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Intra-layer synchronization in duplex networks*

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This paper explores the intra-layer synchronization in duplex networks with different topologies within layers and different inner coupling patterns between, within, and across layers. Based on the Lyapunov stability method, we prove theoretically that the duplex network can achieve intra-layer synchronization under some appropriate conditions, and give the thresholds of coupling strength within layers for different types of inner coupling matrices across layers. Interestingly, for a certain class of coupling matrices across layers, it needs larger coupling strength within layers to ensure the intra-layer synchronization when the coupling strength across layers become larger, intuitively opposing the fact that the intra-layer synchronization is seemly independent of the coupling strength across layers. Finally, numerical simulations further verify the theoretical results.

Keywords: multi-layer network, intra-layer synchronization, Lyapunov stability method

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1. Introduction

Networks are ubiquitous in the real world, such as the Internet, power grid, biological and ecological networks, social networks, and so on. It is a very significant thing to unveil the mechanism behind network collective behaviors, which is still a research focus in network science fields. Synchronization, as one of the collective behaviors, has attracted a great deal of attention,[1–14] since a great deal of work[15] has been published, involving in phase synchronization, complete synchronization, cluster synchronization, generalized synchronization, and even chimera state.[16] Most of the works on synchronization investigated the relationship between synchronization and topological structures, and how to implement synchronization through the controllers or coupling strategies.

Very recently in the last few years, there has been increasing interest in the multi-layer (or multiplex) networks or hypernetworks[22–26] where there are different connectivity patterns between layers, since one kind of networks is always interconnected with other kinds of networks, such as power networks and computer networks, transport networks and communication networks, and the single composite (single connectivity pattern) network model does not well describe the feature of multi-layer networks. Naturally, synchronization in multi-layer networks becomes a hot research topic, and the related works[27–31] appear. Among them, He *et al.*[31] investigated the complete synchronization of multi-agents in multi-layer networks through the Lyapunov stability method, and state that complete synchronization depends on the joint synchronization region and the eigenvalues of each layer. Based on the master stability function method, with the hypothesis that the inner coupling function in multiplex networks is the same in each layer and across layers, some works[29,30] explored the complete synchronization and synchronizability from the view point of network topological structures. Our recent work[32] extended the master stability framework to multiplex networks with different inner coupling functions within and across layers. With it, we studied simultaneously complete, intra-layer, and inter-layer synchronizations, and showed that the inner coupling function across layers has an important impact on the intra-layer synchornization, but it is still completely unclear.

Intra-layer synchronization can be considered to some extent as the mesoscale level between the microscopic level (such as individual behaviors) and macroscopic large-scale level (such as complete synchronization), which has received increasing attention. Gambuzza *et al.*[33] analyzed the intra-layer synchronization of a population of oscillators indirectly coupled through an inhomogeneous medium. Shortly afterwards, Jalan and Singh[34] investigated nodes interactions in one layer affecting the cluster synchronizability of the other layer, and showed that at weak couplings, the multiplexing enhances the cluster synchronizability, while at strong couplings this enhancement depends on the architecture as well as the connection density of the other layer. Rakshit *et al.*[35] explored the intra-layer and inter-layer synchronizations in multiplex networks where the intra-layer coupling interactions are

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switched stochastically, and showed that the higher switching frequency enhances both intra-layer and inter-layer synchronizations. Tang *et al.*[36] studied the impact of the inter-layer coupling function and the intra-layer coupling delay on the intra-layer synchronization, and indicatd that the inter-layer coupling strength neither improves nor weakens the intra-layer synchronizability for certain inter-layer coupling functions. However, these investigations are based on the master stability function method, which is an approach for local stabilization.

These above researches motivate us to focus on the duplex networks where the intra- and inter-layers coupling functions are different, and employ the Lyapunov stability method (a global method) to investigate the intra-layer synchronization. The main contribution in this paper is to give the intra-layer synchronization criterion and the threshold of intra-layer strengths for three classes of inter-layer coupling matrices, and analyze and discuss their influences on the intra-layer synchronization.

The rest of this paper is organized as follows. A duplex network with different inner coupling function within and across layers is introduced in Section 2. In Section 3, the stability analysis based on the Lyapunov stability method is performed, and the corresponding synchronization criteria and thresholds are presented for different types of inner coupling functions across layers. In Section 4, some numerical examples are provided for further verifying the theoretical results obtained in Section 3. Finally, some conclusions and discussions are given in Section 5.

