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Study on the phase transition of the fractal scale-free networks*

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Based on the Ising spin, the phase transition on fractal scale-free networks with tree-like skeletons is studied, where the loops are generated by local links. The degree distribution of the tree-like skeleton satisfies the power-law form $P(k) \sim k^{-\delta}$. It is found that when $\delta \geq 3$, the renormalized scale-free network will have the same degree distribution as the original network. For a special case of $\delta = 4.5$, a ferromagnetic to paramagnetic transition is found and the critical temperature is determined by the box-covering renormalization method. By keeping the structure of the fractal scale-free network constant, the numerical relationship between the critical temperature and the network size is found, which is the form of power law.

Keywords: fractal scale-free network, phase transition, renormalization

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1. Introduction

For complex networks,[1] such as the small-world network,[2,3] the scale-free network,[4,5] and the hierarchical modularity network,[6] they constitute our basic understanding of real world networks. For the scale-free networks, their degree distribution lacks the characteristic scale and satisfies the power-law distribution $P(k) \sim k^{-\gamma}$. The degree of a node means the number of its links to other nodes. Due to the lack of an Euclid length in the length between the nodes, the complex networks lack a Euclid dimension like a regular lattice, or lack a fractal dimension like a fractal system. In recent years, Song *et al.* proposed a fractal dimension of complex networks based on the box-covering method.[7,8] The fractal relation of complex networks is defined as

$$N_{\mathrm{B}}(\ell_{\mathrm{B}}) \sim \ell_{\mathrm{B}}^{-d_{\mathrm{B}}}. \tag{1}$$

Here $N_{\mathrm{B}}(\ell_{\mathrm{B}})$ means the minimum number of boxes covering the whole network. ℓ_{B} stands for the scale of every box, which means every two nodes in the same box have a distance less than ℓ_{B}. d_{B} stands for the fractal dimension. It is found that the World Wide Web,[9] the metabolic network,[7,10] the protein interaction network of *Homo sapiens*,[11] the actor collaboration networks,[4] etc., satisfy the fractal properties, while the Internet,[7] the Barabási–Albert network,[4,7] etc., do not satisfy the fractal properties. Song *et al.* think that the fractal properties originate from the repulsion between the most connected nodes on all length scales.[12]

Besides, there are definitions of other types of dimensions for the complex networks, such as the correlation dimension,[13] which is obtained from the ergodic theory of dynamic systems, or the fractal dimension proposed by Shanker,[14,15] obtained from a scaling relation between the average density and the distance, etc.

For the phase transition of the Ising model in a random scale-free network, the critical temperature has been determined analytically[16,17] and is validated by the numerical method previously in the Barabási–Albert network.[18] For the random scale-free network, with degree distribution $P(k) \sim k^{-\gamma}$, it is found when $2 < \gamma < 3$, the critical temperature depends on both the exponent γ and the network size N. When $\gamma = 3$, the critical temperature only depends on the network size N. When $3 < \gamma < 5$, the critical temperature is independent of both γ and N. When $\gamma \geq 5$, the phase transition is similar to the phase transition of the Ising model in high dimension regular lattices. In the Barabási–Albert network, the critical temperature of the Ising model[18] is close to the analytic results in Refs. [16] and [17]. In the random scale-free networks,[19] the numerical results are consistent with the analytic results.[16,17] In addition, some recent works[20–24] are based on the uncorrelated scale-free networks and the Ising spin and the transverse field Ising model, further verifying the correctness of the theoretically given critical temperature.

The theoretical research in Refs. [16] and [17] is based on the uncorrelated scale-free networks, so there is a difference between the theoretical critical temperature and the actual network critical temperature. In addition, only the case that the degree distribution exponent satisfies $\gamma > 2$ is analyzed in Refs. [16] and [17] and the result of the critical temperature is not given in theory for the case of $\gamma \leq 2$. Al-

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though in Refs. [19] and [23], the authors thought that the scale-free networks reach the ferromagnetic state at any temperature when $\gamma \leq 2$, these uncorrelated scale-free networks do not have the fractal property that they have in the small-world feature. However, the fractal feature and the small-world feature are often contradictory and cannot coexist in scale-free networks.[25] Therefore, in a finite-size fractal scale-free network system, the system can change from ferromagnetic state to paramagnetic state only when the temperature is higher than a certain value; the temperature can be called the critical temperature, and the critical temperature should be higher than the critical temperature of the scale-free network with small-world characteristics. Based on the above analysis, in the next simulation process, we will do a more detailed study.

