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# Controllability of heterogeneous interdependent group systems under undirected and directed topology*

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The controllability problem of heterogeneous interdependent group systems with undirected and directed topology is investigated in this paper. First, the interdependent model of the heterogeneous system is set up according to the difference of individual characteristics. An extended distributed protocol with the external sliding-mode control is designed, under which it is shown that a heterogeneous interdependent group system is controllable when the corresponding communication topology is controllable. Then, using the network eigenvalue method, the driving individuals are determined for a heterogeneous system with undirected topology. Under directed topology, the maximum match method is utilized to confirm the driving individuals. Some sufficient and necessary conditions are presented to assure that the heterogeneous interdependent group system is structurally controllable. Via theoretical analysis, the controllability of heterogeneous interdependent systems is related to the interdependent manner and the structure of the heterogeneous system. Numerical simulations are provided to demonstrate the effectiveness of the theoretical results.

**Keywords:** controllability, heterogeneous group systems, interdependent model, sliding-mode control

**PACS:** 89.75.Fb, 89.75.Hc, 89.70.Hj, 89.75.Da **DOI:** 10.1088/1674-1056/27/10/108901

## 1. Introduction

Due to the great progress in science and technology, distributed coordinated control of group systems has been researched intensively. Many results have been acquired and applied in science and engineering fields such as spacecraft formation flying, data fusion of multi-sensors, cooperative surveillance,[1] multiple mobile robotic systems,[2–4] and so forth.[5–9]

From the control point of view, it is known that controllability is a fundamental characteristic of the structure of the system, which reflects the ability of the input to control the state of the system. For a linear control system, Lin[10] presented the concepts of structure and structural controllability, and some necessary and sufficient conditions are given. Liu *et al.*[11] developed analytical tools to research the controllability of several real networks. Through advisably perturbing the network structure, Wang *et al.*[12] proposed an approach to optimizing the controllability complex networks. Hou *et al.*[13] put forward a method to improve the controllability of a directed network via changing the direction of a small fraction of links. Yan *et al.*[14] introduced the concept of exact controllability, and achieved full control of a network based on the maximum multiplicity to confirm the minimum set of drive nodes required. Moreover, Yuan *et al.*[15] analyzed the exact controllability of multiplex networks, which are multiple-relation networks and multiple-layer networks, and found a small fraction of the interconnections to enhance the controllability. However, in the above research results, the interdependence of directed networks is not considered for real systems such as traffic power networks, heterogeneous multi-agent (group) systems, and so on.

For group systems, the purpose of the controllability problem is to drive all individuals achieving desired ideal states from any initial states just through several driver individuals externally. The study of the controllability of group systems is still challenging owing to more factors such as interaction topologies, control protocols, driver individuals selection, and so on. Tanner[16] put forward firstly the concept of the controllability of multi-agent systems, as well as provided an algebraic necessary and sufficient conditions of the system to be controllable by a leader. Based on that, using controllability, Ji *et al.*[17] studied the formation control of multi-agents networks under leader-follower structure. It is shown that the topology structure of the interconnection graph uniquely confirmed the controllability of multi-agent systems. Wang *et al.*[18] investigated the controllability of multi-agent systems with an undirected graph on the basic of agreement protocols. The controllability condition of high-order dynamic agents networks was provided. Parlangeli *et al.*[19] researched the reachability and observability properties of a network system with the communication graph of cycle or paths. On the basis of simple algebraic rules from number theory, some nec-

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essary and sufficient conditions are given. Under switching topologies, Liu *et al.*[20] proposed graph-theoretic characterizations of the structural controllability for multi-agent systems. Further, Yazicioglu *et al.*[21] studied the controllability of diffusively coupled networks from a graph theoretic perspective. More specifically, a graph topological lower bound on the rank of the controllability matrix was presented, that is applicable to systems. Ji *et al.*[22] put forward a neighbor-based control protocol, and showed that the communication topology determined solely the controllability of a multi-agent system from a graph theory perspective. In the above studies, most of multi-agent models are single-integrator dynamics. Liu *et al.*[23] investigated second-order controllability of multi-agent systems with multiple leaders. Consider the velocity coupling topology, some sufficient and necessary conditions were presented for the controllability of the system with multiple leaders. For improving the controllability of multi-agent systems, Zhao *et al.*[24] presented a leader selection algorithm and a weight adjustment algorithm.

From the above discussions, most of the aforementioned works mainly focus on the controllability of homogeneous multi-agent systems. Nevertheless, to the best of our knowledge, that many results are from a few works on the controllability of heterogeneous group systems. Generally speaking, the heterogeneity is common in nature and engineering systems, for instance, the different gender, occupation, interests of individuals in human social groups, and cooperative tasks of different military equipment. Guan *et al.*[25] studied the controllability of multi-agent systems under the leader-follower framework with directed topology. Guan *et al.*[26] investigated the controllability of both continuous and discrete-time linear heterogeneous multi-agent systems. However, it neglected the evolution and cooperation relationships among subsystems. Chen *et al.*[27,28] investigated the model of interdependent networks and the robustness of interdependent networks in cascading failures from different aspects. Inspired by the above analyses, this paper investigates the controllability problem of heterogeneous interdependent group systems under undirected and directed topology. In the present article, from the point of the difference of individual characteristics, the interdependent model of the heterogeneous group system is structured. The main contributions of the paper are summarized as follows. For heterogeneous group systems under undirected and directed topology, the corresponding methods are used to confirm the driving individuals, which are the network eigenvalue method and the maximum match method. An extended distributed protocol with the external sliding-mode control is presented. In the meantime, some sufficient and necessary conditions are given to ensure that the heterogeneous interdependent group system is structurally controllable. The interdependent manner and the structure of a heterogeneous system impact on the controllability of the heterogeneous interdependent system.

