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Heteroclinic cycles in a new class of four-dimensional discontinuous piecewise affine systems*

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It is a huge challenge to give an existence theorem for heteroclinic cycles in the high-dimensional discontinuous piecewise systems (DPSs). This paper first provides a new class of four-dimensional (4D) two-zone discontinuous piecewise affine systems (DPASs), and then gives a useful criterion to ensure the existence of heteroclinic cycles in the systems by rigorous mathematical analysis. To illustrate the feasibility and efficiency of the theory, two numerical examples, exhibiting chaotic behaviors in a small neighborhood of heteroclinic cycles, are discussed.

Keywords: heteroclinic cycle, chaos, discontinuous piecewise affine system

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1. Introduction

In recent years, chaotic dynamical systems have become a major topic of interest in the nonlinear science. The understanding of chaotic behavior is crucial for the development of chaotic dynamical systems. It has many potential applications in engineering fields, such as biology, mechanical engineering, private communication, and artificial intelligence.[1–13]

Since 1963, Lorenz put his research findings,[14] the first physical evidence of chaos, into the public article, various researchers all over the world sparked off a craze for studying chaotic behaviors. A lot of achievements[15–21] have been made in the chaotic field. However, interesting issues that how to ensure the existence of heteroclinic cycles and how the heteroclinic cycles lead to chaotic behaviors in the high-dimensional dynamical systems are still here. Fortunately, the famous Shil'nikov theorem[22,23] overcomes difficulties and gives part of the answers to that issue. The theorem suggests that the existence of heteroclinic cycles means chaos presenting in a small enough neighborhood of this special orbit. In other words, the Shil'nikov theorem gives a criterion to ensure the existence of chaos by heteroclinic cycles in smooth systems. For the discontinuity systems, because of the discontinuity in the switch manifold, the Shil'nikov theorem cannot be used directly. Meanwhile, the theorem is hard to do in practice, so detecting the appearance of heteroclinic cycles and chaos in high-dimensional systems is still a great challenge now.

However, for some discontinuous piecewise systems (DPSs), we can obtain the analytic expressions for stable manifolds and unstable manifolds of subsystem, and then describe the intersection of these manifolds and switch manifolds, whereby finding a heteroclinic cycle. So scholars and experts have paid their attention to these special DPSs, and there have been several theoretical results. For instance, in 2008, Carmona[24] gave an analytical proof of the existence of a reversible T-point heteroclinic cycle in a continuous piecewise linear version of the widely studied Michelson system. In 2009, Li[25] proved the existence of the heteroclinic cycle in a kind of piecewise linear chaotic system. In 2011, Bao[26] provided a new series method for continuous-time autonomous dynamical systems. In 2014, Leonov[27] formulated and proved a fishing principle for the existence of homoclinic and heteroclinic trajectories. In 2015, based on the heteroclinic Shil'nikov theorem and switching control, Han[28] constructed a kind of multipiecewise linear chaotic system. In 2016, Wu[29] considered a design of chaos generator and gave the existence of heteroclinic cycle and chaos in some classes of three-dimensional (3D) piecewise affine systems (PASs). In 2017, Wang[30] studied a class of 3D PASs and presented succinct sufficient conditions for the existence of three types of heteroclinic cycles by mathematical analysis. Carmona,[31] taking advantage of the reversibility and some geometrical features of a piecewise linear version of the Michelson system, constructed a global problem that includes homoclinic connections and T-point heteroclinic cycles as particular cases. In 2018, Chen[32] investigated the existence of heteroclinic cycles in 3D three-zone PASs with two switching planes. Yang and Lu[33] gave the existence of orbit homoclinic and chaos in four-dimensional (4D) PASs. Wu[34] also considered a class of 4D PASs

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}\boldsymbol{x} + \boldsymbol{a}, & x_1 < 0, \\ \boldsymbol{B}\boldsymbol{x} + \boldsymbol{b}, & x_1 \geq 0, \end{cases}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ both have two differential pairs of conjugate

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complex eigenvalues. Furthermore, the conditions for the existence of heteroclinic cycles were presented, and chaotic invariant set in a small neighborhood of the heteroclinic cycles was obtained by numerical simulation. However, to the best of our knowledge, except for several works, little seems to be known about heteroclinic cycle in 4D DPSs, even in the system governed by the simplest 4D equations.

In this paper, the existence of heteroclinic cycles is established in a new class of 4D two-zone DPASs. By analyzing the stable and unstable manifolds of equilibriums of the subsystems, we describe the intersection of these manifolds. Then, the heteroclinic cycles are achieved accurately. Furthermore, under the help of Shil'nikov theory, a chaotic invariant set is also obtained by numerical simulation.

The rest of this paper is organized as follows. Section 2 introduces a new class of 4D two-zone DPASs, and proposes a criterion, proved by rigorous analysis, to detect the appearance of heteroclinic cycles. Section 3 offers two numerical examples, displaying chaotic behavior, to depict the availability of analytic results. Section 4 discusses the results found in this paper.

2. Existence of heteroclinic cycle

In this section, a new class of 4D DPASs are introduced first. Then the existence of heteroclinic cycle in this system will be shown in detail.

Consider the 4D piecewise affine systems

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}\boldsymbol{x}+\boldsymbol{a}, & 2x_1+x_3<1, \\ \boldsymbol{B}\boldsymbol{x}+\boldsymbol{b}, & 2x_1+x_3\geq 1, \end{cases} \tag{1}$$

where $\boldsymbol{x}=(x_1\ x_2\ x_3\ x_4)^{\mathrm{T}}\in\mathbb{R}^4$, $\boldsymbol{a}$, $\boldsymbol{b}\in\mathbb{R}^4$ are constant vectors, and for 4×4 invertible matrices $\boldsymbol{P}$, $\boldsymbol{Q}$,

$$\boldsymbol{A}=\boldsymbol{P}\boldsymbol{J}_{\boldsymbol{A}}\boldsymbol{P}^{-1}=\boldsymbol{P}\begin{pmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & \gamma_2 \end{pmatrix}\boldsymbol{P}^{-1},$$

$$\boldsymbol{B}=\boldsymbol{Q}\boldsymbol{J}_{\boldsymbol{B}}\boldsymbol{Q}^{-1}=\boldsymbol{Q}\begin{pmatrix} \rho & -\omega & 0 & 0 \\ \omega & \rho & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}\boldsymbol{Q}^{-1}, \tag{2}$$

with α, β, ρ, $\omega>0$, $\lambda_{1,2}$, $\gamma_{1,2}<0$.