The main line of the present study is as follows. Firstly, we construct fractal scale-free networks with tree-like skeletons. Because in reality, scale-free networks, such as the World Wide Web,[9] the metabolic network,[7,10] the protein interaction network of *Homo sapiens*,[11] etc., all have a tree-like skeleton[26] and both satisfy the fractal property of the box-covering renormalization method.[7,8] Secondly, using the box-covering method,[8] we study the influence of different values of δ ($2 < \delta < 5$) on the similarity of the fractal scale-free networks, where δ is the exponent of the degree distribution of the tree-like skeleton. Thirdly, for a special case of $\delta = 4.5$, using the box-covering renormalization method we find the critical temperature of a finite size scale-free network. Fourthly, under the premise of maintaining the degree distribution of the scale-free network, we find the numerical relationship between the critical temperature and the size of the network, as well as the critical temperature under the thermodynamic limit.

The rest of this paper is arranged as follows. In Section 2, we study the self-similarity of fractal scale-free networks. In Section 3, we obtain the critical temperature and calculate the critical temperature under the thermodynamic limit. The conclusion of this paper is given in Section 4.

2. The fractal scale-free network and its self-similarity

The fractal scale-free network with a tree-like skeleton is constructed as follows.

(i) Starting from a few connected nodes, at every timestep, every node born in the previous timestep will generate m offsprings. The generation probability b_m satisfying

$$b_m \sim m^{-\delta}, \tag{2}$$

where $\delta > 0$. The average number of offsprings of every node should satisfy

$$\langle m \rangle = \sum_{m=1}^{R} m b_m = 1. \tag{3}$$

In Eq. (3), R is an integer satisfying $R \gg N$, where N is the size of the network. In order for Eq. (3) to hold, there should be a probability that $p_0 = \sum_{m=1}^{R} b_m < 1$, and the probability can be determined numerically.

(ii) Produce a uniform random number ξ_1, satisfying $0 < \xi_1 < 1$, and endow it to every node born in the previous step. If a node born in the previous timestep having $\xi_1 < p_0$, then it can generate m offsprings. The specific value of m will be determined in the next step.

(iii) Produce another uniform random number ξ_2, and divide the interval $[0,1]$ into R small intervals. The length of each small interval is proportional to $m^{-\delta}$, where $m = 1,2,3,\ldots,R$. If ξ_2 is in the m-th interval, then the number of offsprings generated in step (ii) is m.

(iv) If the total number of offsprings generated by all the nodes in the previous timestep is no less than one, all the existing nodes in the network will generate a link with a common local link probability $p(0 \leq p \leq 1)$, and the link is randomly connected to the next nearest neighbor node. Repeat steps (ii)–(iv), until network size meets the requirement.

The degree distribution of the tree-like skeleton of the scale-free network satisfies $P(k) \sim k^{-\delta}$, which also determines that the scale-free network has a fractal structure.[27] In Ref. [26], the authors used a method to find the fractal relationship that the fractal scale-free network constructed above satisfies. Here we will find the fractal relationship satisfied by the fractal scale-free network using the method in Ref. [8], and compare it with the previous work, and further to find the conditions for the network to meet the fractal structure and the self-similarity.