The rest of this paper is organized as follows. In Section 2, some preliminaries, the problem statement, and the interdependent model of heterogeneous group systems are given. Section 3 presents the main results of this paper, including the controllability of heterogeneous interdependent group systems with undirected and directed topology. Simulation examples are given to verify the effectiveness of theoretical results in Section 4. Finally, Section 5 summarizes the investigation.

**Notations** The following notations used in the paper are fairly standard. $R$ denotes the set of real numbers, $R^n$ denotes the set of $n$-dimensional real vectors, and $R^{m\times n}$ denotes the set of $m\times n$ real matrices.

## 2. Preliminaries and problem statement

### 2.1. Preliminaries of graph theory

In a group system, the information flow between individuals is usually modeled by an interaction digraph $G=(V,E,A)$, where $V=(v_1,v_2,\ldots,v_n)$ is a nonempty node set; $E\subseteq V\times V$ is an edge set; $A=[a_{ij}]\in R^{n\times n}$ is a weighted adjacency matrix with $a_{ij}>0$, if $(v_i,v_j)\in E$, $a_{ij}=0$ otherwise. An edge $e_{ij}=(v_i,v_j)\in E$ represents that the node $v_j$ can receive information from the node $v_i$. If $a_{ij}\neq a_{ji}$, then a graph is called the directed graph, which is called the undirected otherwise. $D=\mathrm{diag}\,(d)\in R^{n\times n}$ is the diagonal degree matrix with $d_i=\sum_{v_j\in V}a_{ij}$. The Laplacian matrix of $G$ is defined as $L=L(G)=[l_{ij}]\in R^{n\times n}$, that is $L=D-A$, where $l_{ii}=\sum_{p\neq i}^{n}a_{ip}$ and $l_{ij}=-a_{ij},\forall i\neq j$. The eigenvalue of $L$ is denoted as $\lambda_1(L),\lambda_2(L),\ldots,\lambda_n(L)$. In the directed graph $G$, if there exists a directed path from node $v_i$ to node $v_j$ for $\forall v_i,v_j$, then the graph is called strongly connected. In addition, $D_{\mathrm{in}}=\mathrm{diag}\,(d)\in R^{n\times n}$ is the in-degree matrix of $G$, and $L=D_{\mathrm{in}}-A$. All the eigenvalues of $L$ have positive real parts defined, and the corresponding eigenvector $\gamma^{\mathrm{T}}$ of zero eigenvalue satisfies $\gamma^{\mathrm{T}}1=1$, where $\gamma\in R^n$.

### 2.2. Problem formulation

For the controllability discussion of group systems, some individuals are taken to play an individual driving role and others are ordinary individual roles. The ordinary individuals follow the neighbor-based law, but the driving individuals are free, which are allowed to arbitrarily select their control input. In such a way, the states of driving individuals are counted as inputs, which is used to control ordinary individuals.

Assume that a heterogeneous group system is made up of $g\geq 2$ subgroups, which is expressed by $\{P,E\}$, where $P=\{P_1,\ldots P_i,\ldots P_b\}$, $E=\{E_{1,1},\ldots,E_{i,j}\ldots E_{b,b}\}$. $P_i$ represents an individual set of subgroup $i$. $E_{i,i}$ represents an edge

set of subgroup $i$, namely, an internal connection of the subgroup. $E_{i,j}$ represents the edge set between subgroup $i$ and subgroup $j$, where $i \neq j$, namely, an external connection of subgroup.

**Definition 1** Provided that an external connection $(p,q) \in E_{i,j}$, where $p \in P_i$, $q \in P_j$, and $i \neq j$, $p$ is called an interdependent individual of $q$ and *vice versa*. In a subgroup $i$, there is an individual without any interdependent individual, which is called an independent individual.

In this paper, we consider the model of a heterogeneous system with two subgroups, which have different characteristics and topology structures, as shown in Fig. 1. It is seen that the interdependent model present clearly the interdependent relationship. The solid and dashed lines denote internal connection lines and external connection lines of subgroups respectively. A parameters $P_r$ is presented to study the effect of the different degree of interdependence for the controllability of heterogeneous interdependent group systems. $P_r$ is composed of $P_{ri}$ and $P_{rj}$, which represent the proportion of interdependent individuals in the subgroup $i, j$, respectively, as follows:

$$\begin{aligned} P_{ri} &= \#\{p \in P_i | (p,q) \in E_{i,j}\}/Ni, \\ P_{rj} &= \#\{q \in P_j | (p,q) \in E_{i,j}\}/Nj, \end{aligned} \tag{1}$$

where $\#\{\mathrm{Q}\}$ denotes the cardinal number of $Q$ set; $Ni$ and $Nj$ denote the number of individuals in the subgroup $i, j$, respectively.

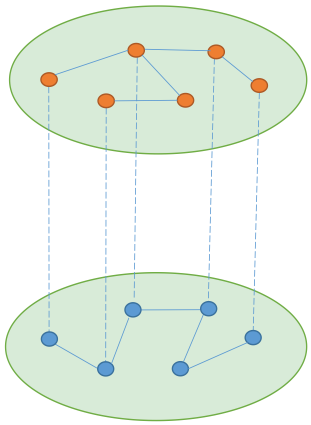


**Fig. 1.** (color online) The interdependent model of heterogeneous group systems.

**Definition 2** For a heterogeneous interdependent group system, if $P_{ri} = P_{rj} = P_r = 1$, then the corresponding interdependent manner is called the entire interdependence, otherwise which is called the partial interdependence.