2. A duplex network model

A duplex network consisting of two layers whose one-to-one connections between layers can be described as

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = f^x(\boldsymbol{x}_i(t)) + c\sum\limits_{j=1}^{m} a_{ij}^x \boldsymbol{H}(\boldsymbol{x}_j - \boldsymbol{x}_i) \\ \qquad\qquad + d\boldsymbol{\Gamma}(\boldsymbol{y}_i - \boldsymbol{x}_i), \\ \dot{\boldsymbol{y}}_i(t) = f^y(\boldsymbol{y}_i(t)) + c\sum\limits_{j=1}^{m} a_{ij}^y \boldsymbol{H}(\boldsymbol{y}_j - \boldsymbol{y}_i) \\ \qquad\qquad + d\boldsymbol{\Gamma}(\boldsymbol{x}_i - \boldsymbol{y}_i), \end{cases} \quad i = 1, 2 \ldots m, \quad (1)$$

where $\dot{\boldsymbol{x}}_i(t) = f^x(\boldsymbol{x}_i(t))$ and $\dot{\boldsymbol{y}}_i(t) = f^y(\boldsymbol{y}_i(t))$ dominate the i-th node of the x-layer and y-layer, respectively. $\boldsymbol{A}^x = (a_{ij}^x)$ and $\boldsymbol{A}^y = (a_{ij}^y)$ are adjacent matrices of x-layer and y-layer, respectively, representing the topological structures within layers. If node i is connected with node j in the same layer, then $a_{ij}^{x(y)} = 1$, $a_{ij}^{x(y)} = 0$ otherwise. c and d are the coupling strengths within and across layers, respectively.

Function $\boldsymbol{H}(\cdot) : R^n \to R^n$ is the inner coupling one within layers, describing the communication pattern between state variables of nodes. $\boldsymbol{\Gamma}(\cdot) : R^n \to R^n$ is the inner coupling function between the node in one layer and its corresponding node in the other layer. They can be the linear or nonlinear functions. For simplicity, $\boldsymbol{H}(\cdot)$ and $\boldsymbol{\Gamma}(\cdot)$ are set as the different linear functions here, namely, $\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{H}\boldsymbol{x}$ and $\boldsymbol{\Gamma}(\boldsymbol{x}) = \boldsymbol{\Gamma}\boldsymbol{x}$. Hereafter, we also call $\boldsymbol{H}$ and $\boldsymbol{\Gamma}$ the inner coupling matrices for convenience. The diagram of duplex networks consisting of x-layer and y-layer is shown in Fig. 1.

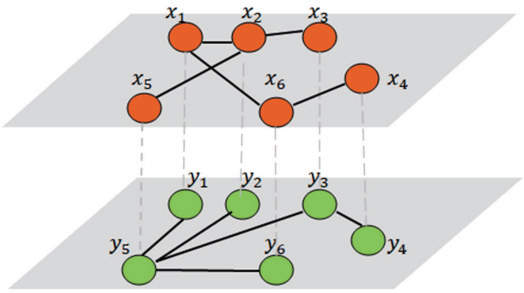


Fig. 1. (color online) A diagram of the duplex network with 6 nodes in each layer.

3. Intra-layer synchronization stability analysis

The so-called intra-layer synchronization means that all the nodes of each layer synchronize to the common state called the intra-layer synchronous state eventually, and the synchronous states are different from each other. The duplex network achieves complete synchronization if the synchronous states are the same. Here, we focus on the nonidentical intra-layer synchronous state in each layer, and theoretically analyze the intra-layer synchronization stability via the Lyapunov stability method.

To begin with, Laplacian matrix $\boldsymbol{L}^{x(y)} = (l_{ij})_{m\times m}$ associated with the adjacent matrix $\boldsymbol{A}^{x(y)}$ is described as

$$l_{ij}^{x(y)} = \begin{cases} -a_{ij}^{x(y)}, & i \neq j, \\ \sum\limits_{j=1, j\neq i}^{m} a_{ij}^{x(y)}, & i = j, \end{cases}$$

which satisfies the dissipative property that $\sum\limits_{j=1}^{m} l_{ij}^{x(y)} = 0$. Taking an example for further illustration, the adjacent matrices within layers of the duplex network shown in Fig. 1.

$$\boldsymbol{A}^x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{A}^y = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

and their Laplacian matrices are

$$\boldsymbol{L}^x = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix},$$

$$\boldsymbol{L}^y = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

With the Laplacian matrices, network (1) can be rewritten as

$$\begin{cases} \dot{\boldsymbol{x}}_i = f^x(\boldsymbol{x}_i) - c\sum_{j=1}^{m} l_{ij}^x \boldsymbol{H}\boldsymbol{x}_j - d\boldsymbol{\Gamma}(\boldsymbol{x}_i - \boldsymbol{y}_i), \\ \dot{\boldsymbol{y}}_i = f^y(\boldsymbol{y}_i) - c\sum_{j=1}^{m} l_{ij}^y \boldsymbol{H}\boldsymbol{y}_j - d\boldsymbol{\Gamma}(\boldsymbol{y}_i - \boldsymbol{x}_i), \end{cases} \quad i = 1,2\ldots m. \tag{2}$$

