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A new class of states of reversible entanglement manipulation under positive partial transpose operations*

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We first study the reversibility for a class of states under the operations which completely preserve the positivity of partial transpose (PPT) and show that the entanglement cost is equal to the distillable entanglement for a rank-two mixed state on the $4\bigotimes 4$ antisymmetric subspace under PPT operations. By using a similar method in finding the irreversibility, we also find that the value of a new efficiently computable additive lower bound $E_\eta(\rho)$ for the asymptotic PPT-relative entropy of entanglement presented in [*Phys. Rev. Lett.* **119**, 180506 (2017)] is equal to the regularized Rains' bound and an upper bound $E_{\mathrm{N}}(\rho)$ for distillable entanglement for these states. Furthermore, we find states on the symmetric subspace satisfying the relation mentioned above, generalize the antisymmetric states and symmetric states in higher dimensions, and give a specific value for distillable entanglement and entanglement cost for these states under the PPT operations.

Keywords: distillable entanglement, entanglement cost, semidefinite programming

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1. Introduction

Quantum entanglement is one of the most important physical resources in quantum mechanics and can be used in quantum information, especially in quantum communication and quantum coding.[1] Based on these results, much work on entanglement has been carried out recently.[2–7]

We study two kinds of fundamental measures for quantum entanglement, including the entanglement cost (E_{C}) and the distillable entanglement (E_{D}). The distillable entanglement[8–12] portrays the maximum efficiency with which one can obtain the maximally entangled states from a bipartite state ρ by local operations and classical communication (LOCC), which means that the distillable entanglement quantifies the optimal efficiency r of transforming $\rho^{\otimes n}$ to rn maximally entangled states. The cost of quantum entanglement[13–15] describes the optimal rate r with which one can obtain the given bipartite state ρ from the maximally entangled states by LOCC alone, which means that the entanglement cost quantifies the efficiency r of transforming rn maximally entangled states to $\rho^{\otimes n}$. Since the LOCC operations are difficult to capture and lead to more difficulties in calculating E_{D} and E_{C}, Rains proposed the positive partial transpose (PPT) operations including LOCC operations,[16] which are more powerful than LOCC operations, such as the creation of any bound entangled state and distilling a non-positive partial transpose (NPT) state, i.e, any state that cannot be created by PPT operations.[17] At the same time, PPT operations can give a more simple classification of states. How to calculate E_{D} remains unknown, and calculating E_{C} for general quantum states is an NP-hard (non-deterministic polynomial-time hard) problem.[14]

A natural idea is to find some bounds[14,15] to estimate the distillable entanglement and the entanglement cost. For distillable entanglement, there are many well-known upper bounds, such as Rains' bound[9] and the relative entropy of entanglement (REE).[18,19] Unfortunately, these well-known bounds are difficult to calculate, and are only easy for some special states.[20,21] Since the distillable entanglement and the entanglement cost are important but difficult to calculate, people attach more importance to finding a method to evaluate them. In Refs. [12], [22], and [23], the semidefinite programming (SDP) bounds for $E_{\mathrm{D,PPT}}$ and $E_{\mathrm{C,PPT}}$ have been proposed, which can be computable and have a lot of interesting characteristics.

For Werner states and orthogonally invariant states, Rains' bound is proved to be equal to the asymptotic relative entropy of entanglement with respect to PPT states.[24,25] It has been shown that E_{D} is equal to E_{C} in the asymptotic limit of many infinitely identical copies of a pure state,[26] i.e.,

$$E_{\mathrm{D}}(|\psi\rangle\langle\psi|) = E_{\mathrm{C}}(|\psi\rangle\langle\psi|) = S(\mathrm{Tr}_B^2(|\psi\rangle\langle\psi|). \quad (1)$$

On one hand, the reversibility under PPT operations means that $E_{\mathrm{D,PPT}}$ is equal to $E_{\mathrm{C,PPT}}$ and the entanglement manipulation between the given states and the maximally entangled states is reversible. On the other hand, it is proved that reversibility no longer holds for mixed states.[27] In fact, the authors gave an example of reversibility under PPT operations in Ref. [17]. In Ref. [28], E_{D} is proved to be strictly less

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than E_C and the regularized Rains' bound is strictly less than a new SDP lower bound $E_\eta(\rho)$ for entanglement cost for a class of rank-two $3\bigotimes 3$ antisymmetric states under PPT operations. There are some states[28] that satisfy $E_\eta(\rho) > R^\infty(\rho)$, while the states we find satisfy $E_\eta(\rho) = R^\infty(\rho)$. At the same time, $E_\eta(\rho) > E_N(\rho)$ for the states in Ref. [28], while we find some states that satisfy $E_\eta(\rho) = E_N(\rho)$, where $E_N(\rho)$ is the logarithmic negativity of a state ρ.