Without loss of generality, consider a heterogeneous interdependent group system with single-integrator dynamic individuals. In order to succinctly present, suppose that the states of all individuals are in a one dimensional space. The dynamics of individuals are given by

$$\dot{x}_i = u_i, \quad i = 1,2,\ldots,n, \tag{2}$$

where $x_i \in R$ and $u_i \in R$ are the state and control input of individual $i$, respectively. Here, individuals include driving individuals and ordinary individuals. The driving individuals are driven by external control inputs, which are denoted by $P_\mathrm{d} = \{v_1, v_2, \ldots v_m\}$. The subset of ordinary individuals are denoted by $P_\mathrm{o} = P/P_\mathrm{d}$. The control input $u_i$ is described as the following form:

$$u_i = \begin{cases} \sum\limits_{j \in N_i} (x_j - x_i) + u_{\mathrm{o}i}, & i \in P_\mathrm{d}, \\ \sum\limits_{j \in N_i} (x_j - x_i), & i \in P_\mathrm{o}, \end{cases} \tag{3}$$

where $u_{\mathrm{o}i} \in R$ is the external control inputs, $N_i$ denotes the neighbor set of individual $i$.

Corresponding to the partition of individuals into driving individuals and ordinary individuals, the Laplacian matrix $L$ can be partitioned as

$$L = \begin{bmatrix} L_\mathrm{o} & L_\mathrm{od} \\ L_\mathrm{do} & L_\mathrm{d} \end{bmatrix},$$

where $L_\mathrm{o}$, $L_\mathrm{d}$ correspond to the indices of ordinary and driving individuals, respectively.

Under the control input (3), the system (2) is written as

$$\dot{x} = -Lx + Bu, \tag{4}$$

where $x = (x_1, x_2, \ldots, x_n)^\mathrm{T} \in R^n$, $u = (u_1, u_2, \ldots, u_m)^\mathrm{T} \in R^m$, and $B \in R^{n \times m}$ are the state vector, the control input vector, and the control matrix, respectively. The eigenvalues of system (4) are denoted as:

$$-\lambda_1(\delta_1, \mu_1), -\lambda_2(\delta_2, \mu_2), \ldots, -\lambda_l(\delta_l, \mu_l), \ \lambda_i \neq \lambda_j, \ \forall i \neq j,$$

where $\delta_i$ and $\mu_i$ denote the algebraic multiplicity and geometric multiplicity, respectively, $i = 1,2,\ldots,l$, $(\delta_1 + \delta_2 + \cdots \delta_l) = n$.

**Definition 3** For the group system (4), when any given initial state $x(0) = x_0$ and final state $x_\mathrm{f}$, there exists an external control $u$ and a finite time $T$ making $x(T) = x_\mathrm{f}$, then the system is controllable.

**Definition 4** For the group system (4) (or the corresponding communication graph), if there is a set of weights to make the system controllable, then the system is called structurally controllable.

## 3. Main results

In this section, based on the model of interdependence, the controllability is discussed for the heterogeneous interdependent group system (or the corresponding communication topology) with undirected and directed topology. Consider the heterogeneous interdependent group system (4), an extended distributed protocol with the external sliding-mode control is

designed, that is the design of external sliding-mode control $u_{oi}$ as follows:

Let $r$ denote the command signal, the position error $e_i$ take as state variable, that is

$$e_i = r - x_i. \tag{5}$$

Then the switching function is described as $s_i = ce_i$. According to the proportional switching control method, the control law $u_{oi}$ is selected as

$$u_{oi} = (\alpha |e_i|)\mathrm{sgn}(s_i),$$

where $\alpha > 0$, $\mathrm{sgn}(\cdot)$ is the sign function. Hence, the proposed extended distributed protocol is rewritten as

$$u_i = \begin{cases} \sum\limits_{j\in N_i} (x_j - x_i) + (\alpha |e_i|)\,\mathrm{sgn}(s_i), & i \in P_{\mathrm{d}}. \\ \sum\limits_{j\in N_i} (x_j - x_i), & i \in P_{\mathrm{o}}. \end{cases} \tag{6}$$

In a heterogeneous group system, the system is sure to be controllable when all individuals are driven by an external control signal. In practical application, it is necessary to reduce energy consumption. Hence, the main question is how to find effectively minimum driving individuals making the group network system controllable.

The key to our theory is based on the two multiplicities and the controllability by the PBH rank condition.[29] Using the nonsingular transformation $z = T^{-1}x$ and $Q = T^{-1}B$, the system (4) is rewritten as the Jordan form

$$\dot{z} = Jz + Qu, \tag{7}$$

where $T^{-1} = (\gamma^1, \gamma^2, \ldots, \gamma^i, \ldots, \gamma^n)$, $J = T^{-1}(-L)T = \mathrm{diag}\,(J_1, J_2, \ldots, J_i \ldots, J_l)$, $l \le n$, $J_i$ is the Jordan block corresponding to the eigenvalue $-\lambda_i$. It is further denoted as the diagonal block matrix by $\mu_i$ Jordan small blocks as follows:

$$J_i = \begin{bmatrix} J_{i1} & & \\ & \ddots & \\ & & J_{i\mu_i} \end{bmatrix}_{\delta_i \times \delta_i}, \quad i = 1, 2, \ldots, l,$$

.

**Theorem 1** For group system (4), there are $m$ driving individuals. The system is controllable when and only when there exists $m$ columns in transformation matrix $T^{-1}$, which are expressed as $\hat{T} = (\gamma^{v_1}, \gamma^{v_2}, \ldots, \gamma^{v_i}, \ldots, \gamma^{v_m})$, and meeting the following two conditions simultaneously.