When network (2) obtains the intra-layer synchronization, then the x-layer and the y-layer tend to the different synchronous states, respectively. Here, suppose $\bar{\boldsymbol{x}} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{x}_i$ is the synchronous solution of the x-layer, and $\bar{\boldsymbol{y}} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{y}_i$ is the one of the y-layer. It follows that

$$\begin{aligned} \dot{\bar{\boldsymbol{x}}} &= \frac{1}{m}\sum_{i=1}^{m} f^x(\boldsymbol{x}_i) - \frac{c}{m}\sum_{i=1}^{m}\sum_{j=1}^{m} l_{ij}^x \boldsymbol{H}\boldsymbol{x}_j + \frac{1}{m}\sum_{i=1}^{m} d\boldsymbol{\Gamma}(\boldsymbol{y}_i - \boldsymbol{x}_i) \\ &= \frac{1}{m}\sum_{i=1}^{m} f^x(\boldsymbol{x}_i) + d\boldsymbol{\Gamma}(\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}), \end{aligned}$$

and similarly,

$$\dot{\bar{\boldsymbol{y}}} = \frac{1}{m}\sum_{i=1}^{m} f^y(\boldsymbol{y}_i) + d\boldsymbol{\Gamma}(\bar{\boldsymbol{x}} - \bar{\boldsymbol{y}}).$$

Due to $\sum_{i=1}^{m}(\boldsymbol{x}_i - \bar{\boldsymbol{x}}) = 0$ and $\sum_{i=1}^{m}(\boldsymbol{y}_i - \bar{\boldsymbol{y}}) = 0$, one can get

$$\sum_{i=1}^{m}(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^{\mathrm{T}}\left[f^x(\bar{\boldsymbol{x}}) - \frac{1}{m}\sum_{j=1}^{m} f^x(\boldsymbol{x}_j)\right] = 0, \tag{3a}$$

$$\sum_{i=1}^{m}(\boldsymbol{y}_i - \bar{\boldsymbol{y}})^{\mathrm{T}}\left[f^y(\bar{\boldsymbol{y}}) - \frac{1}{m}\sum_{j=1}^{m} f^y(\boldsymbol{y}_j)\right] = 0. \tag{3b}$$

Next, some assumptions and lemmas are addressed before giving the main theoretical results.

Assumption 1 Assume the right function $f(\boldsymbol{x})$ of the nodal dynamical system satisfies the Lipschitz condition, i.e., there exists a positive constant ℓ such that

$$\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| \le \ell\|\boldsymbol{x} - \boldsymbol{y}\|.$$

For the functions $f^x(\cdot)$ and $f^y(\cdot)$ in network (2), there exist positive constants a_1 and a_2, respectively, such that

$$\|f^x(\boldsymbol{x}) - f^x(\boldsymbol{y})\| \le a_1\|\boldsymbol{x} - \boldsymbol{y}\| \quad \text{and}$$

$$\|f^y(\boldsymbol{x}) - f^y(\boldsymbol{y})\| \le a_2\|\boldsymbol{x} - \boldsymbol{y}\|.$$

Remark 1 It is easy for continuous-time and bounded dynamical systems to satisfy Assumption 1, such as continuous-time chaotic systems. Assumption 1 implies that the QUAD condition[38] holds, namely

$$(\boldsymbol{x} - \boldsymbol{y})^{\mathrm{T}}(f(\boldsymbol{x}) - f(\boldsymbol{y})) \le \ell(\boldsymbol{x} - \boldsymbol{y})^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{y}).$$

Lemma 1[37] Suppose $h_1, h_2, \ldots, h_m$ and $p_1, p_2, \ldots, p_s$ are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Then the eigenvalues of $\boldsymbol{A} \otimes \boldsymbol{B}$ are $h_i \times p_j$ $(1 \le i \le m, 1 \le j \le s)$ where $\otimes$ represents the knocker product.

Theorem 1 Suppose that $\boldsymbol{H}$ is a positive definite matrix, and $\boldsymbol{\Gamma}$ is a symmetric matrix. If the nodal dynamics $f^x(\cdot)$ and $f^y(\cdot)$ satisfy Assumption 1, then there exist certain coupling strengths c and d, such that network (2) achieves intra-layer synchronization.

Proof Denoting $\boldsymbol{\delta}_i^x = \boldsymbol{x}_i - \bar{\boldsymbol{x}}$ and $\boldsymbol{\delta}_i^y = \boldsymbol{y}_i - \bar{\boldsymbol{y}}$, via Eq. (3), one can get the error system about the intra-layer synchronous solution:

$$\begin{cases} \dot{\boldsymbol{\delta}}_i^x = f^x(\boldsymbol{x}_i) - f^x(\bar{\boldsymbol{x}}) - c\sum_{j=1}^{m} l_{ij}^x \boldsymbol{H}\boldsymbol{\delta}_j^x \\ \qquad - d\boldsymbol{\Gamma}(\boldsymbol{\delta}_i^x - \boldsymbol{\delta}_i^y), \\ \dot{\boldsymbol{\delta}}_i^y = f^y(\boldsymbol{y}_i) - f^y(\bar{\boldsymbol{y}}) - c\sum_{j=1}^{m} l_{ij}^y \boldsymbol{H}\boldsymbol{\delta}_j^y, \\ \qquad - d\boldsymbol{\Gamma}(\boldsymbol{\delta}_i^y - \boldsymbol{\delta}_i^x) \end{cases} \quad i = 1,2,\ldots,m. \tag{4}$$

Choose a Lyapunov candidate function

$$V(t) = V_1(t) + V_2(t), \tag{5}$$

where $V_1(t) = \frac{1}{2}\sum_{i=1}^{m} \boldsymbol{\delta}_i^{x\mathrm{T}}\boldsymbol{\delta}_i^x$ and $V_2(t) = \frac{1}{2}\sum_{i=1}^{m} \boldsymbol{\delta}_i^{y\mathrm{T}}\boldsymbol{\delta}_i^y$.