The numerical results are shown in Figs. 1–3. In these figures, the size of the network is $N = 2000$. When we find the degree distribution of the renormalized network, the box scale takes the value $\ell_B = 2$. When we find the fractal properties of these networks, the value of the box scale changes from $\ell_B = 2$ to the maximum. In Fig. 1(a), it is the degree distribution of the tree-like skeleton of the scale-free network.[27] In Fig. 1(b), the degree distribution of the fractal scale-free network satisfies $P(k) \sim k^{-2.10}$, but the degree is not very large. When the degree is large, the degree distribution will obviously be affected by the finite-size effect. In Fig. 1(c), when the box scale is not very large, the number of the boxes and the scale satisfies the relationship of $N_B(\ell_B) \sim \ell_B^{-2.79}$, which is in agreement with the previous result in Ref. [26]. In Fig. 1(d), when the degree is not very large, the renormalized network satisfies the distribution $P(k') \sim k'^{-1.21}$. By comparing Figs. 1(b) and 1(d), we find that the renormalized scale-free network has a different degree distribution than the non-renormalized scale-free network. So we can say that such a scale-free network is fractal, but not self-similar.

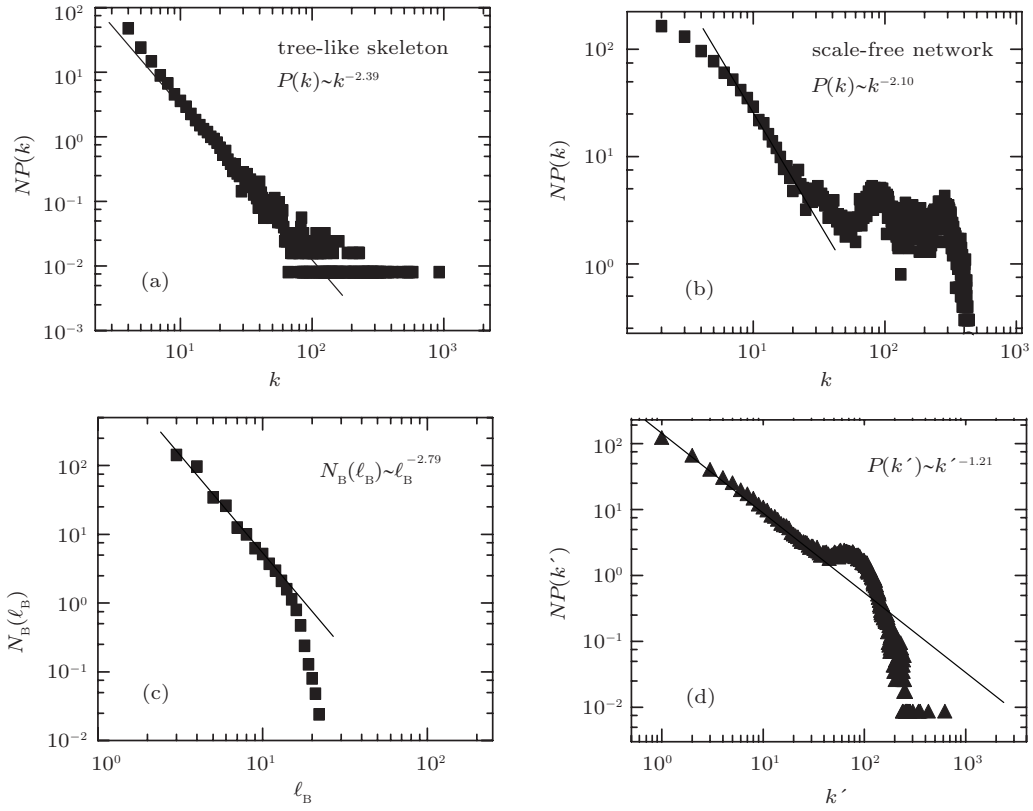


Fig. 1. The tree-like skeleton of the network is created with the probability $b_m \sim m^{-2.3}$, as shown in Eq. (2). (a) Degree distribution of the tree-like skeleton. The solid line has a slope -2.39, which means the distribution satisfies $P(k) \sim k^{-2.39}$. (b) The probability of local links is $p = 0.5$. The degree distribution of the fractal scale-free network satisfies $P(k) \sim k^{-2.10}$. (c) Fractal relation using the box-covering method $N_{\rm B}(\ell_{\rm B}) \sim \ell_{\rm B}^{-2.79}$. If the scale of box is too large, the fractal relationship will be affected by the finite-size effect. (d) The degree distribution of the scale-free network obtained after renormalization by the box-covering method satisfies the relationship $P(k') \sim k'^{-1.21}$.