(i) Suppose that the geometric multiplicity of eigenvalue $-\lambda_i$ is $\mu_i$, with the corresponding Jordan small blocks $J_{i1}, J_{i2} \ldots, J_{i\mu_i}$, then $\mathrm{rank}(U_{\lambda_i}) = \mu_i$, where

$$U_{\lambda_i} = \begin{bmatrix} \gamma_{ri1}^{v_1} & \gamma_{ri1}^{v_2} & \cdots & \gamma_{ri1}^{v_m} \\ \gamma_{ri2}^{v_1} & \gamma_{ri2}^{v_2} & \cdots & \gamma_{ri2}^{v_m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{ri\mu_i}^{v_1} & \gamma_{ri\mu_i}^{v_2} & \cdots & \gamma_{ri\mu_i}^{v_m} \end{bmatrix},$$

$$i = 1, 2, \ldots, l; \eta_1 + \eta_2 + \cdots + \eta_l = l.$$

(ii) In transformation matrix $T^{-1}$, any combination about less than $m$ columns could not satisfy the condition (i). Under the circumstances, individuals $v_1, v_2, \ldots, v_m$ can control the system, which are the column indices of $\hat{T}$.

**Proof** For PBH rank criterion, the discriminant matrix of system (7) is described as

$$[\lambda I - J, Q] = \begin{bmatrix} -\lambda_1 I - J_1 & & & Q_1 \\ & -\lambda_2 I - J_2 & & Q_2 \\ & & \ddots & \vdots \\ & & -\lambda_l I - J_l & Q_l \end{bmatrix}. \tag{8}$$

According to the PBH criterion, the system (4) is controllable when and only when $\mathrm{rank}\,[\lambda I - J, Q] = n$, namely, all $[-\lambda_i I - J_i, Q_i]$, $i = 1, 2, \ldots, l$ are full row rank. The corresponding submatrix is expressed as

$$\begin{bmatrix} -\lambda_i I - J_{i1} & & & Q_{i1} \\ & -\lambda_i I - J_{i2} & & Q_{i2} \\ & & \ddots & \vdots \\ & & -\lambda_i I - J_{i\mu_i} & Q_{i\mu_i} \end{bmatrix},$$

which is full row rank when and only when $\mathrm{rank}(U_{\lambda_i}) = \mu_i$. Therefore, condition (i) is sufficient and necessary for the controllability of system (4). Condition (ii) assures the minimum driving individuals. □

Furthermore, in order to get optimal driving individuals, we use the different methods to select driving individuals for heterogeneous interdependent group systems with undirected and directed topology.

### 3.1. Controllability of heterogeneous interdependent group systems with undirected topology

In this subsection, using the network system eigenvalue method, the driving individuals are chosen for a heterogeneous system with undirected topology. For the undirected communication topology $G$, the network system eigenvalue method is described as follows:

i) To obtain the Laplacian matrix $L$ of the graph $G$.

ii) The eigenvalue of $-L$ is calculated, that is $-\lambda_1, -\lambda_2, \ldots, -\lambda_n$. At the same time, it is need to find the maximum multiplicity $M$ of all eigenvalues and the corresponding eigenvalue $\lambda^M$.

iii) The matrix $(-L) - \lambda^M I_n$ is transformed by the elementary column (or row) transformation to find the maximum linearly independent rows (or columns).

iv) In the column (or row) canonical form, the individuals corresponding to linearly dependent rows (or columns) are driving individuals. The number of driving individuals is equal to the algebraic multiplicity $\delta(\lambda^M)$ of the eigenvalue $\lambda^M$, that is its multiplicity $M$.

Consider a simple group system $(n = 6)$ with undirected communication topology, as shown in Fig. 2(a), the minimum set of driving individuals is identified by the eigenvalue method. The specific process is indicated below:

The corresponding matrix $-L$ and eigenvalues are described as

$$-L=\begin{bmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}, \quad \lambda(-L)=\begin{bmatrix} -0.0000 \\ -1.0000 \\ -1.0000 \\ -3.0000 \\ -3.0000 \\ -6.0000 \end{bmatrix}. \tag{9}$$

Then, we have $\lambda^M=-1.00, \delta(\lambda^M)=2$, and the matrix $(-L)-\lambda^M I_n$ is transformed by the elementary row transformation as the following form

$$(-L)-\lambda^M I_n=\begin{bmatrix} -4 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{10}$$

From Eq. (10), it is known that the individuals corresponding to linearly dependent columns are $v_5, v_6$, which are marked by red.

Moreover, it is known that the proportion of interdependent individuals is $P_{\mathrm{rI}}=1, P_{\mathrm{rII}}=3/4$ through analysis and calculation for the interdependent model in the following Fig. 2(b).

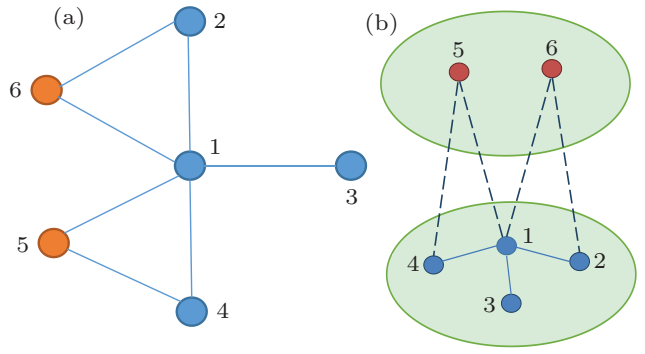



**Fig. 2.** (color online) A heterogeneous group system with undirected topology: (a) the communication topology and (b) the interdependent model.

### 3.2. Controllability of heterogeneous interdependent group systems with directed topology

**Lemma**[11] The minimum drive nodes required to meet the structural controllability of the network are determined by the maximum matching in the network.

Given a bipartite graph $H$, $F$ is a subset of edge set $E$, namely, $F \subseteq E$. If any two edges are not attached to the same vertex in subset $F$, then $F$ is called a match of $H$. In all matches of $H$, the match with the maximum edges is called the maximum match. We use the maximum match method to identify the maximum match, and to get the minimum driving individuals for a heterogeneous system with directed topology. For directed communication topology $G$, $G$ is converted to a bipartite graph $H$, that is $H=(V^+, V^-, \Gamma)$ where $V^+=(v_1^+, v_2^+, \ldots, v_n^+), V^-=(v_1^-, v_2^-, \ldots, v_n^-)$ denote the node set of columns and rows for the state matrix $-L$; $\Gamma=\{(v_i, v_j)|a_{ij}\neq 0\}$ denotes an edge set. The maximum match node set is obtained according to the maximum match method.

