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A new four-dimensional hyperjerk system with stable equilibrium point, circuit implementation, and its synchronization by using an adaptive integrator backstepping control

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This paper reports a new simple four-dimensional (4D) hyperjerk chaotic system. The proposed system has only one stable equilibrium point. Hence, its strange attractor belongs to the category of hidden attractors. The proposed system exhibits various dynamical behaviors including chaotic, periodic, stable nature, and coexistence of various attractors. Numerous theoretical and numerical methods are used for the analyses of this system. The chaotic behavior of the new system is validated using circuit implementation. Further, the synchronization of the proposed systems is shown by designing an adaptive integrator backstepping controller. Numerical simulation validates the synchronization strategy.

Keywords: new hyperjerk chaotic system, stable equilibrium, hidden attractors, adaptive backstepping control, synchronization

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1. Introduction

Three-dimensional (3D) and four-dimensional (4D) chaotic systems are used in several applications as presented in Refs [1]–[3]. Depending on various characteristics, many hyperchaotic and chaotic systems have been reported in the literature.[4–6] Chaotic systems with simple structure are more common in the literature.[7] A simple chaotic system with a jerk structure is considered for the study in Refs [8] and [9]. The structure of a 4D "jerk system" can be defined in a form as

$$\frac{\mathrm{d}^4y}{\mathrm{d}t^4} = J\left(\frac{\mathrm{d}^3y}{\mathrm{d}t^3}, \frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \frac{\mathrm{d}y}{\mathrm{d}t}, y\right), \tag{1}$$

where y is a variable and J is a function of linear and nonlinear terms. The system defined in Eq. (1) can be called a "hyperjerk system".[8] Here, $\mathrm{d}^4y/\mathrm{d}t^4$ is defined as "snap" in the literature[8] and "snap" is also defined as "jounce", "sprite", or "surge" in the literature.[7] It is worth mentioning here that when dealing with chaotic jerk systems, the analysis given in Refs. [10] and [11] may be considered.

Chaotic systems can be divided into two groups. These are (i) systems with self-excited attractors[12–15] and (ii) systems with hidden attractors.[16–21] If the basin of attraction for the strange attractor involves an equilibrium, then that attractor is self-excited. Otherwise, it is hidden.[22–25] According to this definition, strange attractors in chaotic systems with only stable equilibrium points are certainly hidden.[26–30] Other variants of systems with hidden attractors are the systems having no equilibrium point.[31–34] Chaotic systems having infinite equilibria are mostly considered under the category of a hidden attractor in Refs [35]–[39]. The reported 3D and 4D chaotic systems with stable equilibrium points are presented in Table 1. Although many 3D and 4D chaotic systems are reported in the literature, very few jerk and hyperjerk systems are reported.[40–42] The reported hyperjerk systems with various characteristics are considered under the category of self-excited attractors. Recently, some works on coexisting attractors in various types of systems are seen in Refs [43]–[45]. Some chaotic systems are reported with multiple attractors like in Refs. [46]–[48] and unique characteristics in Refs. [49]–[51]. To the best of the authors' knowledge, a hyperjerk chaotic system with stable equilibrium points does not exist in the literature. Therefore, developing a 4D hyperjerk chaotic system having a stable equilibrium point is a worthy motivation of this research work.

The new system exhibits various characteristics like chaotic, periodic, stable, and coexistence of various attractors. Systems with coexisting attractors are termed as multi-stable, which are very important in nature and engineering.[65–70] The proposed hyperjerk system with a stable equilibrium point is

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analyzed using various theoretical and numerical tools.

The synchronization of chaotic systems is another important application of chaotic systems.[71] Various types of synchronization are available in the literature like complete synchronization,[13] anti-synchronization,[13] hybrid synchronization,[71] phase synchronization,[72] lag synchronization,[73] projective synchronization,[74] etc. The effectiveness of a synchronization technique depends on the types of controller designed for it. Different types of controller are used for the synchronization of the hyperchaotic or chaotic systems like nonlinear active control,[13] sliding mode control (SMC),[75] observer-based control,[76] hybrid control techniques,[77] etc. The backstepping control technique is easier and more effective among the above-mentioned controllers. The systems having a strict feedback form are especially more appropriate for the backstepping controller.[78] Therefore, designing a backstepping controller for the synchronization of a strict feedback form chaotic jerk system with unknown parameters is an interesting problem. Motivated by the above discussion, in this paper, an adaptive integrator backstepping controller is designed for the complete synchronization between two identical new hyperjerk systems.