In this paper, we achieve our aims through finding a similar method to that of Ref. [28] to demonstrate a class of $4\bigotimes 4$ antisymmetric states which are reversible under PPT operations. We use the method in constructing the irreversibility[28] while we also obtain the reversibility and show that $E_\eta(\rho)$ is equal to the regularized Rains' bound and $E_N(\rho)$ for this class of states. By using an additive SDP lower bound for the asymptotic REE, we prove that

$$E^\infty_{R,PPT}(\rho) = E_{C,PPT}(\rho) = E_\eta(\rho) = 1. \tag{2}$$

For a class of $4\bigotimes 4$ antisymmetric states which we construct, they also satisfy

$$E_{D,PPT}(\rho) = R^\infty(\rho) = E_N(\rho) = 1. \tag{3}$$

Consequently,

$$E_{D,PPT}(\rho) = E_{C,PPT}(\rho) = E_\eta(\rho) = R^\infty(\rho) = E_N(\rho) = 1. \tag{4}$$

We also generalize this class of states to higher dimensions. Indeed, it is very important that the method we use helps us obtain the specific value of E_D and E_C for a new class of states under PPT operations while E_D and E_C are difficult to calculate. At the same time, the asymptotic reversibility under PPT operations for these states will lead to a unique measure of entanglement which plays a role similar to entropy in thermodynamics and the unique ordering is provided.

This paper is organized as follows. In Section 2, we briefly review some notions related to distillable entanglement and entanglement cost. In Section 3, we give a new class of states which are reversible under PPT operations. In Section 4, we give the conclusion of this paper.

2. Preliminaries

In this section, let us review some notions related to entanglement. We use the symbols A and B to denote the Hilbert spaces. Let $\ell(A)$ denote the set of linear operators over A. A quantum state ρ must satisfy $\rho \geq 0$ and $\mathrm{Tr}(\rho) = 1$. The support of ρ, denoted by $supp(\rho)$, is a subspace spanned by the eigenvectors of ρ with positive eigenvalues.

For a linear operator P on the Hilbert spaces, we define $|P| = \sqrt{P^\dagger P}$ and the trace norm is $\|P\|_1 = \mathrm{Tr}|P|$, where $P^\dagger$ is the conjugate of P. The operator norm $\|P\|_\infty$ is defined as the maximum eigenvalue of $|P|$. A positive semidefinite operator $X_{AB} \in \ell(A\bigotimes B)$ is said to completely preserve positivity of partial transpose if $X^{T_B}_{AB} \geq 0$, i.e.,

$$X_{AB} = |i_A j_B\rangle\langle k_A l_B| \geq 0, \quad X^{T_B}_{AB} = |i_A l_B\rangle\langle k_A j_B| \geq 0. \tag{5}$$

The partial transpose $X^{T_B}_{AB}$ can also be given by Eq. (5).

Semidefinite programming is a very important and useful tool in entanglement theory with a lot of applications.[29–35] The computable SDP upper bound $E_W(\rho)$[22] for distillable entanglement and lower bound $E_M(\rho)$[12] for entanglement cost under PPT operations help us to estimate $E_{D,PPT}$ and $E_{C,PPT}$ better.

The distillable entanglement[36] is defined as follows:

$$E_D(\rho_{AB}) := \sup\{r : \lim_{n\to\infty} \inf_\Lambda \|\Lambda(\rho^{\otimes n}_{AB}) - \Phi(2^{rn})\|_1 = 0\}, \tag{6}$$

where $\Phi(d) = \sum^d_{i,j=1} \frac{1}{d}|ii\rangle\langle jj|$, and Λ ranges over operations including LOCC operations, separable operations (SEP), and operations completely preserving positivity of partial transpose. If Λ ranges over PPT operations, the entanglement distillation is noted as $E_{D,PPT}$.