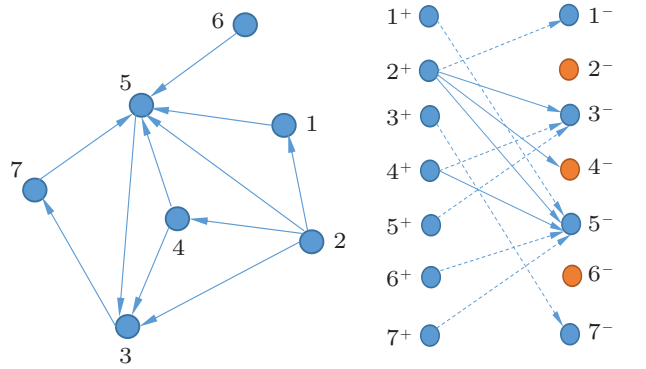



**Fig. 3.** (color online) A heterogeneous group system with directed topology: (a) the communication topology and (b) the corresponding bipartite graph.

Taking into account a group system $(n=7)$ with directed communication topology, as shown in Fig. 3(a), the minimum set of driving individuals is confirmed by the maximum match method. The specific process is stated as follows:

(I) The communication topology is converted to the corresponding bipartite graph for a heterogeneous group system with directed topology, as shown in Fig. 3.

(II) Finding the maximum matching edge set of the corresponding bipartite graph Fig. 3(b). Dash lines denote the maximum matching edge set. It is shown that there are two sets of maximal matching edge sets in Fig. 3(b), which are described as

$$1^+ \to 5^-, 2^+ \to 1^-, 3^+ \to 7^-, 4^+ \to 3^-, 5^+ \to 3^-, 6^+ \to 5^-, 7^+ \to 5^-;$$
$$1^+ \to 5^-, 2^+ \to 4^-, 3^+ \to 7^-, 4^+ \to 3^-, 5^+ \to 3^-, 6^+ \to 5^-, 7^+ \to 5^-.$$

In the first group, $v_1, v_3, v_5$ are matching nodes, and $v_2, v_4, v_6$ are non matching nodes. Similarly, in the second group, $v_3, v_4, v_5$ are matching nodes, and $v_1, v_2, v_6$ are non-matching nodes.

From the above analysis, it is known that the number of drive node sets is two for the directed communication topology in Fig. 3(a). For example, red nodes are drive nodes in the first group as shown in Fig. 3(b). In other words, the individual $2, 4, 6$ are drive individuals for the heterogeneous group system with directed topology. The corresponding interdependent model is the following Fig. 4. It is known that the proportion of interdependent individuals is $P_{\mathrm{rI}} = 1, P_{\mathrm{rII}} = 3/4$ through analysis and calculation for the interdependent model in Fig. 4.

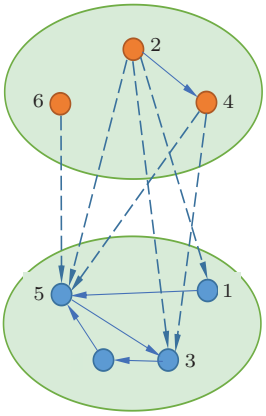



**Fig. 4.** (color online) The corresponding interdependent model.

**Corollary 1** Consider a heterogeneous group system composed of driving individuals and ordinary individuals. When the heterogeneous group system is controllable, the corresponding interdependent manner is the partial interdependence, namely there are connection relationships between some driving individuals and ordinary individuals.

**Proof** Let $\delta_1, \delta_2, \delta_3, \ldots \delta_i, \ldots, \delta_l$ denote the algebraic multiplicity of $-\lambda_1, -\lambda_2, -\lambda_3, \ldots, -\lambda_i, \ldots, -\lambda_l$, and $\mu_1, \mu_2, \mu_3, \ldots, \mu_i, \ldots, \mu_l$ denote the geometric multiplicity of $-\lambda_1, -\lambda_2, -\lambda_3, \ldots, -\lambda_i, \ldots, -\lambda_l$, respectively.

It is known that there exists $1 \le \mu_i \le \delta_i$ for any $-\lambda_i$. Besides, let $\mu = \max\{\mu_1, \mu_2, \mu_3, \ldots, \mu_i, \ldots, \mu_l\}$, then the minimum number of driving individuals $m$ satisfies $\mu \le m \le \sum_{i=1}^{l} \mu_i$. Otherwise, if $m \le \mu$, then $\mathrm{rank}(U_\lambda) < \mu$, namely, $U_{\lambda_i}$ is not a full row rank, which contradicts the condition (i) in Theorem 1. It is obtained that $\mu \le m \le \sum_{i=1}^{l} \mu_i \le \sum_{i=1}^{l} \delta_i = n$ further. All the rest of individuals are ordinary individuals $(n-m)$. Therefore, we have $P_{\mathrm{rI}} = 1$, $P_{\mathrm{rII}} < 1$ for the heterogeneous group system composed of driving individuals and ordinary individuals. □

## 4. Simulation

In this section, some examples are provided to further demonstrate the effectiveness of theoretical results.

**Example 1** Consider a simple group system $(n = 5)$ with undirected communication topology, as shown in Fig. 5(a), the minimum set of driving individuals is determined by the eigenvalue method. The corresponding matrix $-L$ and eigenvalues are described as

$$-L = \begin{bmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \lambda(-L) = \begin{bmatrix} -0.0000 \\ -1.3820 \\ -2.3820 \\ -3.6180 \\ -4.6180 \end{bmatrix}.$$

Then, we have $\lambda^M = -0.00, \delta(\lambda^M) = 1$, and the matrix $(-L) - \lambda^M I_n$ is transformed by the elementary row transformation as the following form

$$(-L) - \lambda^M I_n = \begin{bmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the above analysis, the corresponding linearly dependent columns are $v_4$ or $v_5$, which are marked by red.