Table 1. Hyperchaotic/chaotic systems with stable equilibrium points.

3D, 4D, 5D	Description of the system	References
3D	a chaotic system with one stable equilibrium point	[52]–[57]
	a chaotic system with two stable equilibrium points	[57]–[61]
4D	a hyperchaotic system with one stable equilibrium point	[62]–[65]
	hyperjerk chaotic system having a stable equilibrium point	this work

The rest of the paper is organized as follows. Section 2 describes the dynamics of the new hyperjerk system. Section 3 deals with the dynamical characteristics of the new system using numerical techniques. The circuit implementation of the new system is discussed in Section 4. The synchronization between two identical new hyperjerk systems by an adaptive integrator backstepping controller is shown in Section 5. Results and discussion for the synchronization are shown in Section 6. Finally, the conclusions of the paper are given in Section 7.

2. Dynamics of a new 4D hyperjerk chaotic system with a stable equilibrium point

This section describes the dynamics and basic properties of the new system. A new 4D hyperjerk chaotic system with a stable equilibrium point is considered in this paper and is described in the following Eq. (2).

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = w, \\ \dot{w} = -ax - bw + cxy + dxz + fyw - g, \end{cases} \tag{2}$$

where a, c, d, and g are the parameters and x, y, z, and w are the state variables. The hyperjerk system in Eq. (2) can be represented as

$$\frac{\mathrm{d}^4x}{\mathrm{d}t^4} = \left(-ax - b\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + cx\frac{\mathrm{d}x}{\mathrm{d}t} + \mathrm{d}x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + f\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^3x}{\mathrm{d}t^3} - g\right). \tag{3}$$

The system in Eq. (2) has asymmetry to its coordinates, planes, and spaces. Thus, the system is not invariant under any coordinate transformation. The divergence of the vector field of states of system (2) is given in Eq. (4).

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} = -b + fy. \tag{4}$$

Thus, the system is a conditional dissipative chaotic system, since $\nabla V < 0$ when the average value of $y < b/f$.

The equilibrium points of system (2) are found by equating the time derivative of each state variable to zero. The unique equilibrium point of the new system is $E(x^*, y^*, z^*, w^*) = (-1.74/a, 0, 0, 0)$. The Jacobian matrix of the new system is given in Eq. (5).

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a + cy + dz & cx + fw & dx & fy - b \end{bmatrix}. \tag{5}$$

The characteristic polynomial of the system at equilibrium point E with $b = 1.43$, $c = 2.18$, $d = 6.11$, and $f = 2.65$ is given in Eq. (6).

$$P = \lambda^4 + 1.43\lambda^3 + \left(\frac{10.631399}{a}\right)\lambda^2 + \frac{3.7932}{a}\lambda + a. \tag{6}$$

Eigenvalue plots of system (2) with the variation of parameter a are shown in Fig. 1.

It is apparent from Fig. 1 that the system has all the eigenvalues with the negative real part. Thus, the equilibrium point E of the system is stable for the parameter values $a \in [2.6, 2.75]$ and $b = 1.43$, $c = 2.18$, $d = 6.11$, and $f = 2.65$. Therefore, we say that the system in Eq. (2) satisfies the characteristics to be a hidden attractors chaotic system.[79]