Let the right half part

$$\Sigma^+=\{\boldsymbol{x}\in\mathbb{R}^4\mid 2x_1+x_3>1\},$$

the left half part

$$\Sigma^-=\{\boldsymbol{x}\in\mathbb{R}^4\mid 2x_1+x_3<1\},$$

and the switch mainfold

$$\Sigma=\{\boldsymbol{x}\in\mathbb{R}^4\mid 2x_1+x_3=1\}.$$

It is clear that the left subsystem of system (1) on Σ^-

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{a} \tag{3}$$

has an equilibrium point denoted by $\boldsymbol{p}=(x_{1\boldsymbol{p}}\ x_{2\boldsymbol{p}}\ x_{3\boldsymbol{p}}\ x_{4\boldsymbol{p}})^{\mathrm{T}}=-\boldsymbol{A}^{-1}\boldsymbol{a}$. And the right subsystem of system (1) on Σ^+

$$\dot{\boldsymbol{x}}=\boldsymbol{B}\boldsymbol{x}+\boldsymbol{b} \tag{4}$$

has an equilibrium point denoted by $\boldsymbol{q}=(x_{1\boldsymbol{q}}\ x_{2\boldsymbol{q}}\ x_{3\boldsymbol{q}}\ x_{4\boldsymbol{q}})^{\mathrm{T}}=-\boldsymbol{B}^{-1}\boldsymbol{b}$. Because $\boldsymbol{p}\in\Sigma^-$ and $\boldsymbol{q}\in\Sigma^+$, so we have $2x_{1\boldsymbol{p}}+x_{3\boldsymbol{p}}<1$ and $2x_{1\boldsymbol{q}}+x_{3\boldsymbol{q}}>1$.

For convenience, let $\boldsymbol{P}=(\boldsymbol{\xi}_1\ \ \boldsymbol{\xi}_2\ \ \boldsymbol{\xi}_3\ \ \boldsymbol{\xi}_4)$, $\boldsymbol{Q}=(\boldsymbol{\eta}_1\ \ \boldsymbol{\eta}_2\ \ \boldsymbol{\eta}_3\ \ \boldsymbol{\eta}_4)$ with $\boldsymbol{\xi}_i$, $\boldsymbol{\eta}_i\in\mathbb{R}^4$, $(i=1,2,3,4)$ being the general eigenvectors of matrix $\boldsymbol{A}$ and matrix $\boldsymbol{B}$, respectively. Assume that $\boldsymbol{\psi}_1(t,\boldsymbol{x}_0)$ is a solution of system (3) with initial value $\boldsymbol{x}_0$, and $\boldsymbol{\psi}_2(t,\boldsymbol{y}_0)$ is a solution of system (4) with initial value $\boldsymbol{y}_0$, where

$$\boldsymbol{x}_0=\boldsymbol{\psi}_1(0,\boldsymbol{x}_0)=\boldsymbol{p}+(\boldsymbol{\xi}_1\ \ \boldsymbol{\xi}_2\ \ \boldsymbol{\xi}_3\ \ \boldsymbol{\xi}_4)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{5}$$

$$\boldsymbol{y}_0=\boldsymbol{\psi}_2(0,\boldsymbol{y}_0)=\boldsymbol{q}+(\boldsymbol{\eta}_1\ \ \boldsymbol{\eta}_2\ \ \boldsymbol{\eta}_3\ \ \boldsymbol{\eta}_4)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{6}$$

From systems (3) and (4), we obtain

$$\boldsymbol{\psi}_1(t,\boldsymbol{x}_0)=\mathrm{e}^{\boldsymbol{A}t}(\boldsymbol{x}_0-\boldsymbol{p})+\boldsymbol{p}, \tag{7}$$

$$\boldsymbol{\psi}_2(t,\boldsymbol{y}_0)=\mathrm{e}^{\boldsymbol{B}t}(\boldsymbol{y}_0-\boldsymbol{q})+\boldsymbol{q}. \tag{8}$$

Substituting Eq. (5) into Eq. (7) and Eq. (6) into Eq. (8), we have

$$\begin{aligned}\boldsymbol{\psi}_1(t,\boldsymbol{x}_0) &= \boldsymbol{p}+(\boldsymbol{\xi}_1\ \ \boldsymbol{\xi}_2\ \ \boldsymbol{\xi}_3\ \ \boldsymbol{\xi}_4) \\ &\quad\times\begin{pmatrix} \mathrm{e}^{\alpha t}(x_1\cos\beta t-x_2\sin\beta t) \\ \mathrm{e}^{\alpha t}(x_1\sin\beta t+x_2\cos\beta t) \\ \mathrm{e}^{\gamma_1 t}x_3 \\ \mathrm{e}^{\gamma_2 t}x_4 \end{pmatrix},\end{aligned} \tag{9}$$

$$\begin{aligned}\boldsymbol{\psi}_2(t,\boldsymbol{y}_0) &= \boldsymbol{q}+(\boldsymbol{\eta}_1\ \ \boldsymbol{\eta}_2\ \ \boldsymbol{\eta}_3\ \ \boldsymbol{\eta}_4) \\ &\quad\times\begin{pmatrix} \mathrm{e}^{\rho t}(x_1\cos\omega t-x_2\sin\omega t) \\ \mathrm{e}^{\rho t}(x_1\sin\omega t+x_2\cos\omega t) \\ \mathrm{e}^{\lambda_1 t}x_3 \\ \mathrm{e}^{\lambda_2 t}x_4 \end{pmatrix}.\end{aligned} \tag{10}$$