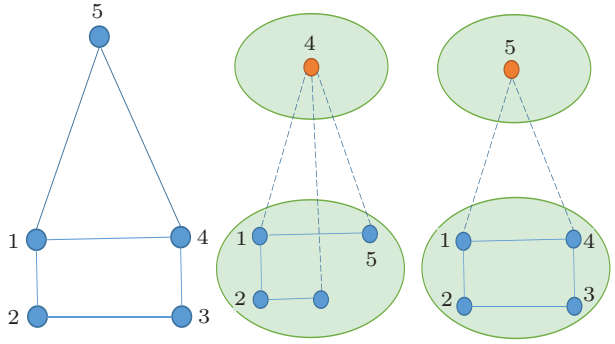



**Fig. 5.** (color online) A heterogeneous group system with undirected topology.

By Eq. (4), the simple group system is described as

$$\dot{x}=\begin{bmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix} x+Bu.$$

When the individual $v_4$ is the driving individual, there are $B=(0,0,0,1,0)^{\mathrm{T}}$, $P_{\mathrm{rI}}=1$, $P_{\mathrm{rII}}=3/4$. According to the controllability criterion, the controllability matrix is $U_{\mathrm{c}}=[B,(-L)B,(-L)^2B,(-L)^3B,(-L)^4B]$, $\mathrm{rank}(U_{\mathrm{c}})=5$. So the system is controllable. The input signal of the system is a unit step signal, that is $r(t)=1$. Set $\alpha=5$, $c=3$, $x(0)=(0.2,0.2,0.2,0.2,0.2)$ in the proposed extended distributed protocol (3) and the switching function $s_i=ce_i$. From the simulation results as shown in Fig. 6, under the action of external sliding-mode control $u_{\mathrm{o}}$, states of individuals can track the input signal $r(t)=1$ after a certain period of time. For example, the adjusting time of individual $v_1$ is $t_{3s}=1.108s$ when the allowable error range is $\Delta=0.05$. It further shows the controllability of the system.

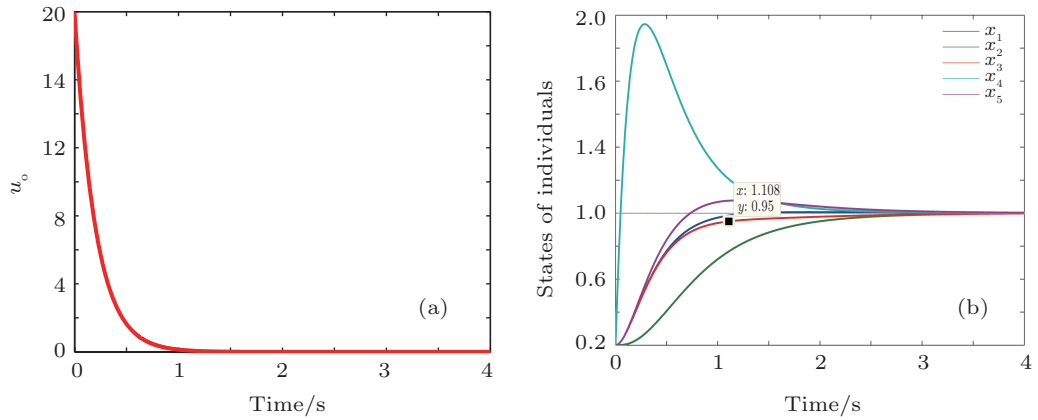



**Fig. 6.** (color online) The external control and states of individuals: (a) the external sliding-mode control $u_{\mathrm{o}}$; (b) $t_{3s}=1.108s$, $\Delta=0.05$.

Similarly, when the individual $v_5$ is the driving individual, there are $B=(0,0,0,0,1)^{\mathrm{T}}$, $P_{\mathrm{rI}}=1$, $P_{\mathrm{rII}}=1/2$. According to the controllability criterion, the controllability matrix is $U_{\mathrm{c}}=[B,(-L)B,(-L)^2B,(-L)^3B,(-L)^4B]$, $\mathrm{rank}(U_{\mathrm{c}})=3<n=5$. So the system is uncontrollable. As shown in Fig. 7, states of individual $v_1,v_2$ are uncontrollable. The adjusting time of individual $v_1$ is $t_{3s}=2.586s$ when the allowable error range is $\Delta=0.05$.

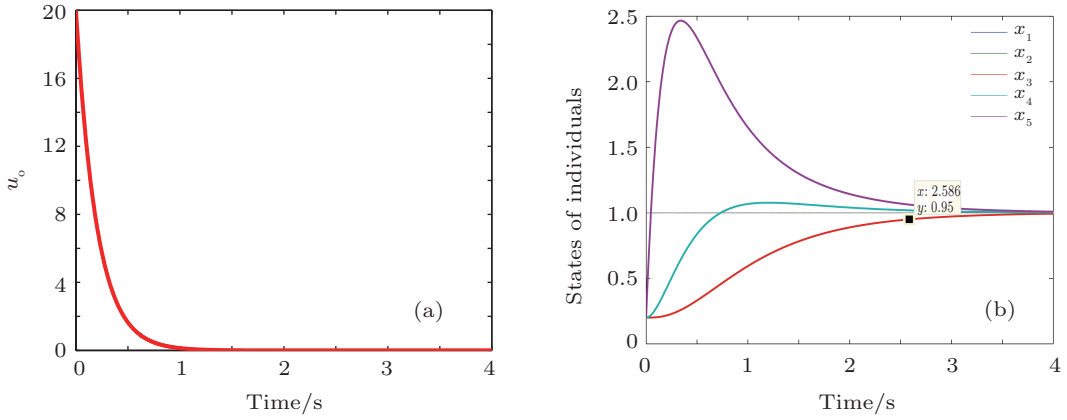



**Fig. 7.** (color online) The external control and states of individuals: (a) the external sliding-mode control $u_{\mathrm{o}}$; (b) $t_{3s}=2.586s$, $\Delta=0.05$.