The entanglement cost[36] is defined by

$$E_C(\rho_{AB}) := \inf\{r : \lim_{n\to\infty} \inf_\Lambda \|\rho^{\otimes n}_{AB} - \Lambda(\Phi(2^{rn}))\|_1 = 0\}, \tag{7}$$

where $\Phi(d) = \sum^d_{i,j=1} \frac{1}{d}|ii\rangle\langle jj|$ and Λ ranges over operations including LOCC operations, PPT operations, and SEP operations. It is clear that

$$E_{D,LOCC}(\rho) \leq E_{D,PPT}(\rho) \leq E_{C,PPT}(\rho) \leq E_{C,LOCC}(\rho). \tag{8}$$

The PPT-relative entropy of entanglement[24,37] is given as a convex optimization

$$E_{R,PPT}(\rho) = \min S(\rho\|\sigma), \quad \text{s.t. } \sigma, \sigma^{T_B} \geq 0, \mathrm{Tr}(\sigma) = 1. \tag{9}$$

The PPT-relative entropy of entanglement is very important in quantum theory; it is a well-known upper bound for distillable entanglement which describes the minimum distance between the given states and the PPT states.[38–40] The asymptotic PPT-relative entropy of entanglement is defined as follows:

$$E^\infty_{R,PPT}(\rho) = \inf_{n\geq 1} \frac{1}{n} E_{R,PPT}(\rho^{\otimes n}). \tag{10}$$

An improved bound for distillable entanglement is Rains' bound which was introduced in Ref. [16] and defined as follows:

$$R(\rho) = \min S(\rho\|\tau), \quad \text{s.t. } \tau \geq 0, \mathrm{Tr}|\tau^{T_B}| \leq 1, \tag{11}$$

where $S(\rho\|\tau) = \mathrm{Tr}(\rho\log\rho - \rho\log\tau)$ is the relative entropy. The regularized Rains' bound was introduced in Ref. [41] which is given as follows:

$$R^\infty(\rho) = \inf_{n\geq 1} \frac{R(\rho^{\otimes n})}{n}. \tag{12}$$

Rains' bound is very important in quantum theory; it describes the minimum distance between the given states and all the possible separable states.

In Ref. [41], it is shown that a lower bound for the PPT-entanglement cost is the asymptotic PPT-relative entropy of entanglement, and the following always holds:

$$E_{\mathrm{D,PPT}}(\rho) \leq R^{\infty}(\rho) \leq E^{\infty}_{\mathrm{R,PPT}}(\rho) \leq E_{\mathrm{C,PPT}}(\rho). \quad (13)$$

In Ref. [28], a new important SDP lower bound for entanglement cost is given as follows:

$$E_{\eta}(\rho) = \max -\log \|Y^{T_B}\|_{\infty}, \quad \text{s.t.} \quad -Y \leq P^{T_B} \leq Y, \quad (14)$$

where P is the projector of ρ, Y^{T_B} is the partial transpose of Y. That work also indicates that $E^{\infty}_{\mathrm{R,PPT}}(\rho) \geq E_{\eta}(\rho)$ for any quantum state ρ and as a consequence

$$E_{\mathrm{C,LOCC}}(\rho) \geq E_{\mathrm{C,PPT}}(\rho) \geq E_{\eta}(\rho). \quad (15)$$

It has been proved that $R^{\infty}(\rho) < E_{\eta}(\rho)$ for a class of states.[28] The relation between $R^{\infty}(\rho)$ and the new SDP lower bound $E_{\eta}(\rho)$ remains a question.