Taking the derivative of Eq. (5) with respect to t along the trajectory of the error system (4), it yields

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t).$$

Using Assumption 1

$$\begin{aligned} \dot{V}_1(t) &= \sum_{i=1}^{m} \boldsymbol{\delta}_i^{x\mathrm{T}}\dot{\boldsymbol{\delta}}_i^x \\ &= \sum_{i=1}^{m} \boldsymbol{\delta}_i^{x\mathrm{T}}[f^x(\boldsymbol{x}_i) - f^x(\bar{\boldsymbol{x}}) \\ &\quad - c\sum_{j=1}^{m} l_{ij}^x \boldsymbol{H}\boldsymbol{\delta}_j^x - d\boldsymbol{\Gamma}(\boldsymbol{\delta}_i^x - \boldsymbol{\delta}_i^y)] \\ &\le \sum_{i=1}^{m} a_1 \boldsymbol{\delta}_i^{x\mathrm{T}}\boldsymbol{\delta}_i^x - c\sum_{i=1}^{m}\sum_{j=1}^{m} l_{ij}^x \boldsymbol{\delta}_i^{x\mathrm{T}}\boldsymbol{H}\boldsymbol{\delta}_j^x \\ &\quad - d\sum_{i=1}^{m} \boldsymbol{\delta}_i^{x\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{\delta}_i^x + d\sum_{i=1}^{m} \boldsymbol{\delta}_i^{x\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{\delta}_i^y, \end{aligned}$$

and similarly

$$\dot{V}_2(t) \le \sum_{i=1}^{m} a_2 \boldsymbol{\delta}_i^{y\mathrm{T}}\boldsymbol{\delta}_i^y - c\sum_{i=1}^{m}\sum_{j=1}^{m} l_{ij}^y \boldsymbol{\delta}_i^{y\mathrm{T}}\boldsymbol{H}\boldsymbol{\delta}_j^y$$

$$-d\sum_{i=1}^{m}\boldsymbol{\delta}_i^{y\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{\delta}_i^{y}+d\sum_{i=1}^{m}\boldsymbol{\delta}_i^{y\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{\delta}_i^{x}.$$

Denote $M=\max\{a_1,a_2\}$, $\boldsymbol{\delta}=[\boldsymbol{\delta}_1^{x\mathrm{T}},\ \boldsymbol{\delta}_2^{x\mathrm{T}},\ \boldsymbol{\delta}_3^{x\mathrm{T}},\ldots,\boldsymbol{\delta}_m^{x\mathrm{T}},\ \boldsymbol{\delta}_1^{y\mathrm{T}},\ \boldsymbol{\delta}_2^{y\mathrm{T}},\ \boldsymbol{\delta}_3^{y\mathrm{T}},\ldots,\boldsymbol{\delta}_m^{y\mathrm{T}}]^{\mathrm{T}}\in R^{2mn}$, $\boldsymbol{L}=\begin{pmatrix}\boldsymbol{L}^x & 0\\ 0 & \boldsymbol{L}^y\end{pmatrix}\in R^{2m\times 2m}$, and $\boldsymbol{Q}=\begin{pmatrix}-I_m & I_m\\ I_m & -I_m\end{pmatrix}$, it yields

$$\begin{aligned}\dot{V}(t) &\le M\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta}-c\boldsymbol{\delta}^{\mathrm{T}}(L\otimes H)\boldsymbol{\delta}-d\boldsymbol{\delta}^{\mathrm{T}}(I_{2m}\otimes\boldsymbol{\Gamma})\boldsymbol{\delta}\\ &\quad+d\boldsymbol{\delta}^{\mathrm{T}}[(I_{2m}\begin{pmatrix}0 & I_m\\ I_m & 0\end{pmatrix})\otimes\boldsymbol{\Gamma}]\boldsymbol{\delta}\\ &= \boldsymbol{\delta}^{\mathrm{T}}[MI_{2mn}-c(\boldsymbol{L}\otimes\boldsymbol{H})+d(\boldsymbol{Q}\otimes\boldsymbol{\Gamma})]\boldsymbol{\delta}. \qquad (6)\end{aligned}$$

Since $\boldsymbol{L}$ is a symmetric and zero row sum matrix due to the fact that $\boldsymbol{L}^x$ and $\boldsymbol{L}^y$ are symmetric and zero row sum matrices, there exists an orthogonal matrix $\boldsymbol{U}\triangleq(\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_{2m})$ such that

$$\boldsymbol{L}=\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{T}},$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix whose diagonal elements being $\lambda_1^x,\lambda_2^x,\ldots,\lambda_m^x,\lambda_1^y,\lambda_2^y,\ldots,\lambda_m^y$, and $0=\lambda_1^x<\lambda_2^x\le\cdots\le\lambda_m^x$ and $0=\lambda_1^y<\lambda_2^y\le\cdots\le\lambda_m^y$ are eigenvalues of $\boldsymbol{L}^x$ and $\boldsymbol{L}^y$, respectively.