According to Eqs. (9) and (10), it is clear that the unstable mainfolds, through $\boldsymbol{p}$, $\boldsymbol{q}$ and denoted by $W^{\mathrm{u}}(\boldsymbol{p})$, $W^{\mathrm{u}}(\boldsymbol{q})$, are two-dimensional invariant mainfolds. The stable invariant mainfolds, through $\boldsymbol{p}$, $\boldsymbol{q}$ and denoted by $W^{\mathrm{s}}(\boldsymbol{p})$, $W^{\mathrm{s}}(\boldsymbol{q})$, also are two dimensional. The concrete forms are as follows:

$$W^{\mathrm{u}}(\boldsymbol{p})=\{\boldsymbol{p}+k_1\boldsymbol{\xi}_1+k_2\boldsymbol{\xi}_2\mid k_1,k_2\in\mathbb{R}\}, \tag{11}$$

$$W^{\mathrm{s}}(\boldsymbol{p})=\{\boldsymbol{p}+k_3\boldsymbol{\xi}_3+k_4\boldsymbol{\xi}_4\mid k_3,k_4\in\mathbb{R}\}, \tag{12}$$

$$W^{\mathrm{u}}(\boldsymbol{q})=\{\boldsymbol{q}+k_1'\boldsymbol{\eta}_1+k_2'\boldsymbol{\eta}_2\mid k_1',k_2'\in\mathbb{R}\}, \tag{13}$$

$$W^{\mathrm{s}}(\boldsymbol{q}) = \{\boldsymbol{q} + k_3'\boldsymbol{\eta}_3 + k_4'\boldsymbol{\eta}_4 \mid k_3', k_4' \in \mathbb{R}\}, \tag{14}$$

with k_1, k_2, k_3, k_4 and k_1', k_2', k_3', $k_4' \in \mathbb{R}$.

The following theorem establishes a criterion for the existence of heteroclinic cycle of 4D piecewise affine system (1).

Theorem 1 If there exist real constants k_i, k_i', $(i = 1, 2, 3, 4)$ such that the following conditions (i)–(iii) hold:

(i) $\boldsymbol{p}_1 = \boldsymbol{p} + k_1\boldsymbol{\xi}_1 + k_2\boldsymbol{\xi}_2 = \boldsymbol{q} + k_3'\boldsymbol{\eta}_3 + k_4'\boldsymbol{\eta}_4 \in \Sigma$,

(ii) $\boldsymbol{q}_1 = \boldsymbol{q} + k_1'\boldsymbol{\eta}_1 + k_2'\boldsymbol{\eta}_2 = \boldsymbol{p} + k_3\boldsymbol{\xi}_3 + k_4\boldsymbol{\xi}_4 \in \Sigma$;
$\gamma_1 k_3(2\xi_{31} + \xi_{33}) + \gamma_2 k_4(2\xi_{41} + \xi_{43}) < 0$,
$\lambda_1 k_3'(2\eta_{31} + \eta_{33}) + \lambda_2 k_4'(2\eta_{41} + \eta_{43}) > 0$,
$-\alpha(1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}) - \beta[k_1(2\xi_{21} + \xi_{23})$
$-k_2(2\xi_{11} + \xi_{13})] < 0$,
$-\rho(1 - 2x_{1\boldsymbol{q}} - x_{3\boldsymbol{q}}) - \omega[k_1'(2\eta_{21} + \eta_{23})$
$-k_2'(2\eta_{11} + \eta_{13})] > 0$,

where ξ_{ij}, η_{ij} are the j-th coordinate of $\boldsymbol{\xi}_i$, $\boldsymbol{\eta}_i$, $(i, j = 1, 2, 3, 4)$;

(iii) $$\mathrm{e}^{-\alpha T_1}U_1\frac{\beta}{\sqrt{\alpha^2+\beta^2}} < 1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}},$$
$$\mathrm{e}^{-\rho T_2}U_2\frac{-\omega}{\sqrt{\rho^2+\omega^2}} > 1 - 2x_{1\boldsymbol{q}} - x_{3\boldsymbol{q}},$$

where

$$\begin{aligned}
u_1 &= k_1(2\xi_{21} + \xi_{23}) - k_2(2\xi_{11} + \xi_{13}),\\
u_2 &= k_1'(2\eta_{21} + \eta_{23}) - k_2'(2\eta_{11} + \eta_{13}),\\
U_1 &= \sqrt{(1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}})^2 + u_1^2},\\
U_2 &= \sqrt{(1 - 2x_{1\boldsymbol{q}} - x_{3\boldsymbol{q}})^2 + u_2^2},
\end{aligned}$$

$$T_1 = \begin{cases} \dfrac{1}{\beta}\arctan\dfrac{\beta}{\alpha} + \dfrac{1}{\beta}\arcsin\dfrac{1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}}{U_1} + \dfrac{\pi}{\beta}, & u_1 \geq 0,\\ \dfrac{1}{\beta}\arctan\dfrac{\beta}{\alpha} + \dfrac{1}{\beta}\arctan\dfrac{1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}}{u_1} + \dfrac{2\pi}{\beta}, & u_1 < 0, \end{cases}$$

$$T_2 = \begin{cases} \dfrac{1}{\omega}\arctan\dfrac{\omega}{\rho} - \dfrac{1}{\omega}\arcsin\dfrac{1 - 2x_{1\boldsymbol{q}} - x_{3\boldsymbol{q}}}{U_2} + \dfrac{\pi}{\omega}, & u_2 \geq 0,\\ \dfrac{1}{\omega}\arctan\dfrac{\omega}{\rho} - \dfrac{1}{\omega}\arctan\dfrac{1 - 2x_{1\boldsymbol{q}} - x_{3\boldsymbol{q}}}{u_2} + \dfrac{2\pi}{\omega}, & u_2 < 0. \end{cases}$$

Then system (1) has a heteroclinic cycle Γ connecting equilibrium points $\boldsymbol{p}$ and $\boldsymbol{q}$, which intersect Σ at $\boldsymbol{p}_1$, $\boldsymbol{q}_1$ transversally.