In Ref. [42], the logarithmic negativity of a state ρ is given as follows:

$$E_{\mathrm{N}}(\rho) = \log \|\rho^{T_B}\|_1. \quad (16)$$

Then the following conclusion is drawn:

$$E^{(1)}_{0,\mathrm{D,PPT}}(\rho) \leq E_{\mathrm{D,PPT}}(\rho) \leq R^{\infty}(\rho) \leq E_{\mathrm{N}}(\rho). \quad (17)$$

$E_{\eta}(\rho)$ is an SDP lower bound for the PPT-entanglement cost and $E_{\mathrm{N}}(\rho)$ is an SDP upper bound for the PPT-distillable entanglement. The relation between $E_{\eta}(\rho)$ and $E_{\mathrm{N}}(\rho)$ is unknown.

In Refs. [12] and [22], the one-copy deterministic PPT-distillable entanglement of ρ is defined by

$$E^{(1)}_{0,\mathrm{D,PPT}}(\rho) = -\log W_0(\rho), \quad (18)$$

where

$$W_0(\rho) = \min \|R^{T_B}\|_{\infty}, \ \text{s.t.} \ P_{AB} \leq R \leq I_{AB}, \quad (19)$$

where P_{AB} is the projector of ρ, I_{AB} is an identity matrix. The strong dual SDP is given by

$$E^{(1)}_{0,\mathrm{D,PPT}}(\rho) = \max_{R} -\log_2 \|R^{T_B}\|_{\infty}, \quad \text{s.t.} \ P_{AB} \leq R \leq I_{AB}. \quad (20)$$

3. Main results

In this section, we first consider a class of bipartite states supported on the $4 \bigotimes 4$ antisymmetric subspace.

Proposition 1 If a state can be written as follows:

$$\rho = p|V_1\rangle\langle V_1| + (1-p)|V_2\rangle\langle V_2| \ \ (0 < p < 1), \quad (21)$$

$$|V_1\rangle = \frac{1}{\sqrt{2}}(|ab\rangle - |ba\rangle), \quad (22)$$

$$|V_2\rangle = \frac{1}{\sqrt{2}}(|cd\rangle - |dc\rangle), \quad (23)$$

where a, b, c, and d are all different, then it will have

$$E_{\mathrm{C,PPT}}(\rho) = E^{\infty}_{\mathrm{R,PPT}}(\rho) = E_{\eta}(\rho) = 1. \quad (24)$$

Proof In the first part, we also choose

$$\begin{aligned} Y \ = \ & 1/2(|aa\rangle\langle aa| + |bb\rangle\langle bb| + |cc\rangle\langle cc| + |dd\rangle\langle dd| \\ & + |ab\rangle\langle ab| + |ba\rangle\langle ba| + |cd\rangle\langle cd| + |dc\rangle\langle dc|). \end{aligned} \quad (25)$$

We have the projection onto $supp(\rho)$ as follows:

$$P = |V_1\rangle\langle V_2| + |V_2\rangle\langle V_2|. \quad (26)$$

Then, it holds that

$$\begin{aligned} Y - P^{T_B} \geq 0, \\ Y + P^{T_B} \geq 0. \end{aligned} \quad (27)$$

Y is a feasible solution to Eq. (14). It is obvious that

$$E_{\eta}(\rho) \geq -\log \|Y^{T_B}\|_{\infty} = -\log \frac{1}{2} = 1. \quad (28)$$

Because a $4 \bigotimes 4$ maximally entangled state can efficiently prepare an exact copy of ρ by LOCC operations, we will find that $E_{\mathrm{C,LOCC}}(\rho) \leq 1$ and

$$\begin{aligned} 1 \ \leq \ & E_{\eta}(\rho) \leq E^{\infty}_{\mathrm{R,PPT}}(\rho) \leq E_{\mathrm{C,PPT}}(\rho) \\ & \leq E_{\mathrm{C,LOCC}}(\rho) \leq 1. \end{aligned} \quad (29)$$

Then we will have $E_{\mathrm{C,PPT}}(\rho) = E^{\infty}_{\mathrm{R,PPT}}(\rho) = E_{\eta}(\rho) = 1$.

Applying $E_{\eta}(\rho)$, we obtain that $E_{\mathrm{C,PPT}}(\rho) = E^{\infty}_{\mathrm{R,PPT}}(\rho) = E_{\eta}(\rho) = 1$ and it is very significant that we use the bounds for the entanglement cost to get $E_{\mathrm{C,PPT}}(\rho) = 1$ for the states we mentioned above. Whether all the states satisfy $E_{\mathrm{C,PPT}}(\rho) = E^{\infty}_{\mathrm{R,PPT}}(\rho)$ remains a question.