**Example 2** In subsection 3.1, the number of individuals is $n=6$ for the heterogeneous interdependent group system, by Eq. (4), which is described as

$$\dot{x}=\begin{bmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & 0 & -2 \end{bmatrix} x+Bu. \tag{11}$$

From Subsection 3.1, we know that $\lambda^M = -1.00, \delta(\lambda^M) = 2$, and individuals $v_5, v_6$ are driving individuals, namely, $B = (0,0,0,0,1,1)^{\mathrm{T}}$.

To structure the transformation matrix $T$, $T$ and the corresponding inverse matrix $T^{-1}$ are respectively written as

$$T = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -2 & -2 & 1 \\ 1 & -1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} -0,1667 & 0.0333 & 0.0333 & 0.0333 & 0.0333 & 0.0333 \\ 0 & 0 & 0 & -0.5 & 0.5 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.5 \\ 0 & -0.2 & -0.2 & 0.3 & 0.3 & -0.2 \\ 0 & 0.3 & -0.2 & -0.2 & -0.2 & 0.3 \\ 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 & 0.1667 \end{bmatrix}.$$

Using the nonsingular transformation $z = T^{-1}x$ and $Q = T^{-1}B$, the system (11) is rewritten as the Jordan form

$$\dot{z} = Jz + Qu,$$

where $J = T^{-1}(-L)T = \mathrm{diag}(-6,-3,-3,-1,-1,0)$, $Q = (0.0667, 0.5, 0.5, 0.1, 0.1, 0.3333)^{\mathrm{T}}$. According to Theorem 1, we know that the heterogeneous interdependent group system is controllable.

Set $r(t) = 1$, $\alpha = 5$, $c = 3$, $x(0) = (0.2, 0.3, 0.2, 0.3, 0.2, 0.4)$. From simulation results as shown in Fig. 8, states of individuals can track the input signal $r(t) = 1$ under the action of external sliding-mode control $u_{\mathrm{o}}$.

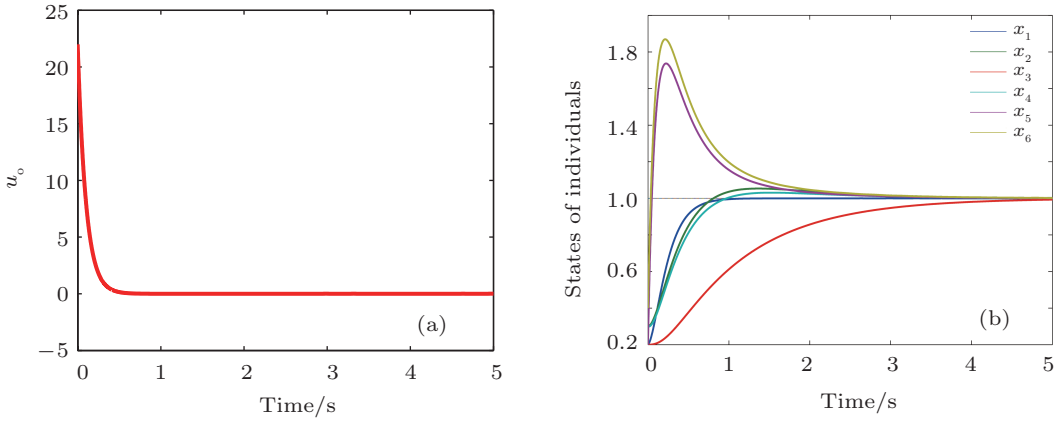



**Fig. 8.** (color online) The external control and states of individuals: (a) the external sliding-mode control $u_{\mathrm{o}}$, (b) the states of individuals.

**Example 3** In Subsection 3.2, consider a heterogeneous interdependent group system with directed communication topology, the number of individuals is $n = 7$. The corresponding Laplacian matrix $L$ and eigenvalues of $-L$ are written as

$$L = D_{\mathrm{in}} - A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 5 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \lambda(-L) = \begin{bmatrix} -0.000 \\ -0.000 \\ -1.000 \\ -1.000 \\ -0.8851 \\ -3.2541 \\ -4.8608 \end{bmatrix}.$$

Hence, we have $\lambda^M = 0$, $\mu(\lambda^M) = 2$. By Eq. (4), the system is described as

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} x + Bu, \tag{12}$$

where $B=(0,1,0,1,0,1,0)^{\mathrm{T}}$ or $B=(1,1,0,0,0,1,0)^{\mathrm{T}}$.

The transformation matrix $T$ is structured as

$$T=\begin{bmatrix} -0.0769 & 1.0769 & 5 & -1 & 0 & 0 & 0 \\ -0.0769 & 1.0769 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.1149 & -2.2541 & 0.4699 \\ -0.0769 & 1.0769 & -1 & 0 & 0 & 0 & 0 \\ 0.1538 & 0.8462 & 1 & 0 & 0.2430 & 0.5728 & -0.8743 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & -0.1217 \end{bmatrix}.$$

The corresponding inverse matrix $T^{-1}$ is written as

$$T^{-1}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0.9286 & 0 & 0 & 0 & 0.0714 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 6 & 0 & -5 & 0 & 0 & 0 \\ 0.9240 & -6.9913 & 0.4369 & 4.7261 & 0.1062 & -0.1200 & 0.9240 \\ 0.1166 & 0.0875 & -0.4587 & 0.3200 & -0.2627 & 0.0807 & 0.1166 \\ 0.3332 & 0.1548 & -0.1791 & 0.3796 & -1.2864 & 0.2646 & 0.3332 \end{bmatrix}.$$

Using the nonsingular transformation $z=T^{-1}x$ and $Q=T^{-1}B$, the system (12) is rewritten as the Jordan form

$$\dot{z}=Jz+Qu,$$

where $J=T^{-1}(-L)T=\mathrm{diag}\,(0,0,-1,-1,-0.8851,-4.8608,-3.2541)$, $Q=(1,1,0,1,-2.3911,0.4883,0.7990)^{\mathrm{T}}$ or

$$Q=(1,1,1,5,-6.1932,0.2848,0.7527)^{\mathrm{T}}.$$

It is known that the heterogeneous interdependent group system is controllable by Theorem 1.