Proof To prove Theorem 1, we only need to show that the following conditions (H_1)–(H_6) hold:

(H_1) $\boldsymbol{p}_1 \in L_1 \cap L_4$, $\boldsymbol{q}_1 \in L_2 \cap L_3$;

(H_2) the negative orbit of $\boldsymbol{p}_1$ satisfies $\{\psi_1(t, \boldsymbol{p}_1) \mid t < 0\} \subset \Sigma^-$;

(H_3) the positive orbit of $\boldsymbol{q}_1$ satisfies $\{\psi_1(t, \boldsymbol{q}_1) \mid t > 0\} \subset \Sigma^-$;

(H_4) the negative orbit of $\boldsymbol{q}_1$ satisfies $\{\psi_2(t, \boldsymbol{q}_1) \mid t < 0\} \subset \Sigma^+$;

(H_5) the positive orbit of $\boldsymbol{p}_1$ satisfies $\{\psi_2(t, \boldsymbol{p}_1) \mid t > 0\} \subset \Sigma^+$;

(H_6)

$$(2\ 0\ 1\ 0)(\boldsymbol{A}\boldsymbol{p}_1 + \boldsymbol{a}) > 0,\ \ (2\ 0\ 1\ 0)(\boldsymbol{A}\boldsymbol{q}_1 + \boldsymbol{a}) < 0,$$
$$(2\ 0\ 1\ 0)(\boldsymbol{B}\boldsymbol{p}_1 + \boldsymbol{b}) > 0,\ \ (2\ 0\ 1\ 0)(\boldsymbol{B}\boldsymbol{q}_1 + \boldsymbol{b}) < 0,$$

where $L_1 = W^{\mathrm{u}}(\boldsymbol{p}) \cap \Sigma$, $L_2 = W^{\mathrm{s}}(\boldsymbol{p}) \cap \Sigma$, $L_3 = W^{\mathrm{u}}(\boldsymbol{q}) \cap \Sigma$, $L_4 = W^{\mathrm{s}}(\boldsymbol{q}) \cap \Sigma$, and $L_1 \cap L_4 \neq \emptyset$, $L_2 \cap L_3 \neq \emptyset$.

The condition (i) in Theorem 1 is equivalent to the condition (H_1), which can be obtained from simple analysis. In the following we further show condition (H_2) holding.

Let

$$g_1(t) = \boldsymbol{c}'(\psi_1(-t, \boldsymbol{p}_1) - \boldsymbol{p}), \tag{15}$$

where $\boldsymbol{c}' = (2\ 0\ 1\ 0)$ and ψ_1 is defined as Eq. (9). According to $\boldsymbol{p}_1 \in \text{Ł}_1$, one obtains

$$g_1(t) = \mathrm{e}^{-\alpha t}U_1\sin(-\beta t + \theta_1), \tag{16}$$
$$g_1'(t) = \mathrm{e}^{-\alpha t}U_1\Big(-\alpha\sin(\beta t + \theta_1) - \beta\cos(\beta t + \theta_1)\Big), \tag{17}$$

where

$$\sin\theta_1 = \frac{1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}}{U_1},$$
$$\cos\theta_1 = \frac{k_1(2\xi_{21} + \xi_{23}) - k_2(2\xi_{11} + \xi_{13})}{U_1}.$$

If $g_1(t) < 1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}$ for arbitrary $t > 0$, it is clear that the negative orbit of $\boldsymbol{p}_1$ satisfies $\{\psi_1(t, \boldsymbol{p}_1) \mid t < 0\} \subset \Sigma^-$. That means we simply need to prove $g_1(t) < 1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}$ for all $t > 0$. In fact, by Eq. (16) and the third inequation in condition (ii) in Theorem 1, we obtain

$$\begin{aligned}
g_1(0) &= 1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}} > 0,\\
g_1'(0) &= -\alpha(1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}) - \beta[k_1(2\xi_{21} + \xi_{23})\\
&\quad - k_2(2\xi_{11} + \xi_{13})] < 0.
\end{aligned}$$

This means $g_1(t) < 1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}$ for $t \in (0, \varepsilon)$ with sufficient small ε. Let $T' = -\frac{1}{\alpha}\ln\frac{\varepsilon}{U_1}$ with sufficiently small ε, then $g_1(t) < 1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}$ for $t \in [T', +\infty)$. We further show $g_1(t) < 1 - 2x_{1\boldsymbol{p}} - x_{3\boldsymbol{p}}$ for $t \in I = [\varepsilon/2, T' + \frac{2\pi}{\beta}]$.

Let $g_1'(t_0) = 0$, $t_0 > 0$, then t_0 satisfies

$$\tan(-\beta t_0 + \theta_1) = -\frac{\beta}{\alpha}, \tag{18}$$

i.e., either

$$\sin(-\beta t_0 + \theta_1) = -\frac{\beta}{\sqrt{\alpha^2+\beta^2}},$$
$$\cos(-\beta t_0 + \theta_1) = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \tag{19}$$

or

$$\sin(-\beta t_0 + \theta_1) = \frac{\beta}{\sqrt{\alpha^2+\beta^2}},$$

$$\cos(-\beta t_0+\theta_1)=-\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}. \tag{20}$$

Moreover, from Eq. (17), we have

$$\begin{aligned} g_1''(t) &= \mathrm{e}^{-\alpha t}U_1\big((\alpha^2-\beta^2)\sin(-\beta t+\theta_1)\\ &\quad +2\alpha\beta\cos(-\beta t+\theta_1)\big). \end{aligned} \tag{21}$$

When equation (19) holds for $t_0\in I$, we have $g_1''(t_0)>0$, which means that t_0 is a local minimum point of $g_1(t)$. Similarly, when equation (20) holds for $t_0\in I$, we obtain $g_1''(t_0)<0$, which means that t_0 is a local maximum point of $g_1(t)$.

