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TOPICAL REVIEW — 80th Anniversary of Northwestern Polytechnical University (NPU)

Some new advance on the research of stochastic non-smooth systems*

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Due to the extensive applicability in real life, the non-smooth system with random factors attracted much attention in past two decades. A lot of methods and techniques have been proposed to research these systems by scholars. In this paper, we will summarize some new research advance on the stochastic non-smooth systems. The existing results about the stochastic vibro-impact system, the stochastic friction system, and the stochastic hysteretic system are introduced respectively. Some conclusions and outlook are given at the end.

Keywords: non-smooth system, random factor, vibro-impact system, friction system, hysteretic system

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1. Introduction

The non-smooth systems exist commonly in real life. Different from the smooth system, the non-smooth factors may lead a simple equation to bear a complicated response. Besides the non-smooth factors, the stochastic factors also exist ubiquitously in real life and often affect the fundamental dynamics of the systems. So the study on the dynamical system with both non-smooth factor and random factor will be of great research significance. In this paper, we will summarize the recent researches on some stochastic non-smooth systems, which include the vibro-impact system, the friction system, and the hysteretic system. The schematic diagrams of these systems are shown in Fig. 1.

From Fig. 1(a), we can see that the vibro-impact system means that the components of a system collide with each other or with the rigid obstacles. The jumps of the velocity can induce the non-smoothness, which is described as the discontinuities of the system state or its differential. Just because of this discontinuity, the response of the system will be very difficult to obtain, but the behavior of the system shows very interesting phenomena. The equation of the rigid vibro-impact system can be given by

$$\begin{aligned} &\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}+\boldsymbol{f}(\boldsymbol{x},\dot{\boldsymbol{x}})=\boldsymbol{F}(t), && x_1<\delta,\\ &\dot{x}_{1+}=-r\dot{x}_{1-}, && x_1=\delta. \end{aligned} \tag{1a}$$

If the impact surface is an elastic obstacle, this non-smooth system will be an elastic one, whose equation is

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}+\boldsymbol{g}(\boldsymbol{x},\dot{\boldsymbol{x}})=\boldsymbol{F}(t),$$

$$\boldsymbol{g}(\boldsymbol{x},\dot{\boldsymbol{x}})=\begin{cases}\boldsymbol{g_1}(\boldsymbol{x},\dot{\boldsymbol{x}}), & x_1>\delta,\\ \boldsymbol{g_2}(\boldsymbol{x},\dot{\boldsymbol{x}}), & x_1\le\delta.\end{cases} \tag{1b}$$

Here, $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are the nondimensional mass, damping, and stiffness matrixes, respectively, $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}$, and $\boldsymbol{F}=(F_1,F_2,\ldots,F_n)^{\mathrm{T}}$ is the external random excitation. If the value of x_1 reaches δ, the impact will happen. The subscripts "minus" and "plus" refer to the values just before and after the impact. $\boldsymbol{f}$ and $\boldsymbol{g}$ are the nonlinear functions of the state variables. If the system is driven by random noises, a stochastic vibro-impact system will be born.

Figure 1(b) shows the sketch of the friction system. The physical model of this system consists of some blocks supported by a moving belt, and the contact surface is rough. Each block is connected to a fixed spring, and the neighboring blocks are also connected by a spring. The moving belt makes the blocks show stick motion because of the rough surface. When the force generated by the spring on the block exceeds the force of friction, the block will do slip motion. The switch between the stick motion and the slip motion makes this system a non-smooth one. And the response of the system will be quite complicated. The equation of this motion is given by

$$\boldsymbol{M}\ddot{\boldsymbol{x}}+\boldsymbol{C}\dot{\boldsymbol{x}}+\boldsymbol{K}\boldsymbol{x}+\boldsymbol{Z}=\boldsymbol{F}(t), \tag{2}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are the nondimensional mass, damping, and stiffness matrixes, respectively, and $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}$. $\boldsymbol{Z}$ is the friction force vector which can be described by the Coulomb model, hysteretic model, Dahl model, and so on. $\boldsymbol{F}(t)$ is the external random excitation vector.