We can also evaluate the PPT-distillable entanglement of ρ by using the upper bound $E_{\mathrm{N}}(\rho)$ for the regularized Rains' bound and the one-copy deterministic PPT-distillable entanglement $E^{(1)}_{0,\mathrm{D,PPT}}(\rho)$.

Proposition 2 For the states given in Proposition 1,

$$E_{\mathrm{D,PPT}}(\rho) = R^{\infty}(\rho) = \log 2 = 1. \quad (30)$$

Proof As mentioned in Section 2, the logarithmic negativity of a quantum state ρ is an upper bound for the regularized Rains' bound. If we prove that $E^{(1)}_{0,\mathrm{D,PPT}}(\rho) = 1 = E_{\mathrm{N}}(\rho)$, we can obtain the result we want.

Firstly, we can get the one-copy deterministic PPT-distillable entanglement of ρ from Eq. (18):

$$E^{(1)}_{0,\mathrm{D,PPT}}(\rho) = \max -\log_2 \|R^{T_B}\|_{\infty}, \quad \text{s.t.} \ P \leq R \leq I. \quad (31)$$

Here P is the projection onto $supp(\rho)$.

Now, we choose

$$R = P. \tag{32}$$

It is easy to check $P \leq R \leq I$, which means P is a feasible solution to Eq. (34).

We can calculate that

$$\|P^{T_B}\|_\infty = \frac{1}{2}. \tag{33}$$

Therefore,

$$E^{(1)}_{0,\mathrm{D},\mathrm{PPT}}(\rho) \geq -\log\|P^{T_B}\|_\infty = -\log\frac{1}{2} = 1. \tag{34}$$

Secondly, we find that

$$\|\rho^{T_B}\|_1 = 2. \tag{35}$$

Then

$$E_\mathrm{N}(\rho) = \log||\rho^{T_B}||_1 = \log 2 = 1. \tag{36}$$

As a result, we have

$$E_\mathrm{N}(\rho) = E^{(1)}_{0,\mathrm{D},\mathrm{PPT}}(\rho) = \log 2 = 1. \tag{37}$$

Finally, we obtain that $E_{\mathrm{D},\mathrm{PPT}}(\rho) = R^\infty(\rho) = E_\mathrm{N}(\rho) = \log 2 = 1$.

Applying the upper bound $E_\mathrm{N}(\rho)$ and the lower bound $E^{(1)}_{0,\mathrm{D},\mathrm{PPT}}(\rho)$ for the distillable entanglement, we obtain that $E_{\mathrm{D},\mathrm{PPT}}(\rho) = R^\infty(\rho) = E_\mathrm{N}(\rho) = \log 2 = 1$, while it is difficult to calculate the specific value of $E_{\mathrm{D},\mathrm{PPT}}(\rho)$. It remains a question whether all the states satisfy $E_{\mathrm{D},\mathrm{PPT}}(\rho) = R^\infty(\rho)$, and the relation between $R^\infty(\rho)$ and the new SDP lower bound $E_\eta(\rho)$ also remains a problem. It is more important that $R^\infty(\rho) = E_\eta(\rho)$ and $E_\mathrm{N}(\rho) = E_\eta(\rho)$ for the states we find.

By Proposition 1 and Proposition 2, we have shown a class of special states which are reversible under PPT operations and satisfy $E_\mathrm{N}(\rho) = R^\infty(\rho) = E_\eta(\rho)$.

Corollary 1 If a state can be written as follows:

$$\rho = p|V_1\rangle\langle V_1| + (1-p)|V_2\rangle\langle V_2| \quad (0 < p < 1), \tag{38}$$

$$|V_1\rangle = \frac{1}{\sqrt{2}}(|ab\rangle + |ba\rangle), \tag{39}$$

$$|V_2\rangle = \frac{1}{\sqrt{2}}(|cd\rangle + |dc\rangle), \tag{40}$$

where a, b, c, and d are all different, then it will have

$$\begin{aligned} E_{\mathrm{C},\mathrm{PPT}}(\rho) &= E_{\mathrm{D},\mathrm{PPT}}(\rho) = R^\infty(\rho) = E_\mathrm{N}(\rho) \\ &= E_\eta(\rho) = 1. \end{aligned} \tag{41}$$