Let $\boldsymbol{\alpha}=(\boldsymbol{U}^{\mathrm{T}}\otimes I_n)\boldsymbol{\delta}$ and denote $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_1^{\mathrm{T}},\boldsymbol{\alpha}_2^{\mathrm{T}},\cdots\boldsymbol{\alpha}_{2m}^{\mathrm{T}})^{\mathrm{T}}$ where $\boldsymbol{\alpha}_i$ is an n-dimensional vector, and take $c=c_0+1$ where

$$c_0=\frac{1}{\lambda_2(\boldsymbol{\Lambda}\otimes\boldsymbol{H})}(M+d\lambda_{\max}((\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\boldsymbol{\Gamma})),$$

we can get

$$\begin{aligned}\dot{V}(t) &\le M\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}-c\boldsymbol{\alpha}^{\mathrm{T}}(\boldsymbol{\Lambda}\otimes\boldsymbol{H})\boldsymbol{\alpha}+d\boldsymbol{\alpha}^{\mathrm{T}}((\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\boldsymbol{\Gamma})\boldsymbol{\alpha} \qquad (7a)\\ &\le (M-c\lambda_2(\boldsymbol{\Lambda}\otimes\boldsymbol{H})+d\lambda_{max}((\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\boldsymbol{\Gamma}))\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} \qquad (7b)\\ &= -\lambda_2(\boldsymbol{\Lambda}\otimes\boldsymbol{H})\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}=-\lambda_2(\boldsymbol{\Lambda}\otimes\boldsymbol{H})\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta}. \qquad (7c)\end{aligned}$$

where $\lambda_2(\boldsymbol{\Lambda}\otimes\boldsymbol{H})$ is the smallest nonzero eigenvalue of $\boldsymbol{\Lambda}\otimes\boldsymbol{H}$, and $\lambda_{max}((\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\boldsymbol{\Gamma})$ is the largest eigenvalue of $(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\boldsymbol{\Gamma}$.

According to LaSalle's invariance principle,[39] the network (4) or network (2) achieves synchronization, and the proof is completed.

Note 1 The inequality (7b) is based on the fact that $\boldsymbol{\alpha}_1=0$ and $\boldsymbol{\alpha}_{m+1}=0$ correspond to $\lambda_1^x=0$ and $\lambda_1^y=0$, respectively. In fact, the characteristic vector $\boldsymbol{u}_1=\frac{1}{\sqrt{m}}(1,1,\ldots,1,0,0,\ldots,0)^{\mathrm{T}}$ corresponding to $\lambda_1^x=0$, and

$$\begin{aligned}\boldsymbol{\alpha}_1 &= (\boldsymbol{u}_1^{\mathrm{T}}\otimes I_n)\boldsymbol{\delta}=\frac{\boldsymbol{\delta}_1^x+\boldsymbol{\delta}_2^x+\cdots+\boldsymbol{\delta}_m^x}{\sqrt{m}}\\ &= \frac{1}{\sqrt{m}}\left(\sum_{i=1}^{m}\boldsymbol{x}_i-m\bar{\boldsymbol{x}}\right)=0.\end{aligned}$$

Similarly, $\boldsymbol{u}_{m+1}=\frac{1}{\sqrt{m}}(0,0,\ldots,0,1,1,\ldots,1)^{\mathrm{T}}$ corresponding to $\lambda_1^y=0$, and $\boldsymbol{\alpha}_{m+1}=(\boldsymbol{u}_{m+1}^{\mathrm{T}}\otimes I_n)\boldsymbol{\delta}=0$.