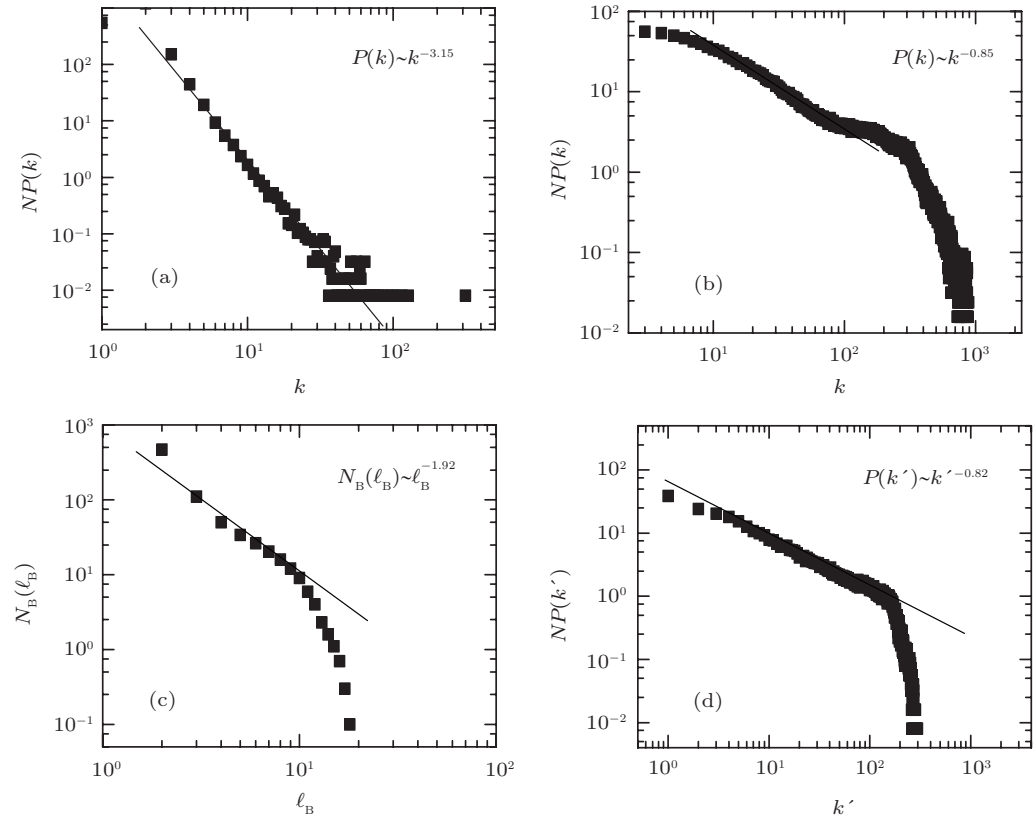


Fig. 2. The skeleton of the scale-free network is created with probability $b_m \sim m^{-3.0}$ (a) Degree distribution of the tree-like skeleton satisfies $P(k) \sim k^{-3.15}$. (b) Degree distribution of the network, with local link probability $p = 0.5$, satisfies $P(k) \sim k^{-0.85}$. (c) The fractal relation that the box number and the box scale satisfy $N_{\rm B}(\ell_{\rm B}) \sim \ell_{\rm B}^{-1.92}$. (d) Degree distribution of the renormalized network satisfies $P(k') \sim k'^{-0.82}$.

When the tree-like skeleton of the scale-free network is generated with probability $b_m \sim m^{-\delta}$, as shown in Eq. (2), and $\delta = 3.0$ and $\delta = 4.5$, we find in Figs. 2 and 3, the renormalized scale-free network and the non-renormalized scale-free network have almost the same degree distribution. It can be said that these scale-free networks are both fractal and self-similar at this time. In addition, we find a significant difference when $\delta \geq 3$, comparing the degree distributions of the tree-like skeletons with the scale-free networks in Figs. 2 and 3. Because when $\delta \geq 3$, the degree distribution of the tree-like skeleton will no longer play a leading role in the degree distribution of the scale-free network, and the local link probability will play a leading role.