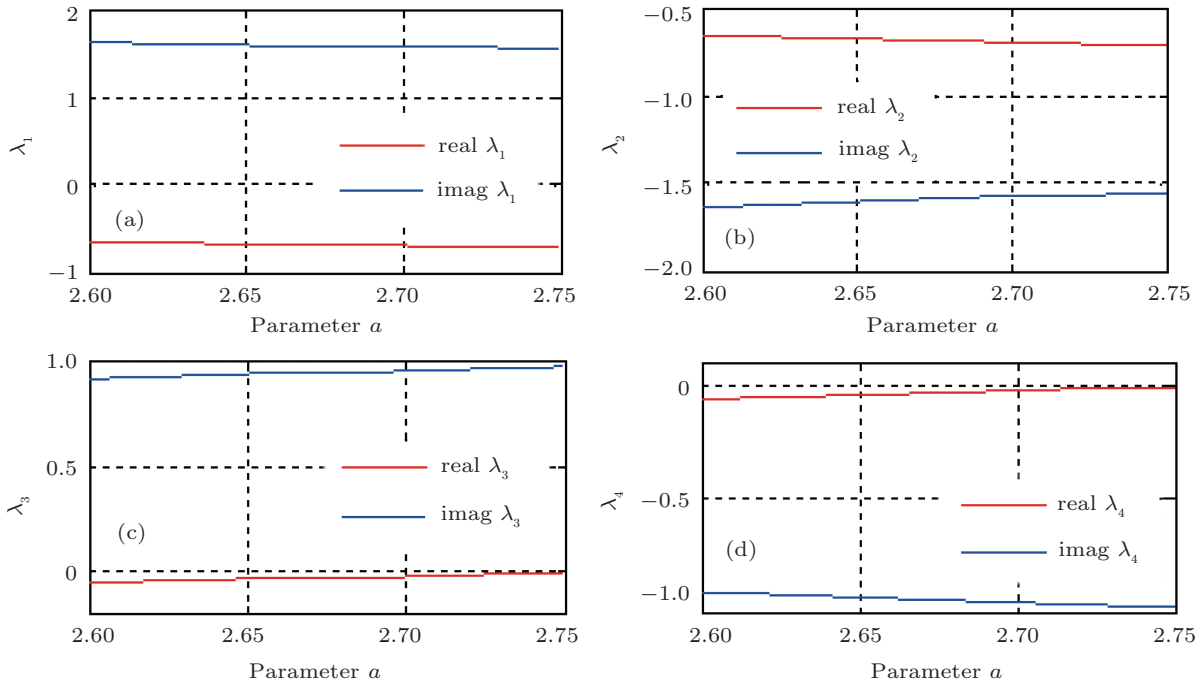


Fig. 1. (color online) Eigenvalue plot of system (2) with the variation of $a \in [2.6, 2.75]$ and $b = 1.43$, $c = 2.18$, $d = 6.11$, and $f = 2.65$.

3. Dynamical analysis

Lyapunov spectrums and bifurcation diagrams of the system with the variation of parameter a (and keeping all other parameters fixed) are shown in Fig. 2. The Lyapunov spectrum of the system is plotted by finding Lyapunov exponents using the Wolf algorithm proposed in Ref. [80] with fixed initial conditions $(x(0), y(0), z(0), w(0)) = (0, 0, -2, 0)$. The bifurcation diagram of the system is calculated by using the continuation method with increasing values of bifurcation parameter a and starting initial conditions $(x(0), y(0), z(0), w(0)) = (0, 0, -2, 0)$. Projections of strange attractors of system (2) with $a = 2.73$, $b = 1.43$, $c = 2.18$, $d = 6.11$, $f = 2.65$, $g = 1.74$, and initial conditions $(x(0), y(0), z(0), w(0) = (0, 0, -2, 0)$ are shown in Fig. 3(a). The periodic and stable nature of the system with $a = 2.61$ and $c = 2.77$ (keeping other parameters fixed) are shown Figs. 3(b) and 3(c), respectively.