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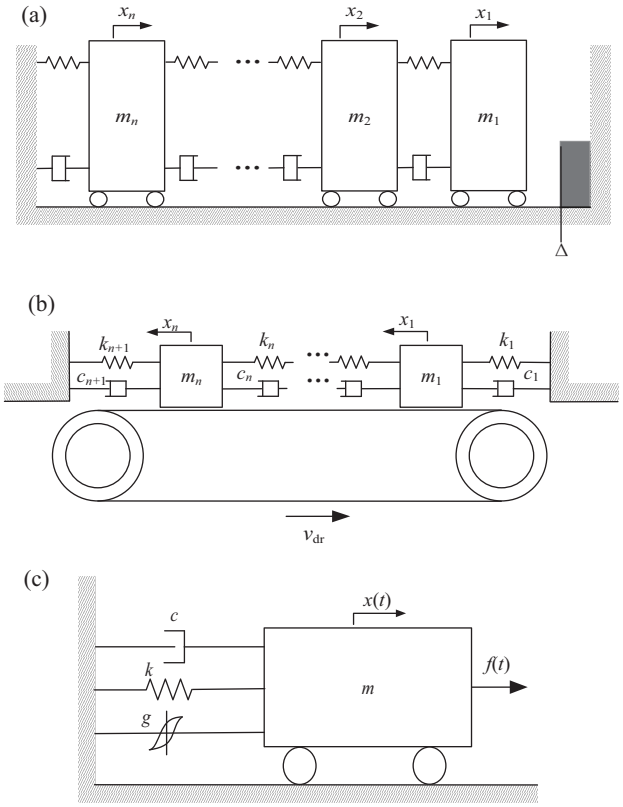


Fig. 1. Schematic diagrams of (a) the multi-degree-freedom vibro-impact system, (b) the multi-degree-freedom vibrate system with friction, and (c) the single-degree-of-freedom nonlinear hysteretic system.

Figure 1(c) manifests the dynamic model schematic of a single-degree-of-freedom nonlinear hysteretic system. It should be noted that the hysteretic forces depend on not only the instantaneous deformation but also the history of deformation. Various analytical models have been proposed for representing the hysteretic constitutive relationship, all of which have the cusp nature and need to be described by the non-smooth auxiliary differential equation that makes the system a non-smooth one. The governing equation of motion of the system is given as follows:

$$\ddot{x}+2c\dot{x}+kx+g(x,\dot{x})=f(t), \tag{3}$$

where x denotes the nondimensional displacement, c is the viscous damping coefficient, k is the linear stiffness, g denotes the nonlinear hysteretic restoring force governed by the specific hysteresis models, and $f(t)$ represents an external random excitation.

2. The stochastic vibro-impact system

The motion of the vibro-impact system with random factors exists in a wide variety of engineering applications. The equations of the motion are discontinuous and strongly nonlinear, so such systems can generate very rich dynamic behaviors.[12] The researches on stochastic vibro-impact system begin at about middle of last century. As early as 1972, Nayak did some works on this field.[3] In the Refs. [4] and [5], Jing obtained the exact/approximation responses of a single-degree-of-freedom vibro-impact system excited by Gaussian white noise. Lin *et al.* proposed a novel clearance estimation approach for a randomly excited vibro-impact system.[6] For a rattling system, Feng *et al.* used the non-Gaussian closure technique to establish a mean map to describe the stochastic impulsive system. And the random chaos was investigated in Ref. [7]. Feng also studied a class of random vibro-impact system by defining a mean vibro-impact Poincare map. And two examples were investigated to show the complex nonlinear behaviors in these systems, including bifurcation and chaos.[8] Luo did many works on the nonlinear dynamics of the vibro-impact systems, including many types of bifurcations and the routes to chaos.[9] In Ref. [10] , Wouw *et al.* researched the response of a strongly nonlinear beam vibro-impact system under both broad and small banded Gaussian excitations numerically. And their simulations are corresponding with the experimental results.

By using the stochastic averaging methods, Huang *et al.* studied a multi-degree-of-freedom vibro-impact system under white noise excitations.[11] In the same year, Dimentberg gave a review on the random vibrations with impacts. Some analytical methods and specific results were presented in their paper.[12] In Ref. [13] , Namachchivaya developed an averaging approach to study the dynamics of a vibro-impact system excited by random perturbations. Xu *et al.* also used this averaging approach to analyze the responses of a vibro-impact system excited by some different types of noises.[14–19] And the stochastic bifurcations were also explored in these papers by analyzing the probability density functions of the responses. Besides the ordinary vibro-impact system, Xu also investigated the response of some especial stochastic vibro-impact systems using stochastic averaging method, such as the system with viscoelasticity,[20–22] bilateral barriers,[23] fractional derivative damping,[24–26] Poisson noise,[27] colored noise,[28] and so on.[29–32] The averaging method can also be utilized to study the transient response of some stochastic vibro-impact systems.[33] For the vibro-impact systems with inelastic contact interaction, Xu *et al.* determined the system response through the stochastic averaging and energy dissipation balance technique. They also discussed the vibro-impact system using the equivalent nonlinearization technique.[34–36]