If $k_1(2\xi_{21}+\xi_{23})-k_2(2\xi_{11}+\xi_{13})\geq 0$, we obtain that the local maximum points t_{0k}[35] of $g_1(t)$ satisfy

$$\begin{aligned} t_{0k} &= \frac{1}{\beta}\arctan\frac{\beta}{\alpha}+\frac{1}{\beta}\arcsin\frac{1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}}}{U_1}+\frac{k\pi}{\beta}\in I,\\ &\quad k=1,2,\ldots,[\beta T'\pi]=k_0. \end{aligned}$$

If $k_1(2\xi_{21}+\xi_{23})-k_2(2\xi_{11}+\xi_{13})<0$, we obtain that the local maximum points t_{0k} of $g_1(t)$ satisfy

$$\begin{aligned} t_{0k} &= \frac{1}{\beta}\arctan\frac{\beta}{\alpha}+\frac{1}{\beta}\arctan\frac{1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}}}{k_1(2\xi_{21}+\xi_{23})-k_2(2\xi_{11}+\xi_{13})}\\ &\quad +\frac{(2+k)\pi}{\beta}\in I,\ \ k=0,1,\ldots,[\beta T'\pi]=k_0. \end{aligned}$$

Then the local maximum value is given by

$$g_1(t_{0k})=U_1\,\mathrm{e}^{-\alpha t_{0k}}\frac{\beta}{\alpha^2+\beta^2}. \tag{22}$$

According to $\alpha>0$ and the first inequation in condition (iii) of Theorem 1, it yields $g_1(t_{0k_0})<\cdots<g_1(t_{01})<1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}}$. Then $g_1(t)<1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}}$ on $t\in I$. Therefore (H_2) holds.

By the same method, (H_4) also can be proved if the fourth inequation in condition (ii) and the second inequation in condition (iii) of Theorem 1 hold.

Next (H_3), $\{\psi_1(t,\boldsymbol{q}_1)\mid t>0\}\subset\Sigma^-$, will be shown. Let

$$g_2(t)=\boldsymbol{c}'(\psi_1(t,\boldsymbol{q}_1)-\boldsymbol{p}).$$

Then $\{\psi_1(t,\boldsymbol{q}_1)\mid t>0\}\subset\Sigma^-$ will hold if $g_2(t)<1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}}$ for $t>0$. Thus $g_2(t)<1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}}$ for $t>0$ will be shown in following.

According to Eq. (9) and $\boldsymbol{q}_1\in Ł_2$, we have

$$\begin{aligned} g_2(t) &= k_3(2\xi_{31}+\xi_{33})\,\mathrm{e}^{\gamma_1 t}+k_4(2\xi_{41}+\xi_{43})\,\mathrm{e}^{\gamma_2 t},\\ g_2'(t) &= \mathrm{e}^{\gamma_1 t}h(t), \end{aligned} \tag{23}$$

where

$$h(t)=\gamma_1 k_3(2\xi_{31}+\xi_{33})+\gamma_2 k_4(2\xi_{41}+\xi_{43})\,\mathrm{e}^{(\gamma_2-\gamma_1)t}.$$

From expression (23) and the first in equation in condition (ii) of Theorem 1, we have

$$g_2(0)=1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}},\ \ g_2'(0)<0,$$

$$\begin{aligned} &h(0)=\gamma_1 k_3(2\xi_{31}+\xi_{33})+\gamma_2 k_4(2\xi_{41}+\xi_{43})<0,\\ &g_2(t)\to 0\ \text{ for }\ t\to+\infty. \end{aligned}$$

Since function $h(t)$ is monotonous, $h(t)=0$ has only one solution at most for $t>0$.

If $h(t)=0$ has no solution for $t>0$, owing to $h(0)<0$, we have $h(t)<0$ for $t>0$, which implies $g_2'(t)<0$ for $t>0$. Hence $g_2(t)<1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}}$ for $t>0$ as shown in Fig. 1(a). Alternatively, $h(t)=0$ has unique solution $\bar{t}>0$, owing to $h(0)<0$,

$$h(t)<0\ \text{ for }\ 0<t<\bar{t}\ \text{ and }\ h(t)>0\ \text{ for }\ t>\bar{t},$$

which implies $g_2'(t)<0$ for $0<t<\bar{t}$, and $g_2'(t)>0$ for $t>\bar{t}$. Combining with $g_2(t)\to 0$ for $t\to+\infty$, we always have $g_2(t)<1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}}$ for $t>0$ as shown in Fig. 1(b). Thus condition (H_3) is proved. Similarly, througth the second inequation in condition (ii), we also can prove the condition (H_5).

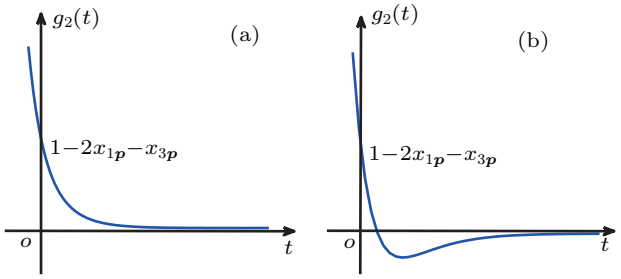


Fig. 1. (color online) The plot of $g_2(t)$ for $t>0$ when (a) $h(t)=0$ has no solution, (b) $h(t)=0$ has a unique solution.