Remark 2 The intra-layer synchronous state is described by

$$\begin{cases}\dot{\bar{\boldsymbol{x}}}=\dfrac{1}{m}\displaystyle\sum_{i=1}^{m}f^x(\bar{\boldsymbol{x}})+d\boldsymbol{\Gamma}(\bar{\boldsymbol{y}}-\bar{\boldsymbol{x}}),\\ \dot{\bar{\boldsymbol{y}}}=\dfrac{1}{m}\displaystyle\sum_{i=1}^{m}f^y(\bar{\boldsymbol{y}})+d\boldsymbol{\Gamma}(\bar{\boldsymbol{x}}-\bar{\boldsymbol{y}}).\end{cases} \qquad (8)$$

Actually, the intra-layer synchronization is complete synchronization when $\bar{\boldsymbol{x}}=\bar{\boldsymbol{y}}$. However, the different nodal dynamics between two layers, namely $f^x\ne f^y$, can guarantee nonoccurrence of compete synchronization. Otherwise, the complete synchronization solution $\boldsymbol{s}$ satisfies $\dot{\boldsymbol{s}}=\frac{1}{m}\sum_{i=1}^{m}f^x(\boldsymbol{s})$ and $\dot{\boldsymbol{s}}=\frac{1}{m}\sum_{i=1}^{m}f^y(\boldsymbol{s})$ at the same time, which contradict each other. However, when $f^x\ne f^y$, whether complete synchronization takes place or not depends on the inter-layer coupling strength d and matrix $\boldsymbol{\Gamma}$.

Corollary 1 If the inner coupling matrix across layers $\boldsymbol{\Gamma}$ is a positive (definite) semi-definite matrix, and taking the inner coupling matrix within layers $\boldsymbol{H}=I_n$, then the network (2) achieves intra-layer synchronization when

$$c>\tilde{c}_0\triangleq\frac{M}{\min\{\lambda_2^x,\lambda_2^y\}}.$$

Proof Since $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U}$ are the same eigenvalues 0, -2, as those of $\boldsymbol{Q}$, according to Lemma 1, $(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\boldsymbol{\Gamma}$ is a negative semi-definite matrix, and $\lambda_2(\boldsymbol{\Lambda}\otimes\boldsymbol{H})=\lambda_2(\boldsymbol{\Lambda})\lambda_{\min}(\boldsymbol{H})=\min\{\lambda_2^x,\lambda_2^y\}$. From the inequality (7b), it follows that

$$\dot{V}(t)\le\min\{\lambda_2^x,\lambda_2^y\}(\tilde{c}_0-c)\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}=\min\{\lambda_2^x,\lambda_2^y\}(\tilde{c}_0-c)\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta}.$$

The rest of the proof is easily completed.

Remark 3 It can be found from the threshold $\tilde{c}_0$ that the coupling strength c for intra-layer synchronization depends on the nodal dynamics (M) and the intra-layer topology structure with weaker connectivity (the smaller of λ_2^x and λ_2^y).

Corollary 2 If Γ is a negative-definite (or semi-definite) matrix, and taking $\boldsymbol{H}=I_n$, then the network (2) achieves intra-layer synchronization when

$$c>c_0^*\triangleq\frac{M-2d\lambda_{\min}(\boldsymbol{\Gamma})}{\min\{\lambda_2^x,\lambda_2^y\}}.$$

Proof According to Lemma 1, $(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\Gamma$ is a positive semi-definite matrix, and $\lambda_{\max}((\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\boldsymbol{\Gamma})=-2\lambda_{\min}(\boldsymbol{\Gamma})$. The rest of the proof is easily completed.

Remark 4 In this case of negative-definite (or semi-definite) $\boldsymbol{\Gamma}$, besides both factors of nodal dynamics and intra-layer topological structures, the coupling strength c within layers also depends on the coupling strength d across layers. Specifically, a large c is required to draw the nodes within layers into the identical state when the strength d across layers becomes large, please refer to Fig. 5. This is very interesting. Intuitively, the intra-layer synchronization seems to have nothing to do with the coupling strength d and the inner coupling pattern $\boldsymbol{\Gamma}$ across layers.

Remark 5 In general, a positive-definite matrix separates the states trajectories due to its repulsive force. Therefore, to realize mere intra-layer synchronization without inter-layer synchronization, the matrix $\boldsymbol{\Gamma}$ should be taken as the negative-definite one such that $(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U})\otimes\boldsymbol{\Gamma}$ is a positive-definite one. In other words, one should select an appropriate $\boldsymbol{\Gamma}$ such that the synchronous state of x-layer is unsynchronized with the one of y-layer.

4. Numerical simulations

To show the effectiveness of the theoretical results, we take the Lorenz chaotic system as network nodal dynamics in both layers:

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \sigma(x_2 - x_1), \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_1(\beta - x_3) - x_2, \\ \dfrac{\mathrm{d}x_3}{\mathrm{d}t} = x_2 x_1 - \gamma x_3, \end{cases} \tag{9}$$

where $\sigma = 10$, $\beta = 28$, and $\gamma = 8/3$. Since the Lorenz chaotic system is global Lipschtiz continuous and bounded, one can get the positive constant $M = 50$ by estimating the ultimate bound of the Lorenz chaotic system.

The duplex network is composed of x-layer and y-layer, and each layer has 100 Lorenz chaotic oscillators, respectively. The x-layer is a BA scale-free network generated from the random graph with the initial condition of $m_0 = 4$. The y-layer is an NW small-world network generated from the ring network with the connection probability of $p = 0.1$. After fixing the intra-layer topologies, the smallest eigenvalues of Laplacian matrices are easily obtained. Here, $\lambda_2^x = 0.5772$, $\lambda_2^y = 2.2402$, and the intra-layer inner coupling matrix $\boldsymbol{H} = \mathrm{diag}\{1,1,1\}$.