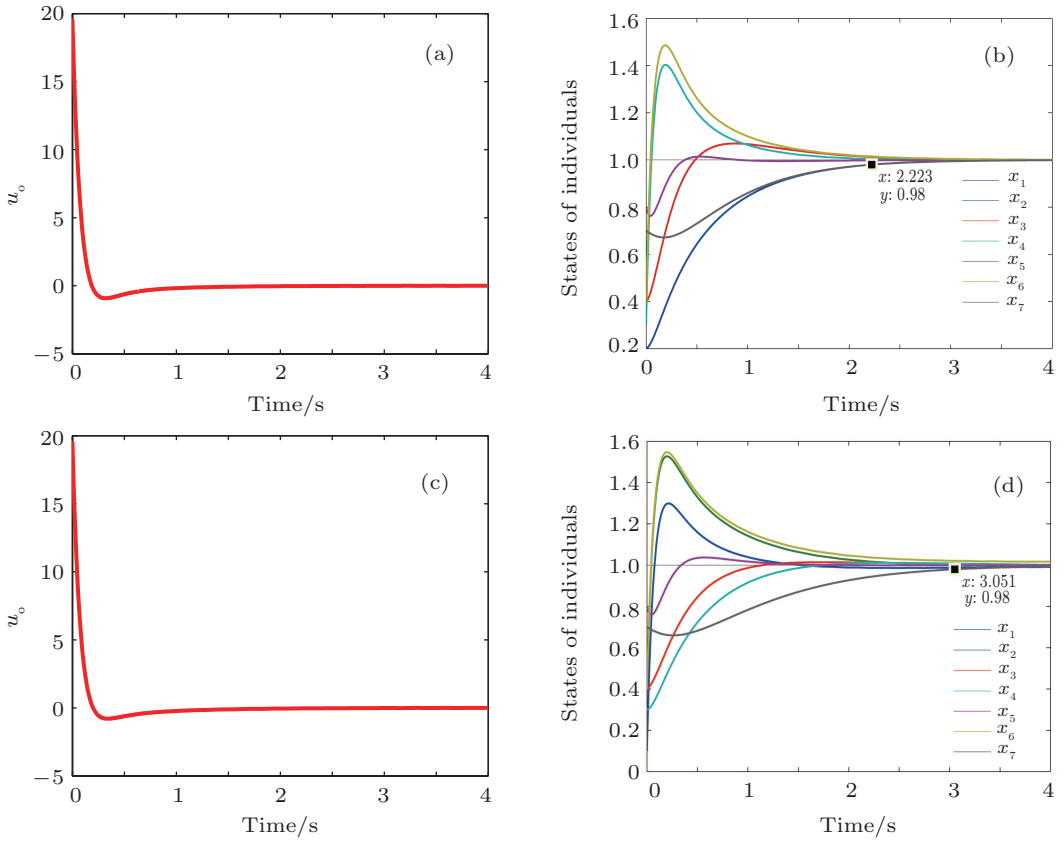



**Fig. 9.** (color online) The external control and states of individuals: (a) The external sliding-mode control $u_{\mathrm{o}}$; (b) $t_{7s}=2.223s$, $\Delta=0.02$, $B=(0,1,0,1,0,1,0)^{\mathrm{T}}$; (c) the external sliding-mode control $u_{\mathrm{o}}$; (d) $t_{7s}=3.051s$, $\Delta=0.02$, $B=(1,1,0,0,0,1,0)^{\mathrm{T}}$.

Set $r(t) = 1$, $\alpha = 5$, $c = 3$, and

$$x(0) = (0.1, 0.38, 0.4, 0.3, 0.8, 0.4, 0.7).$$

As shown in Fig. 9, states of individuals can track the input signal $r(t) = 1$ under the action of external sliding-mode control $u_o$. It further shows that the group system is controllable. When driving individuals are $v_2, v_4, v_6$, the adjusting time of individual $v_7$ is $t_{7s} = 2.223s$ with the allowable error range $\Delta = 0.02$, as shown in Fig. 9(b). When driving individuals are $v_1, v_2, v_6$, there is $t_{7s} = 3.051s$ with the allowable error range $\Delta = 0.02$, as shown in Fig. 9(d).

## 5. Conclusion

In this paper, the controllability problem of heterogeneous interdependent group systems with undirected and directed topology has been investigated. In the light of the difference of individual characteristics, the interdependent model of the heterogeneous system has been structured. Then, an extended distributed protocol has been proposed, which has the external sliding-mode control. For heterogeneous systems with undirected topology, the driving individuals have been confirmed by the network eigenvalue method. Under directed topology, the maximum match method has been used to identify the driving individuals. Meanwhile some sufficient and necessary conditions have been presented to ensure that the heterogeneous interdependent group system has been structurally controllable. The relationship of the interdependent manner and the controllability has been analyzed on heterogeneous systems. In addition, the effectiveness of theoretical results has been showed via numerical simulations. In the future, the controllability of high-order heterogeneous interdependent group systems can be further researched.

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