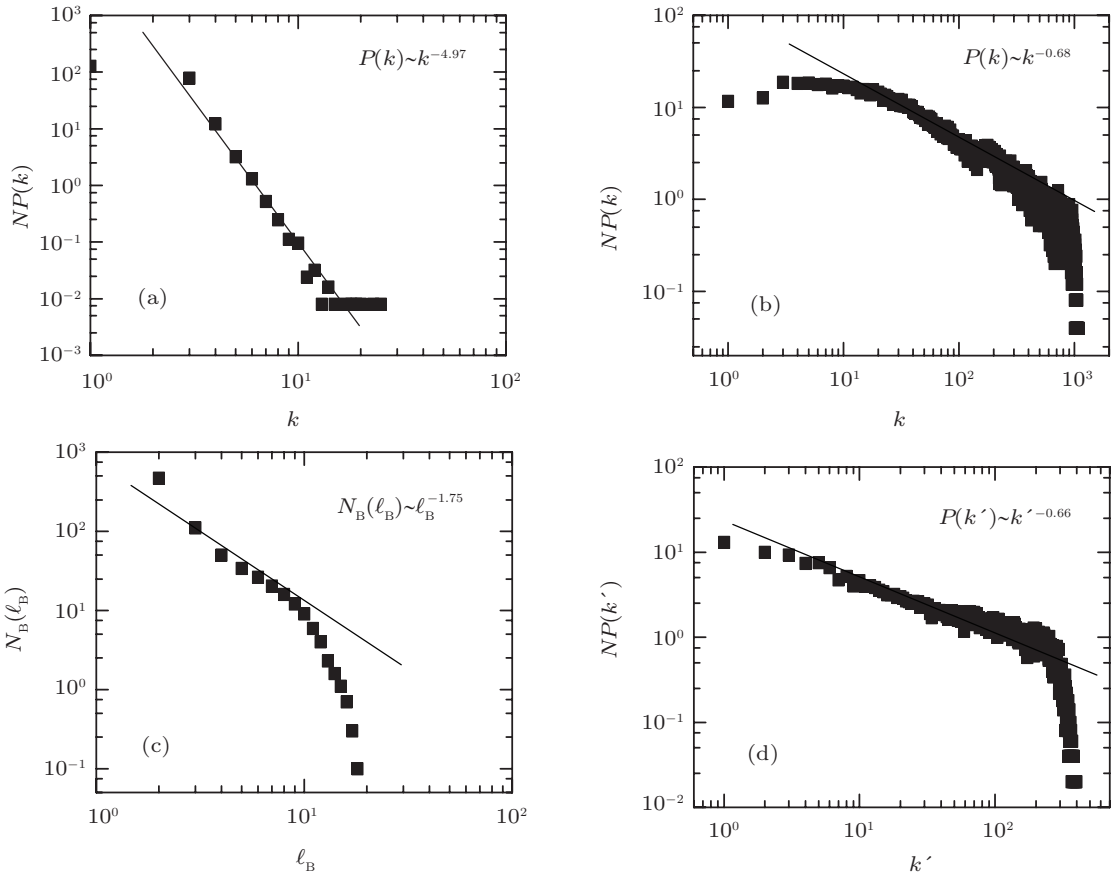


Fig. 3. The tree-like skeleton of the network is generated with probability $b_m \sim m^{-4.5}$. (a) Degree distribution of the skeleton satisfies $P(k) \sim k^{-4.97}$. (b) Degree distribution of the network, with local link probability $p = 0.5$, satisfies $P(k) \sim k^{-0.68}$. (c) Fractal relation that the box number and the box scale satisfy $N_{\rm B}(\ell_{\rm B}) \sim \ell_{\rm B}^{-1.75}$. (d) Degree distribution of the renormalized network satisfies $P(k') \sim k'^{-0.66}$.