To measure network synchronization, we define the intra-layer synchronization (sync.) error of x-layer and y-layer:

$$E^X = \frac{1}{m}\sum_{i=1}^{m} ||\boldsymbol{x}_i - \bar{\boldsymbol{x}}||_\infty,\ E^Y = \frac{1}{m}\sum_{i=1}^{m} ||\boldsymbol{y}_i - \bar{\boldsymbol{y}}||_\infty,$$

and the complete synchronization error:

$$E^{XY} = \frac{1}{2m}\sum_{i=1}^{2m} ||\boldsymbol{z}_i - \bar{\boldsymbol{z}}||_\infty,$$

where $\boldsymbol{z} = (\boldsymbol{z}_i)_{2m\times 1} = (\boldsymbol{x}_1^{\mathrm{T}}, \boldsymbol{x}_2^{\mathrm{T}}, \ldots, \boldsymbol{x}_m^{\mathrm{T}}, \boldsymbol{y}_1^{\mathrm{T}}, \boldsymbol{y}_2^{\mathrm{T}}, \ldots, \boldsymbol{y}_m^{\mathrm{T}})^{\mathrm{T}}$, and

$$\bar{\boldsymbol{z}} = \frac{1}{2m}\sum_{i=1}^{2m} \boldsymbol{z}_i.$$

For further verification of the theoretical results, we select different inter-layer inner coupling matrices $\boldsymbol{\Gamma}$ for numerical simulations. Specifically, we take $d = 0.1$, and $c = 88$ (which satisfies $c > \tilde{c}_0 = 50/0.5772 = 86.625$) for the cases of $\boldsymbol{\Gamma} = \mathrm{diag}(1,1,0)$ and $\boldsymbol{\Gamma} = \mathrm{diag}(0,0,1)$; $c = 91 (> c_0^* = 52/0.5772 = 90.0901)$ for the cases of $\boldsymbol{\Gamma} = \mathrm{diag}(-1,-1,-1)$ and $\boldsymbol{\Gamma} = \mathrm{diag}(-1,1,1)$. Figures 2–5 show that each layer in the duplex network converges to the respective synchronous state, which verifies our theoretical results.

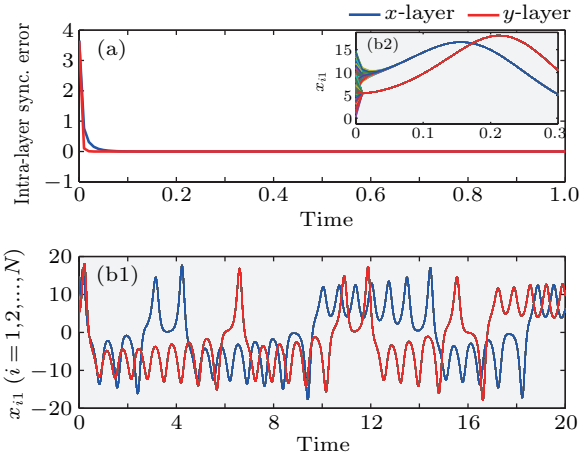


Fig. 2. (color online) The intra-layer synchronization error (a) and x variables (b) of time, here $\Gamma = \mathrm{diag}\{1,1,0\}$, $c = 88$, and $d = 0.1$. The small picture (b2) inserted in panel (a) is the magnification of one segment in panel (b1).

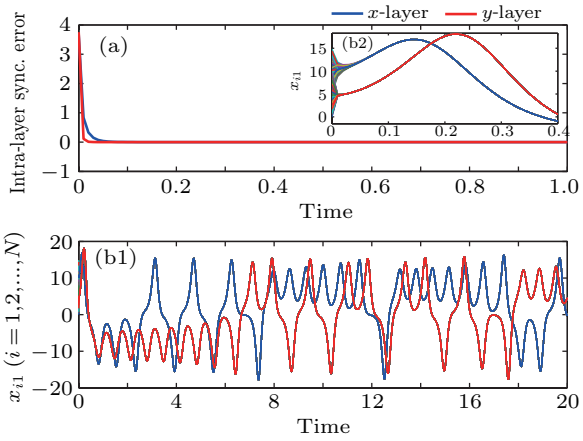


Fig. 3. (color online) The intra-layer synchronization error (a) and x variables (b) of time, here $\Gamma = \mathrm{diag}\{0,0,1\}$, $c = 88$, and $d = 0.1$. The small picture (b2) inserted in panel (a) is the magnification of one segment in (b1).