The vibro-impact system with random parameters often appears in reality, thus the related research is significantly important. In Refs. [37] and [38] , by defining a mean constraint plane and the mean jump equation, Xu *et al.* used the polynomial approximation to explore the period-doubling bifurcation and grazing bifurcation of some stochastic vibro-impact systems. They proved that the polynomial approximation is valid to stochastic vibro-impact systems. And Wang applied this

scheme to investigate the elastic vibro-impact oscillator, the period doubling bifurcation was also found in this system.[39] Then they investigated the effect of the random parameter on the basins and attractors of the elastic vibro-impact system, as well as the discontinuous system,[40,41] and observed that the random parameter may change the basins and attractors tremendously..

The top Lyapunov exponent is an important indicator to study the nonlinear dynamics, both for the deterministic condition and the stochastic condition. In Ref. [42], Xu proposed a method to calculate the top Lyapunov exponent for vibro-impact system under the Gaussian white noise perturbation. And the bifurcations were also discussed by this criterion. They also developed the Melnikov method for a general nonlinear vibro-impact system.[43] Kumar *et al.* studied the stochastic bifurcation in a vibro-impact system excited by white noise through calculating the Lyapunov exponent and Shannon entropy, respectively.[44] In Ref. [45], the principal resonance response of a stochastic elastic vibro-impact system with time delayed feedback was considered using the multiple scales method, the effect of the system parameter on the response was discussed in detail.

Chaos control is a significant problem for nonlinear dynamical system. In Refs. [46] and [47], the authors discussed a controlled vibr-impact system. Using the damping control law, they suppressed the chaos to periodic orbit successfully. And they also found that this control scheme can generate chaos in the system. The robustness of the control law to the random perturbations was considered as well. Additionally, the impulsive control is a special method to control chaos in nonlinear system.[48] It makes the controlled system a non-smooth one. Based on this feature, Wang developed this method to accomplish the chaos control in vibro-impact system with random factors. The key point is to implement the impulses just when the impact occurs.[49]

As we know, the numerical method is a very crucial aspect to understand the nonlinear systems. In Ref. [50] , Iourtchenko showed us a response probability density function of a stochastic vibro-impact system excited by additive Gaussian white noise by numerical investigation. Based on the cell mapping method, Gan *et al.* researched a vibro-impact oscillator under multiple harmonic and bounded-noise excitations. Some kinds of stochastic attractors were shown in their works.[51,52] Naess developed the path integration technique to the non-smooth stochastic system and obtained the probability density function of this system.[53] Zhu used the exponential–polynomial closure method to get the probability density function for a stochastic vibro-impact system, and examined the effectiveness of this method by some examples.[54,55] Kumar *et al.* used the finite element method to solve the FPK equation, then got the responses of the stochastic vibro-impact systems.[56] By defining an impact-to-impact mapping, a novel numerical scheme was used to obtain the probability density function of the stochastic vibro-impact systems based on the cell mapping method in Ref. [57] . And the authors used this strategy to discuss the evolution of the stochastic responses.

3. The stochastic friction system

Friction, which is one of the typical non-smooth factors, plays an important role in wide fields, such as physics, biology, mechanical and civil engineering. Friction phenomenon gives rise to strongly non-smooth nonlinearity and significantly changes the performance of the original system. Therefore, it is imperative to investigate the dynamical behaviors of friction systems. According to whether the mathematical description contains auxiliary differential equations, the phenomenological friction models can be classified into two classes, i.e., static class and dynamic class. By far, the research on random dynamic behaviors of non-smooth systems with friction is very few due to the complexity of non-smoothness and randomness.