3. Critical temperature

In the fractal scale-free network, each node is given an Ising spin, which takes the value $+1$ or -1, so the network at this point can represent a complex ferromagnetic system. Without an external field, the Hamiltonian of the system is defined as

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \tag{4}$$

where $J > 0$, and σ_i stands for the spin of node i. So we get the following relation,

$$-\beta H = K \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \tag{5}$$

where $\beta = 1/kT$, $K = J/kT$, and k represents the Boltzmann constant.

First, we find the critical point of the ferromagnetic system with fractal and scale-free network structure. Here we select the Binder's fourth-order cumulant to find the critical point, defined as follows:

$$U(T) = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}. \tag{6}$$

Here $m = (1/N)\sum_{i=1}^{N} \sigma_i$. The average values in Eq. (6) are over different network realizations and different equilibrium spin states. For the renormalized network system, we can get the corresponding $U(T)$. If the fractal scale-free network is self-similar, then it means for $U(T)$ the renormalized system will have a common cross point with the original system, and the cross point should be independent of the scale of the spin block. According to the box-covering renormalization method, since each node in the network is given a spin, each box corresponds to a spin block. Since the correlation length is infinite at the critical point, $U(T)$ should be independent of the scale of spin block, which is why there is a common cross

point between the original system and the renormalized systems. Through the numerical results in Section 2, we know the fractal scale-free network is self-similar when the exponent $\delta \geq 3$. So in the next study, we study the special case of $\delta = 4.5$. We get the critical point of the phase transition shown in Fig. 4. In Fig. 4, we let $K = J/kT$, and select $J/k = 1$, in order to better obtain a numerically simulated figure.

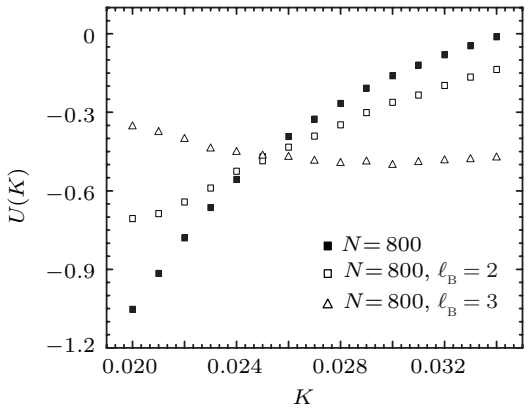


Fig. 4. The relationship between the Binder's cumulant $U(K)$ and the parameter K ($K = J/kT$). The size of the initial network is $N = 800$ (solid square ■), and the tree-like skeleton satisfies the relation $P(k) \sim k^{-4.5}$, while the local link probability is $p = 0.8212$. For the renormalized networks, the block scales are $\ell_B = 2$ (square □) and $\ell_B = 3$ (triangle △) respectively. It is found that the critical point is $K_c = 0.025(2)$. The numerical results are the average over 2×10^5 network realizations and renormalizations. For every network realization, there are 10 times of spin flipping processes. In every spin flipping process, there are up to 10^4 timesteps of spin flipping through the whole system.

Then, taking into account the process of network growth, all nodes in the network will increase a link with the common probability at each timestep. Therefore, the local link probabilities of different sized networks must be different. By the numerical method, we obtain a number of similar fractal scale-free networks, their degree distributions are shown in Fig. 5.