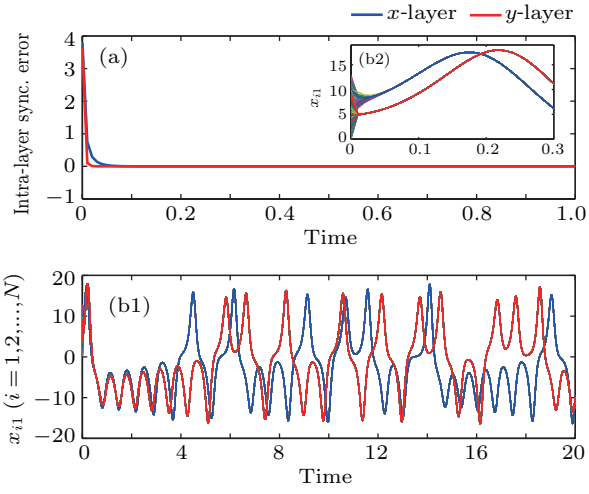


Fig. 4. (color online) The intra-layer synchronization error (a) and x variables (b) of time, here $\Gamma = \mathrm{diag}\{-1,1,1\}$, $c = 91$, and $d = 0.1$. The small picture (b2) inserted in panel (a) is the magnification of one segment in panel (b1).

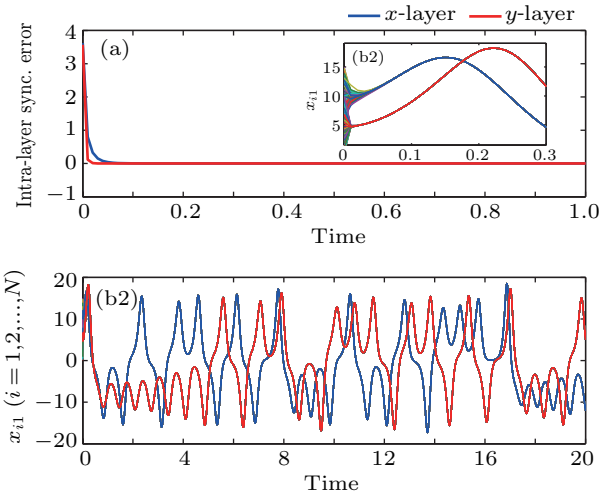


Fig. 5. (color online) The intra-layer synchronization error (a) and x variables (b) of time, here $\Gamma = \mathrm{diag}\{-1,-1,-1\}$, $c = 91$, and $d = 0.1$. The small picture (b2) inserted in panel (a) is the magnification of one segment in panel (b1).

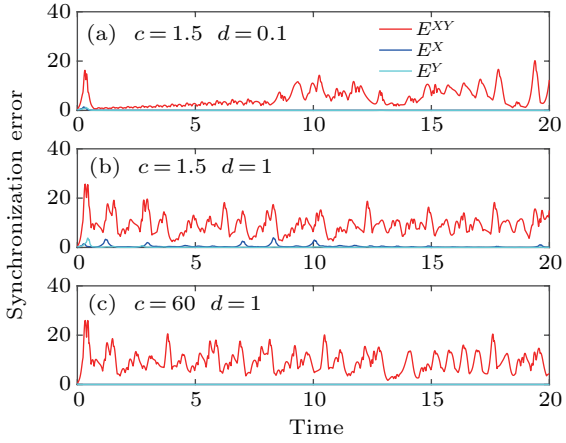


Fig. 6. (color online) Synchronization error of time for the duplex network with 50 nodes in each layer, $\Gamma = \mathrm{diag}\{-1,-1,-1\}$ and different combinations of c and d. E^{XY}, E^X, and E^Y represent complete synchronization, synchronization of x-layer and y-layer, respectively. It indicates that a larger c is necessary for realizing intra-layer synchronization when d increases from 0.1 to 1.

5. Conclusions and discussions

In summary, focusing on the duplex network where there are different topological structures in each layer and different inner coupling functions between, within, and across layers, we have theoretically studied the intra-layer synchronization based on the Lyapunov stability method, and give some synchronization criteria as well as the threshold of coupling strengths within layers. The threshold depends not only on the intra-layer topologies and nodal dynamics, but also on the inter-layer coupling strength. When fixing nodal dynamics and taking a positive-definite matrix as the inner coupling one across layers, the threshold of intra-layer coupling strengths is determined by the intra-layer topologies, i.e., the smaller of nonzero smallest eigenvalues of the intra-layer Laplacians. More interesting, for negative-definite (even indefinite) coupling matrices across layers, the inter-layer coupling strength has an influence on the intra-layer synchronization. In particular, a large inter-layer coupling strength needs a large intra-layer coupling strength to drive the oscillators in each layer into their respective synchronous states. Our results also show that when the nodal dynamics between two layers is the same, the negative-definite (even indefinite) coupling matrix across layers may ensure no inter-layer but mere intra-layer synchronization, coinciding with the result in Ref. [32] analyzed by the master stability function method. Further theoretical investigation on why some coupling matrices across layers can guarantee mere intra-layer synchronization but others cannot is interesting. The results also can work for multiplex (more than two-layer) networks.

However, what happens if the delay or directed topology is taken account into, and what is the influence of delays and directed connections on the intra-layer synchronization? As these problems remain unsolved, it is worth further study.

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