As the simplest static model, the dry friction model was extensively adopted.[58–60] Brouwers[61] investigated the nonlinearly damped response of a marine riser subject to random waves. Pollak and Berezhkovskii[62] simplified the one-dimensional stochastic equation of motion for a particle in the presence of space- and time-dependent friction to an exact Fokker–Planck equation. Feng[63,64] adopted a mean Poincaré map to establish the discrete model for random systems with one friction interface, and showed that random perturbations may break the limit cycle, and lead to chaos. Subsequently, she extended this discrete model to two or more friction interfaces of random systems.[65] In Ref. [66] , the dynamic behavior of a disc brake subjected to a follower frictional force governed by a constant or random friction coefficient was studied. Gaus *et al.*[67] analyzed the stochastic bifurcation behavior of a mass-on-belt system by the randomization to friction coefficient. Through the equivalent nonlinearization technique, Wang *et al.*[68] studied the stationary response of the single-degree-of-freedom oscillator with hyperbolic tangent friction. Based on the stochastic averaging method, Xu *et al.* found that the dry friction can induce the P-bifurcation of stationary probability density of the system with viscoelastic forces under Gaussian colored noise excitation[69] and promote the reliability of the system under the combination of additive and multiplicative random excitations.[70]

In the aspect of numerical method, Sun[71] researched the stochastic response of Coulomb friction system by applying the generalized cell mapping method, which is based on the short-time Gaussian approximation. Baule[72] investigated the

stick–slip motion of solids with dry friction subject to random vibrations and an external field by using the path integral method.

It has been known that the first passage failure is a crucial issue for dynamical system. Chen and Wolfram provided an analytical solution to the first-passage time problem of Brownian motion with dry friction[73] by using the eigenfunction decomposition and solving the backward Kolmogorov equation.[74]

In addition, some authors considered the influence of Coulomb friction on the energy harvester. Tian *et al.*[75] studied the optimal load resistance of a randomly excited nonlinear electromagnetic energy harvester with Coulomb friction by utilizing equivalent nonlinear method. Furthermore, the important influence of the unavoidable friction induced by the mechanical bearings in electromagnetic energy harvester was proved experimentally.[76]

The dynamic friction models such as Dahl, Bliman-Sorine, LuGre as well as atomic scale and fractal models have been developed to describe some special phenomena, such as pre-displacement, rate-dependence, and hysteresis.[77–80] Unlike the static models, the dynamic friction models must be described by an auxiliary first-order non-smooth differential equation with respect to certain auxiliary variable which induces the higher dimension of governing equations with respect to the response of dynamic friction systems. By far, few attention has been paid to the random response of the dynamic friction system due to the complexity induced by the auxiliary non-smooth differential equation and the random excitation. Wang *et al.* studied the stochastic responses of Duffing system with Dahl friction by introducing the associated generalized harmonic transformations and quasi-linearization technique.[81] Recently, Jin and Wang established an approximately analytical procedure to evaluate the random response of LuGre frictional system.[82]

4. The stochastic hysteretic system

The hysteretic characterizes a wide variety of nonlinear physical systems endowed with memory. Because of the extensively existence in the mechanical and structural systems, such as ferromagnetism, plasticity, porous media filtration, and pseudo-elasticity of smart materials,[83–87] it has been drawn enormous attention from the researchers. Several models such as bilinear model, Bouc–Wen model, Jenkins–Iwan model, Masing model, Duhem model, and Preisach model have been proposed for representing the hysteretic constitutive relationship.[88–92] Generally, it is extremely difficult to analytically determine the exact responses of hysteretic systems with random noise, thus some approximate solution techniques have been developed.

By the stochastic averaging method, the stochastic responses of various hysteretic models under different random excitations have been extensively studied. For the Bouc-Wen model, a quasi-conservative stochastic averaging was proposed by Noori *et al.*[93] to research the stationary probability density and the first passage problem of the Bouc–Wen–Baber hysteretic system under Gaussian white noise excitation. Meanwhile, Lin *et al.*[94] considered the stochastic response of the system with Bouc–Wen hysteretic through the stochastic averaging of energy envelope. Considering the Duhem hysteretic model, the stationary response of this hysteretic system subjected to non-white random excitation was investigated through the stochastic averaging of energy envelope.[95,96] Jin *et al.*[97] obtained the semi-analytical solution of transient responses of the Duhem hysteretic system under random seismic excitation by the combination of the Galerkin method and the stochastic averaging method based on the generalized harmonic functions. Furthermore, the bounded control strategy to minimize the transient responses was designed. The Preisach model, which can describe hysteresis nonlinearity with nonlocal memory, has been adopted in various fields of engineering mechanics. The stochastic averaging procedure for Preisach hysteretic systems has been developed by Spanos *et al.*[98] and Wang.[99] They firstly obtained the equivalent nonlinear system by using the harmonic balance technique. Subsequently, it was adopted to study the first-passage failure problem of single-degree-of-freedom Preisach hysteretic system in Ref. [10]. The method of stochastic averaging of energy envelope was also applied to predict the response of bilinear hysteretic systems subjected to Gaussian white noise excitation.[101,102] And recently, it was extended to the Poisson white noise excitation by Zeng.[103]

