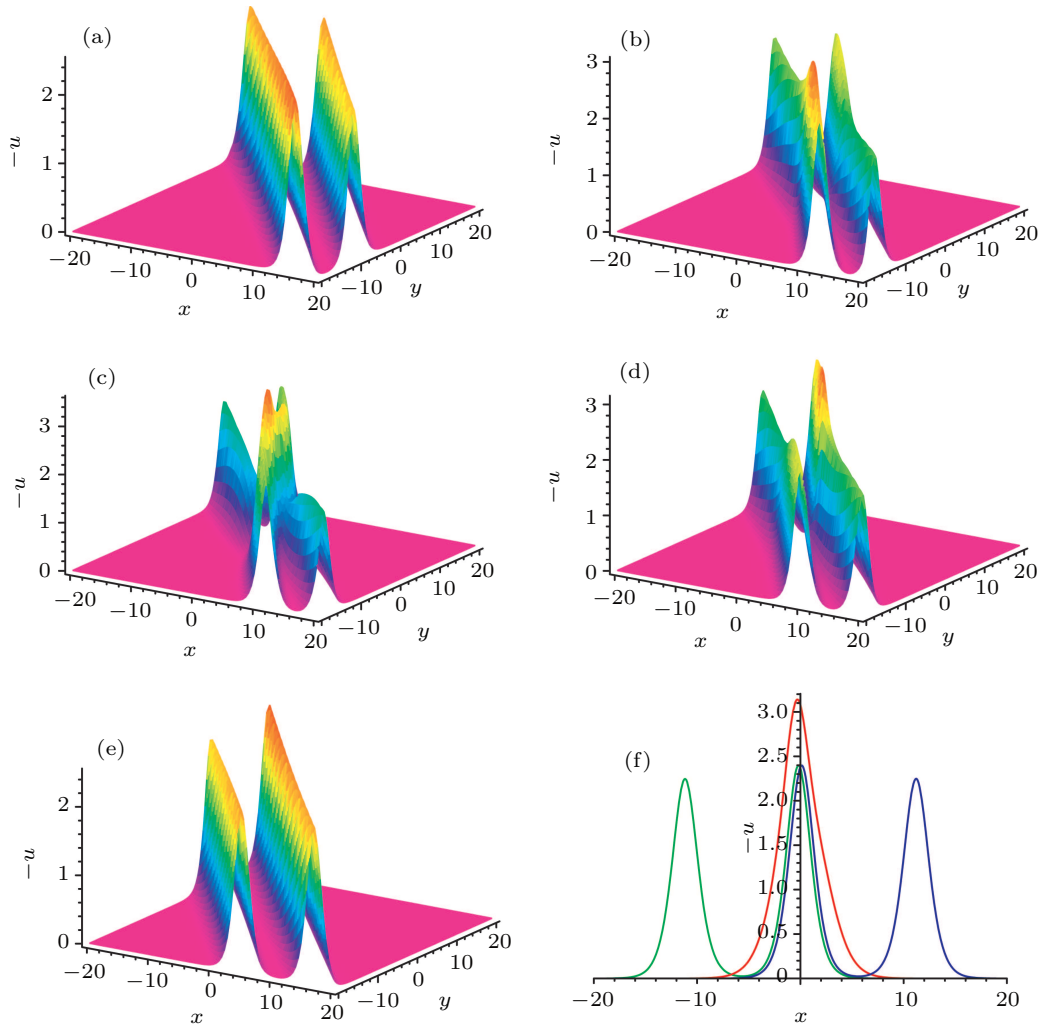


Fig. 7. (color online) The evolution plots of instanton solution with small amplitude at (a) $t=-20$, (b) $t=-5$, (c) $t=0$, (d) $t=5$, and (e) $t=20$, respectively. Panel (f) is the wave height in $y=0$ for $t=-20$ in blue, $t=0$ in red and $t=20$ in green.

For the general amplitude, we obtain the instanton solution to the NEE. If we choose $a_1=7$, $x_0=0$ with other parameters invariant in Eq. (63), an instanton solution is found. Figure 7 exhibits the instanton solution at times (a) $t=-20$, (b) $t=-5$, (c) $t=0$, (d) $t=5$, and (e) $t=20$, respectively. Panel (f) is the wave height in $y=0$ for $t=-20$ in blue, $t=0$ in red, and $t=20$ in green.

6. Summary and discussions

In summary, a $(3+1)$-dimensional nonlinear evolution equation (NEE) is studied in this paper. The bilinear formalism is constructed by binary Bell polynomials theories, and then a lump solution is found to the special NEE for $z=x$. With the help of the obtained lump solution, a more general form of lump solution is found. The more general form of lump equations possesses seven arbitrary parameters and four constraint conditions. Furthermore, by cutting the lump by the induced solitons, the lumpoff and instanton/rogue wave solutions are constructed.

It should be emphasized that the soliton is induced by the lump itself. A special dispersive relation is found for the soliton being determined by the lump. If the lump does not exist, the soliton will also disappear.

By cutting a lump between two off-times, the instanton/rogue wave solution is found. If the lump reaches a large amplitude, it will become a rogue wave, or be an instanton for general amplitudes. It is concluded that the amplitude of the wave height is related to both lump and the induced solitons. Because the visible two-solitons show all the information of the lump, it is possible to predict the exitance of a rogue wave.

It is interesting that our method is effective for other nonlinear system which provides a possible way to find this kind of rogue waves in physical systems due to the connection of the NEE and other systems. Meanwhile, the progress in understanding the physics of the instanton/rogue wave phenomenon and development of adequate mathematical models is very significant because it is possible to find instanton/rogue wave in physical systems.

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