According to the inequalities in conditions (ii) and (iii) of Theorem 1, we have

$$\begin{aligned} &(2\ 0\ 1\ 0)(\boldsymbol{A}\boldsymbol{q}_1+\boldsymbol{a})\\ &= \gamma_1 k_3(2\xi_{31}+\xi_{33})+\gamma_2 k_4(2\xi_{41}+\xi_{43})<0,\\ &(2\ 0\ 1\ 0)(\boldsymbol{B}\boldsymbol{p}_1+\boldsymbol{b})\\ &= \lambda_1 k_3'(2\eta_{31}+\eta_{33})+\lambda_2 k_4'(2\eta_{41}+\eta_{43})>0,\\ &(2\ 0\ 1\ 0)(\boldsymbol{B}\boldsymbol{q}_1+\boldsymbol{b})\\ &= \rho(1-2x_{1\boldsymbol{q}}-x_{3\boldsymbol{q}})\\ &\quad +\omega[k_1'(2\eta_{21}+\eta_{23})-k_2'(2\eta_{11}+\eta_{13})]<0,\\ &(2\ 0\ 1\ 0)(\boldsymbol{A}\boldsymbol{p}_1+\boldsymbol{a})\\ &= \alpha(1-2x_{1\boldsymbol{p}}-x_{3\boldsymbol{p}})\\ &\quad +\beta[k_1(2\xi_{21}+\xi_{23})-k_2(2\xi_{11}+\xi_{13})]>0, \end{aligned}$$

which imply condition (H_6). Therefore, the proof of Theorem 1 is completed.

Remark 1 According to the results in Refs. [35]–[38], suppose that conditions (i)–(iii) of Theorem 1 hold, the eigenvalues of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy $\frac{\alpha\rho}{\gamma\lambda}<1$ with $\gamma=\max\{\gamma_1,\ \gamma_2\}$ and $\lambda=\max\{\lambda_1,\ \lambda_2\}$, then system (1) has a chaotic invariant set in a sufficiently neighborhood of Γ.

3. Numerical simulation

In this section, two numerical tests are given to illustrate the effectiveness of our main results.

Example 1 Consider the following 4D piecewise affine systems

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}\boldsymbol{x}+\boldsymbol{a}, & 2x_1+x_3<1, \\ \boldsymbol{B}\boldsymbol{x}+\boldsymbol{b}, & 2x_1+x_3\geq 1, \end{cases} \tag{24}$$

where

$$\boldsymbol{A} = \begin{pmatrix} -3.4 & 5 & 12.8 & -12.8 \\ -1.25 & -2 & -10 & 10 \\ 2.3 & 2.5 & -1.6 & -18.4 \\ 1.5 & 0 & -3 & -17 \end{pmatrix},$$

$$\boldsymbol{B} = \begin{pmatrix} -8 & -7 & 20 & 0 \\ 0 & -15 & 0 & 0 \\ 10 & 0 & -8 & -40 \\ -5 & -1.75 & 10 & 12 \end{pmatrix},$$

$$\boldsymbol{a} = (1 \ \ -10.5 \ \ 20.5 \ \ 20)^{\mathrm{T}},$$

$$\boldsymbol{b} = (-5 \ \ 15 \ \ 48 \ \ -18.25)^{\mathrm{T}}.$$

By simple calculations, we obtain the equilibrium points

$$\boldsymbol{p} = (-2 \ \ 1 \ \ 0 \ \ 1)^{\mathrm{T}}$$

of subsystem $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}+\boldsymbol{a}$,

$$\boldsymbol{q} = (1 \ \ 1 \ \ 1 \ \ 1.25)^{\mathrm{T}}$$

of subsystem $\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{x}+\boldsymbol{b}$. There exist invariant matrices

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & -0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 1 \\ 0 & 0 & 0.25 & 1 \end{pmatrix},$$

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 \end{pmatrix}.$$

We can respectively convert matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ to Jordan matrices

$$\boldsymbol{J}_{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0.5 & -5 & 0 & 0 \\ 5 & 0.5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -20 \end{pmatrix},$$

$$\boldsymbol{J}_{\boldsymbol{B}} = \boldsymbol{Q}^{-1}\boldsymbol{B}\boldsymbol{Q} = \begin{pmatrix} 2 & -10 & 0 & 0 \\ 10 & 2 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -15 \end{pmatrix}.$$

Choose $k_1 = 0$, $k_2 = 2$, $k_3 = 11/9$, $k_4 = 1/9$ and $k_1' = -8/9$, $k_2' = 1/3$, $k_3' = 0$, $k_4' = -1/2$, we have

$$\boldsymbol{p}_1 = (0 \ \ 0 \ \ 1 \ \ 1)^{\mathrm{T}}, \quad \boldsymbol{q}_1 = \left(\frac{4}{9} \ \ 1 \ \ \frac{1}{9} \ \ \frac{17}{12}\right)^{\mathrm{T}}.$$

Therefore, it is easy to see that

$$\gamma_1 k_3(2\xi_{31}+\xi_{33}) + \gamma_2 k_4(2\xi_{41}+\xi_{43}) = -80/3 < 0,$$
$$\lambda_1 k_3'(2\eta_{31}+\eta_{33}) + \lambda_2 k_4'(2\eta_{41}+\eta_{43}) = 30 > 0,$$
$$\mathrm{e}^{-\alpha T_1}U_1\frac{\beta}{\sqrt{\alpha^2+\beta^2}} - 1 + 2\boldsymbol{x}_{1\boldsymbol{p}} + \boldsymbol{x}_{3\boldsymbol{p}} = -2.3192 < 0,$$
$$\mathrm{e}^{-\rho T_2}U_2\frac{-\omega}{\sqrt{\rho^2+\omega^2}} - 1 + 2\boldsymbol{x}_{1\boldsymbol{q}} + \boldsymbol{x}_{3\boldsymbol{q}} = 0.4586 > 0,$$
$$-\alpha(1-2\boldsymbol{x}_{1\boldsymbol{p}}-\boldsymbol{x}_{3\boldsymbol{p}}) - \beta[k_1(2\xi_{21}+\xi_{23})$$
$$-k_2(2\xi_{11}+\xi_{13})] = -2.5 < 0,$$
$$-\rho(1-2\boldsymbol{x}_{1\boldsymbol{q}}-\boldsymbol{x}_{3\boldsymbol{q}}) - \omega[k_1'(2\eta_{21}+\eta_{23})$$
$$-k_2'(2\eta_{11}+\eta_{13})] = 286/9 > 0.$$