Besides, in early research, the equivalent linearization technique was employed to study the response of the bilinear hysteretic system to random excitation.[88,104] Lutes[105] proposed a technique for seeking an equivalent nonlinear non-hysteretic system whose stationary statistics can be found in a closed form and predicted the response statistics of a bilinear hysteretic oscillator. The second-order statistics have been approximately determined by Ni *et al.*[106] and Ying *et al.*[107] for the stochastic response of the Preisach hysteretic systems based on the switching probability analysis and the Gaussian approximation of the response. Liu *et al.*[108] derived the approximate closed solution of the stationary probability density function of Bouc–Wen hysteresis system excited by Gaussian white noise using the iterative weighted residual method.

In addition to the above approximation theoretical methods, some numerical methods have also been applied to this problem. By the path integral method, Naess and Moe[109] evaluated the probability density function of an oscillator with bilinear hysteresis driven by white noise. By adapting the clas-

sical algorithm by Wolf and co-workers, Luciano[11] discussed the Lyapunov exponent estimation of nonlinear hysteretic systems. Rosario[111] researched the stochastic resonance in magnetic systems described by Preisach hysteresis system employing a numerical method. A wavelet based nonlinear random vibration theory has been proposed to obtain the responses of a tank-liquid system supported on bilinear hysteretic soil medium. Li[112] developed a probability density evolution method for responses analysis of nonlinear stochastic structures with bilinear hysteretic restoring forces, subjected to harmonic excitation and seismic excitation.

From the control perspective, some smart materials which exhibit significant hysteresis are often utilized for vibration control. Nonlinear hysteresis effects can cause inaccuracy in open-loop control and instability in close-loop control, thus it is necessary to develop the control strategy for the system with hysteresis. A sliding mode controller and an optimal polynomial controller have been proposed by Yang *et al.* to control hysteretic structures under earthquake excitation.[113,114] Based on the stochastic averaging method and the stochastic dynamical programming principle, Zhu[115] proposed a strategy for the random vibration control of Bouc–Wen hysteretic system under random excitations. And Wang[116] established the optimal control of Preisach hysteretic systems subjected to a stochastic excitation. Besides, an optimal bound feedback control for minimizing the first-passage failure of a hysteretic system was considered by Li.[117] An adaptive control scheme for vibration suppression of a flexible string system with input hysteresis was designed by He.[118] Li[119] also developed a physical nonlinear polynomial stochastic optimal control strategy for hysteretic structural systems.

5. Conclusions and outlook

Stochastic non-smooth systems play a prominent role in a range of application areas, including mechanics, biology, electronic circuits, and many other engineering fields. Numerous works on stochastic non-smooth models have been reported. Special attention has been paid to the dynamics both through the theoretical analysis and numerical results.

For some typical non-smooth systems with random noise, the different types of non-smooth factors can represent the varied discontinuities. In the previous works, the main ideas are to deal with the approximation of non-smooth events. The reduced equivalent smooth system is then investigated by using smooth theories. These methods can reveal the ordinary dynamical behavior, but also may lose some novel dynamical phenomena caused by non-smoothness, such as zero behavior, inflated bifurcations, chattering dynamics, and stick-slip motion. The development of new theory frame and technique is strongly desired to reveal the nature of non-smooth events. This emerging and challenging topic is one of the open areas.

In the numerical aspects, the numerical solver for the stochastic ordinary differential equation (ODE) with discontinuity is the basic problem. The key issue is the numerical location of discontinuous instants. The non-smooth systems not only are described as stochastic discontinuous ODE, but also appear in the models of complementarity systems, constrained dynamical systems, and so on. This promising subject can be developed by applying the cross amalgamation of multiple research fields.

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