Example 1 For state

$$\rho = \frac{1}{2}(|V_1\rangle\langle V_1| + |V_2\rangle\langle V_2|), \tag{42}$$

where

$$|V_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{43}$$

$$|V_2\rangle = \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle), \tag{44}$$

the following equation holds:

$$E_{\mathrm{C},\mathrm{PPT}}(\rho) = E_{\mathrm{D},\mathrm{PPT}}(\rho) = R^\infty(\rho) = E_\mathrm{N}(\rho) = E_\eta(\rho) = 1. \tag{45}$$

Firstly, choosing $Y = \frac{1}{2}[|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22| + |33\rangle\langle 33| + |01\rangle\langle 01| + |10\rangle\langle 10| + |23\rangle\langle 23| + |32\rangle\langle 32|]$, we find that $P^{T_B} = \frac{1}{2}[|01\rangle\langle 01| + |10\rangle\langle 10| + |23\rangle\langle 23| + |32\rangle\langle 32|] - \frac{1}{2}[|11\rangle\langle 00| + |00\rangle\langle 11| + |22\rangle\langle 33| + |33\rangle\langle 22|]$, $Y - P^{T_B} = \frac{1}{2}[(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) + (|22\rangle + |33\rangle)(\langle 22| + \langle 33|)]$, and $Y + P^{T_B} = \frac{1}{2}[(|00\rangle - |11\rangle)(\langle 00| - \langle 11|) + (|22\rangle - |33\rangle)(\langle 22| - \langle 33|) + 2|10\rangle\langle 10| + |01\rangle\langle 01| + |23\rangle\langle 23| + |32\rangle\langle 32|]$. Noting that $Y \pm P^{T_B} \geq 0$, we can find that Y is a feasible solution to $E_\eta(\rho)$. It is clear that $\|Y^{T_B}\|_\infty = \frac{1}{2}$. Thus, $E_\eta(\rho) \geq -\log\|Y^{T_B}\|_\infty = -\log\frac{1}{2} = 1$. It is obvious that a maximally entangled state can efficiently prepare an exact copy of ρ by LOCC. By using Eqs. (15) and (13), we can obtain that $1 \leq E_\eta(\rho) \leq E^\infty_{\mathrm{R},\mathrm{PPT}}(\rho) \leq E_{\mathrm{C},\mathrm{PPT}}(\rho) \leq E_{\mathrm{C},\mathrm{LOCC}}(\rho) \leq 1$. Finally, it is easy to obtain that $\|\rho^{T_B}\|_1 = 2$ and $\|P^{T_B}\|_\infty = \frac{1}{2}$. As a result, $E_{\mathrm{D},\mathrm{PPT}}(\rho) = E_{\mathrm{C},\mathrm{PPT}}(\rho) = R^\infty(\rho) = E_\eta(\rho) = E_\mathrm{N}(\rho) = 1$.

Next, we find a class of states supported on the $N \bigotimes N$ antisymmetric subspace satisfying reversibility and $E_\mathrm{N}(\rho) = R^\infty(\rho) = E_\eta(\rho)$.

Proposition 3 We can find a rank-t ($t \leq n$) state supported on the $N \bigotimes N$ ($N \geq 2n, n \geq 2$) antisymmetric subspace satisfying reversibility and $E_\eta(\rho) = R^\infty(\rho) = E_\mathrm{N}(\rho) = 1$.