It means system (24) satisfies the conditions (i)–(iii) in Theorem 1. So the system (24) exists a heteroclinic cycle Γ

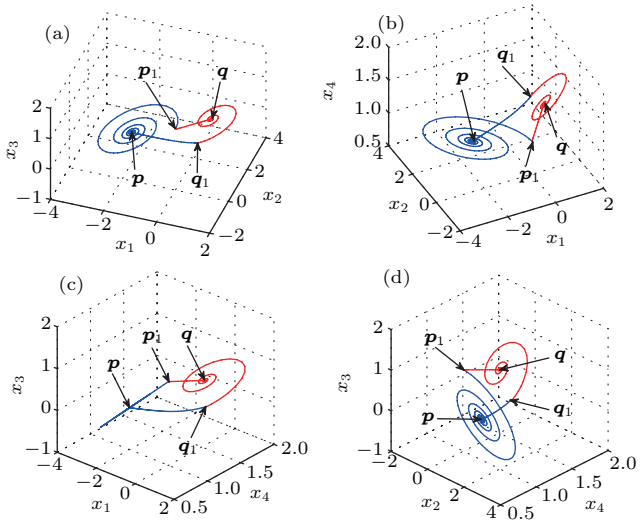


Fig. 2. (color online) The 3D projections of Γ for system (24) in (a) x_1–x_2–x_3 space, (b) x_1–x_2–x_4 space, (c) x_1–x_3–x_4 space, (d) x_2–x_3–x_4 space.

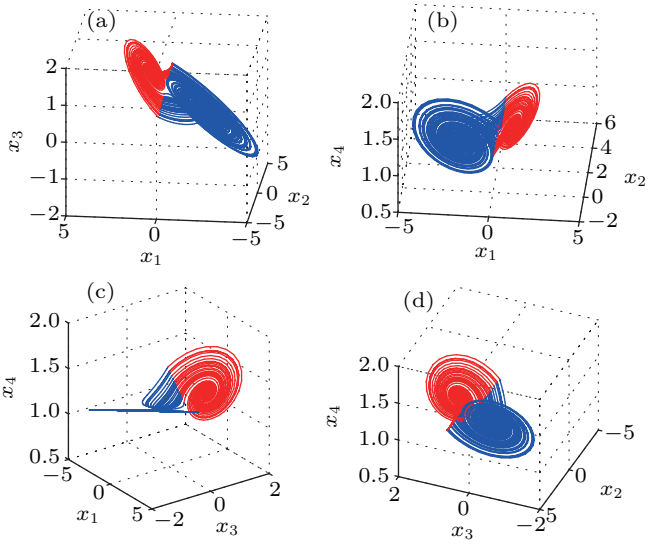


Fig. 3. (color online) The 3D projections of chaotic invariant set for system (24) in (a) x_1–x_2–x_3 space, (b) x_1–x_2–x_4 space, (c) x_1–x_3–x_4 space, (d) x_2–x_3–x_4 space.

connecting equilibrium $\boldsymbol{p}$ and $\boldsymbol{q}$, which intersects Σ transversally at points $\boldsymbol{p}_1$ and $\boldsymbol{q}_1$.

Moreover, the eigenvalues of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy $\frac{\alpha\rho}{\gamma\lambda} = 0.025 < 1$. Thus, by Remark 1, the system (24) has a chaotic invariant set in an adequately small neighborhood of Γ. The 3D projections of Γ and chaotic invariant set of system (24) are shown in Figs. 2 and 3, respectively.

Example 2 Consider the following piecewise affine system:

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}\boldsymbol{x} + \boldsymbol{a}, & 2x_1 + x_3 < 1, \\ \boldsymbol{B}\boldsymbol{x} + \boldsymbol{b}, & 2x_1 + x_3 \geq 1, \end{cases} \tag{25}$$

where

$$\boldsymbol{A} = \begin{pmatrix} 12.4 & 10 & -42.6 & -12.2 \\ -20 & 0.2 & 19.8 & 10 \\ 0 & 0 & -10 & 0 \\ 24.4 & 10 & -44.6 & -24.2 \end{pmatrix},$$

$$\boldsymbol{B} = \begin{pmatrix} 21 & 20 & -71.5 & -20.5 \\ -40 & 0.5 & 44.5 & 20 \\ 0 & 0 & -15 & 0 \\ 41 & 20 & 76.5 & -40.5 \end{pmatrix},$$

$$\boldsymbol{a} = (-38,\ 9.8,\ 10,\ -96)^{\mathrm{T}},$$

$$\boldsymbol{b} = (9,\ 24.5,\ -15,\ 84)^{\mathrm{T}}.$$

The equilibrium points of system (25) are

$$\boldsymbol{p} = (-2,\ 2,\ 1,\ -7)^{\mathrm{T}},\ \boldsymbol{q} = (3,\ 0,\ -1,\ 7)^{\mathrm{T}}$$

of subsystems $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{a}$ and $\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{x} + \boldsymbol{b}$, respectively. There exist invariant matrices

$$\boldsymbol{P} = \boldsymbol{Q} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Convert matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ to Jordan matrices respectively, we have

$$\boldsymbol{J}_{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0.2 & -10 & 0 & 0 \\ 10 & 0.2 & 0 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & -12 \end{pmatrix},$$

$$\boldsymbol{J}_{\boldsymbol{B}} = \boldsymbol{Q}^{-1}\boldsymbol{B}\boldsymbol{Q} = \begin{pmatrix} 0.5 & -20 & 0 & 0 \\ 20 & 0.5 & 0 & 0 \\ 0 & 0 & -15 & 0 \\ 0 & 0 & 0 & -20 \end{pmatrix}.$$

Similarly, according to Theorem 1, choose $k_1 = 0, k_2 = 2, k_3 = -2, k_4 = 7$ and $k_1' = 0, k_2' = -2, k_3' = 2, k_4' = -7$, we have

$$\boldsymbol{p}_1 = (0\ \ 2\ \ 1\ \ -5)^{\mathrm{T}},\quad \boldsymbol{q}_1 = (1\ \ 0\ \ -1\ \ 5)^{\mathrm{T}}.$$