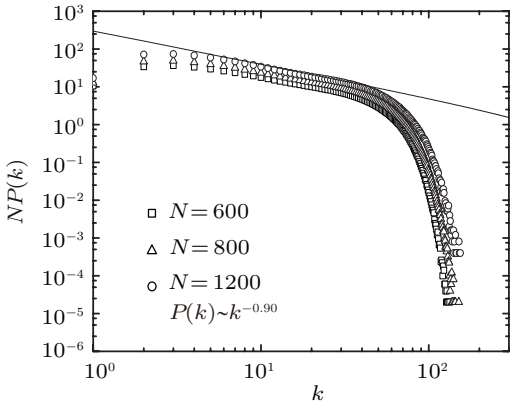


Fig. 5. Degree distribution of several similar fractal scale-free networks. The degree distribution of the tree-like skeleton satisfies $P(k) \sim k^{-4.5}$. The sizes of these networks are $N = 600$, 800, and 1200 respectively, and the local link probabilities of these networks are respectively $p = 0.0951$, 0.0821, and 0.0660. The network degree distribution satisfies the power-law form $P(k) \sim k^{-\gamma}$. From the slope of the line in the figure, we get $\gamma = 0.90$. The numerical results are the average over 2×10^5 network realizations.

Next, we find the fractal dimension of those similar networks in Fig. 5. Using the greedy coloring algorithm in the box-covering renormalization method,[8] the numerical results we obtained are shown in Fig. 6, and the resulting fractal dimension is $d_B = 1.25$.

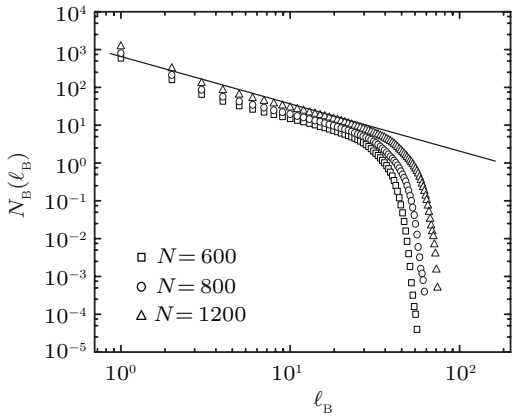


Fig. 6. The fractal relationship that these fractal scale-free networks satisfy is $N_B(\ell_B) \sim \ell_B^{d_B}$. The slope of the line in the figure is 1.25, which means that the fractal dimension is $d_B = 1.25$.

Finally, when the degree distribution of the fractal-free scale network is maintained and the degree distribution of the tree-like skeleton remains unchanged, we find out the numerical relationship between the critical temperature and the system size, and the relationship between the local link probability and the system size, as shown in Fig. 7. Based on Fig. 7(a), these similar fractal scale-free networks can only be tree-like networks under the thermodynamic limit. Based on the results of Fig. 7(b), we can judge that the critical temperature of the system is infinite at the thermodynamic limit.

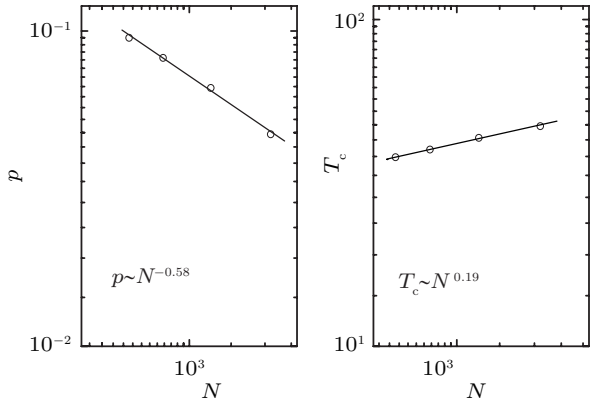


Fig. 7. In panel (a), the local link probability and the network size satisfy the approximate relationship $p \sim N^{-0.58}$. In panel (b), the critical temperature T_c (with $J/k = 1$) and the network size satisfies the approximate relationship $T_c \sim N^{0.19}$.

4. Conclusion

Based on the Ising spin, we study the phase transition of the complex ferromagnetic system with fractal and scale-free network structure. The fractal scale-free networks have a common tree-like skeleton whose degree distribution satisfies $P(k) \sim k^{-\delta}$. We find that when $\delta \geq 3$, the renormalized networks will maintain an unchanged degree distribution. In

addition, loops in the network are generated by random local links. Using the box-covering renormalization method, we find the critical temperature of the system. While keeping the network structure unchanged, we find the numerical relationship between the critical temperature and the network size, and find that the critical temperature of the network is infinite under the thermodynamic limit.

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