Proof For $\rho = p_1|V_1\rangle\langle V_1| + p_2|V_2\rangle\langle V_2| + \cdots + p_t|V_t\rangle\langle V_t|$ $(0 < p_i < 1)$, where

$$|V_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{46}$$

$$|V_2\rangle = \frac{1}{\sqrt{2}}(|23\rangle - |32\rangle), \tag{47}$$

$$\cdots$$

$$|V_t\rangle = \frac{1}{\sqrt{2}}(|2t-1, 2t-2\rangle - |2t-2, 2t-1\rangle), \tag{48}$$

choosing

$$\begin{aligned} Y = \frac{1}{2}[&|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22| + |33\rangle\langle 33| + \cdots \\ &+ |2t-1\rangle\langle 2t-1| + |01\rangle\langle 01| + |10\rangle\langle 10| + \cdots \\ &+ |2t-2, 2t-1\rangle\langle 2t-2, 2t-1| \\ &+ |2t-1, 2t-2\rangle\langle 2t-1, 2t-2|], \end{aligned} \tag{49}$$

we find that

$$\begin{aligned} P^{T_B} = \frac{1}{2}[&|01\rangle\langle 01| + |10\rangle\langle 10| + \cdots \\ &+ |2t-2, 2t-1\rangle\langle 2t-2, 2t-1| \\ &+ |2t-1, 2t-2\rangle\langle 2t-1, 2t-2| \\ &- (|00\rangle\langle 11| + |11\rangle\langle 00| + \cdots \end{aligned}$$

$$+|2t-2,2t-2\rangle\langle 2t-1,2t-1|$$
$$+|2t-1,2t-1\rangle\langle 2t-2,2t-2|)]. \quad (50)$$

Noting that $Y \pm P^{T_B} \geq 0$, we can find that Y is a feasible solution to $E_\eta(\rho)$. It is clear that

$$\|Y^{T_B}\|_\infty = \frac{1}{2}. \quad (51)$$

Thus,

$$E_\eta(\rho) \geq -\log \|Y^{T_B}\|_\infty = -\log \frac{1}{2} = 1. \quad (52)$$

Because a $4\bigotimes 4$ maximally entangled state can efficiently prepare an exact copy of ρ by LOCC operations, we will find that $E_{\mathrm{C,LOCC}}(\rho) \leq 1$ and

$$1 \leq E_\eta(\rho) \leq E^{\infty}_{\mathrm{R,PPT}}(\rho) \leq E_{\mathrm{C,PPT}}(\rho)$$
$$\leq E_{\mathrm{C,LOCC}}(\rho) \leq 1. \quad (53)$$

It is easy to check that $\|\rho^{T_B}\|_1 = 2$ and $\|P^{T_B}\|_\infty = \frac{1}{2}$. Combining all the above, we have as a consequence that $E_{\mathrm{D,PPT}}(\rho) = E_{\mathrm{C,PPT}}(\rho) = E_\eta(\rho) = R^\infty(\rho) = E_{\mathrm{N}}(\rho) = 1$.

Corollary 2 We can find a rank-t ($t \leq n$) state supported on the $N\bigotimes N$ ($N \geq 2n, n \geq 2$) symmetric subspace satisfying reversibility and $E_\eta(\rho) = R^\infty(\rho) = E_{\mathrm{N}}(\rho) = 1$.

Remark We find that if there exists a feasible solution such that $E^{(1)}_{0,\mathrm{D,PPT}}(\rho) \geq 1$ for these states, they will satisfy reversibility. It means that we must find an R satisfying $P \leq R \leq I$ and $\|R^{T_B}\|_\infty = \frac{1}{2}$.

4. Conclusion

In this paper, we study the entanglement cost and distillable entanglement to obtain reversibility and $E_{\mathrm{N}}(\rho) = E_\eta(\rho) = R^\infty(\rho)$ on some conditions. As a result, we find a class of rank-two $4\bigotimes 4$ antisymmetric states and symmetric states under PPT operations and extend them to a class of states in high dimensions satisfying reversibility and $E_{\mathrm{N}}(\rho) = E_\eta(\rho) = R^\infty(\rho)$. Indeed, we also calculate $E_{\mathrm{D,PPT}}(\rho)$ and $E_{\mathrm{C,PPT}}(\rho)$ for these states. It also remains unknown whether all the states satisfy $E_\eta(\rho) \geq R^\infty(\rho)$ and $E_\eta(\rho) \geq E_{\mathrm{N}}(\rho)$. The asymptotic reversibility of a class of operations will lead to a unique measure of entanglement which plays a role similar to the entropy in thermodynamics and the unique ordering is provided for some entangled states. In future work, we want to find a way to classify the states with respect to reversibility and irreversibility under PPT operations, and we will attach more importance to the measure of entanglement.

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