Therefore, it is easy to see that

$$\gamma_1 k_3(2\xi_{31} + \xi_{33}) + \gamma_2 k_4(2\xi_{41} + \xi_{43}) = -80/3 < 0,$$

$$\lambda_1 k_3'(2\eta_{31} + \eta_{33}) + \lambda_2 k_4'(2\eta_{41} + \eta_{43}) = 30 > 0,$$

$$\mathrm{e}^{-\alpha T_1} U_1 \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} - 1 + 2\boldsymbol{x}_{1\boldsymbol{p}} + \boldsymbol{x}_{3\boldsymbol{p}} = -2.3192 < 0,$$

$$\mathrm{e}^{-\rho T_2} U_2 \frac{-\omega}{\sqrt{\rho^2 + \omega^2}} - 1 + 2\boldsymbol{x}_{1\boldsymbol{q}} + \boldsymbol{x}_{3\boldsymbol{q}} = 0.4586 > 0,$$

$$-\alpha(1 - 2\boldsymbol{x}_{1\boldsymbol{p}} - \boldsymbol{x}_{3\boldsymbol{p}}) - \beta[k_1(2\xi_{21} + \xi_{23})$$

$$-k_2(2\xi_{11} + \xi_{13})] = -2.5 < 0,$$

$$-\rho(1 - 2\boldsymbol{x}_{1\boldsymbol{q}} - \boldsymbol{x}_{3\boldsymbol{q}}) - \omega[k_1'(2\eta_{21} + \eta_{23})$$

$$-k_2'(2\eta_{11} + \eta_{13})] = 286/9 > 0.$$

It means that system (25) satisfies the conditions (i)–(iii) in Theorem 1. From Theorem 1, we can also see that system (25) exists a heteroclinic cycle Γ connecting equilibrium points $\boldsymbol{p}$ and $\boldsymbol{q}$, which intersects Σ transversally at points $\boldsymbol{p}_1$ and $\boldsymbol{q}_1$.

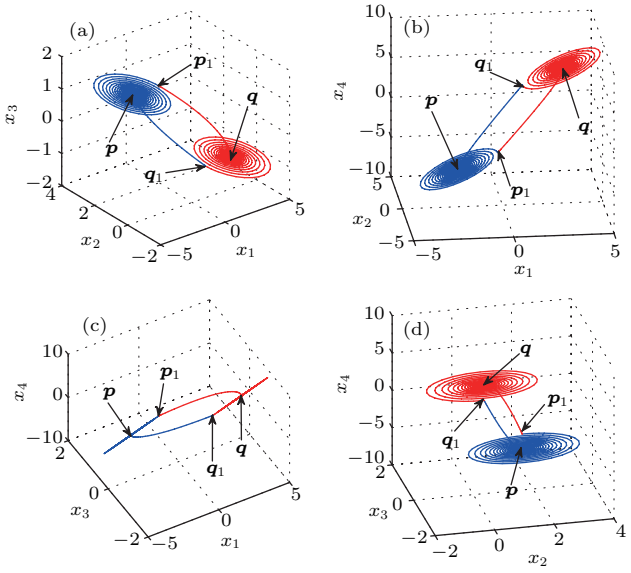


Fig. 4. (color online) The 3D projections of Γ for system (25) in (a) x_1–x_2–x_3 space, (b) x_1–x_2–x_4 space, (c) x_1–x_3–x_4 space, (d) x_2–x_3–x_4 space.

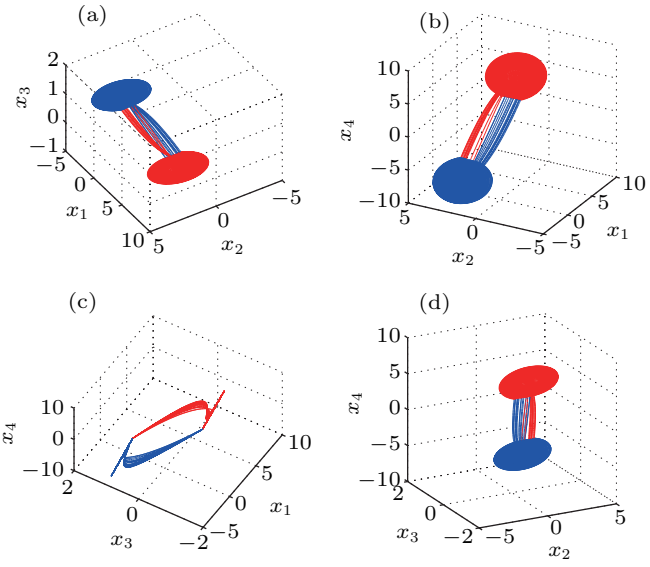


Fig. 5. (color online) The 3D projections of chaotic invariant set for system (25) in (a) x_1–x_2–x_3 space, (b) x_1–x_2–x_4 space, (c) x_1–x_3–x_4 space, (d) x_2–x_3–x_4 space.

In addition, the eigenvalues of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy $\frac{\alpha\rho}{\gamma\lambda} = 0.025 < 1$. Thus, by Remark 1, the system (25) has a chaotic invariant set in an adequately small neighborhood of Γ. The 3D projections of Γ and chaotic invariant set of system (25) are shown in Figs. 4 and 5, respectively.

4. Conclusion

We have constructed a new class of 4D two-zone DPASs. The criterion, ensuring that 4D two-zone DPASs can possess heteroclinic cycles to spiral saddle-foci, is given here. From Shil'nikov type theory, it is clear that such 4D piecewise affine systems display chaotic behaviors. The chaotic invariant sets, in a neighborhood of heteroclinic cycles, are also worked out by numerical method. This makes it a good method to analyze the heteroclinic cycles and chaotic invariant sets of 4D piecewise affine systems. Two numerical examples are used to demonstrate the efficiency of our theoretical results.

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