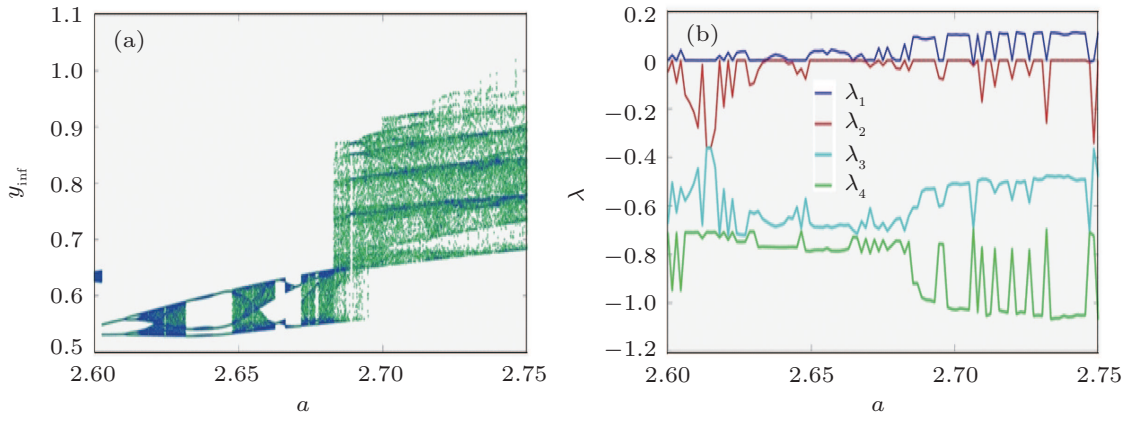


Fig. 2. (color online) Bifurcation diagram and Lyapunov spectrum with the variation of parameter $a \in [2.6, 2.75]$ and $b = 1.43$, $c = 2.18$, $d = 6.11$, $f = 2.65$, and $g = 1.74$.

The new hyperjerk system shows the coexistence of various attractors. The coexistence of chaotic and stable attractors, the coexistence of chaotic and period-1 attractors, and the coexistence of chaotic and period-2 attractors with $a = 2.60$, $b = 1.43$, $c = 2.18$, $d = 6.11$, $f = 2.65$, and $g = 1.74$ are shown in Figs. 4(a), 4(b), and 4(c), respectively, for different initial conditions. Further, the coexistence of period-1 and period-2 attractors, the coexistence of period-1 and stable attractors, and the coexistence of chaotic and period-2 attractors with the same parameters and different initial conditions are shown in Figs. 5(a), 5(b), and 5(c), respectively. Basin of attraction of the new hyperjerk system in the plane $z = w = 0$ is shown in Fig. 6.

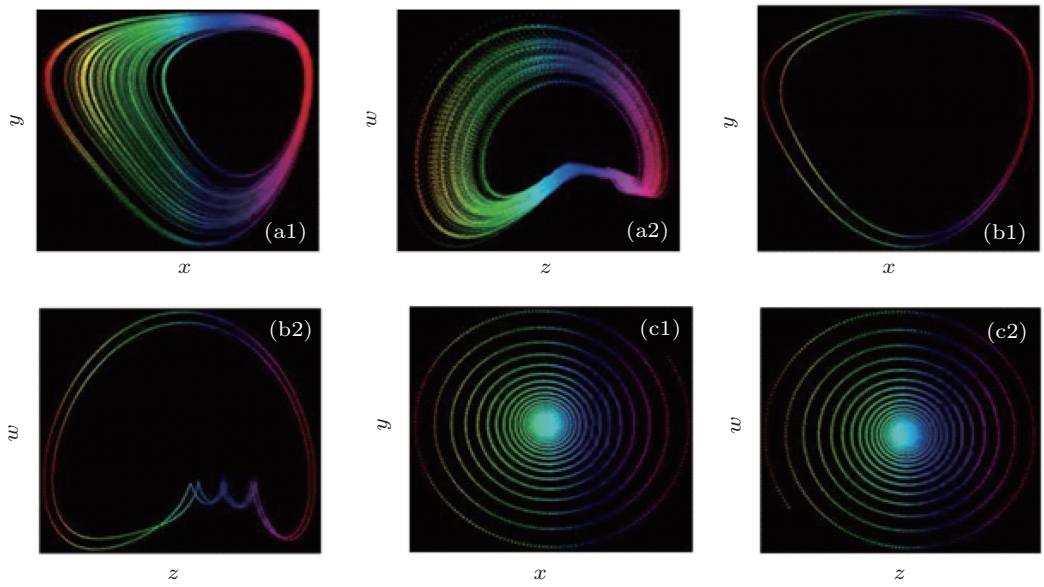


Fig. 3. (color online) (a1) and (a2) Chaotic attractor with $a = 2.73$, $b = 1.43$, $c = 2.18$, $d = 6.11$, $f = 2.65$, and $g = 1.74$, (b1) and (b2) periodic attractors $a = 2.61$, $b = 1.43$, $c = 2.18$, $d = 6.11$, $f = 2.65$, and $g = 1.74$, and (c1) and (c2) stable behavior with $a = 2.73$, $b = 1.43$, $c = 2.28$, $d = 6.11$, $f = 2.65$, and $g = 1.74$ of system (2).

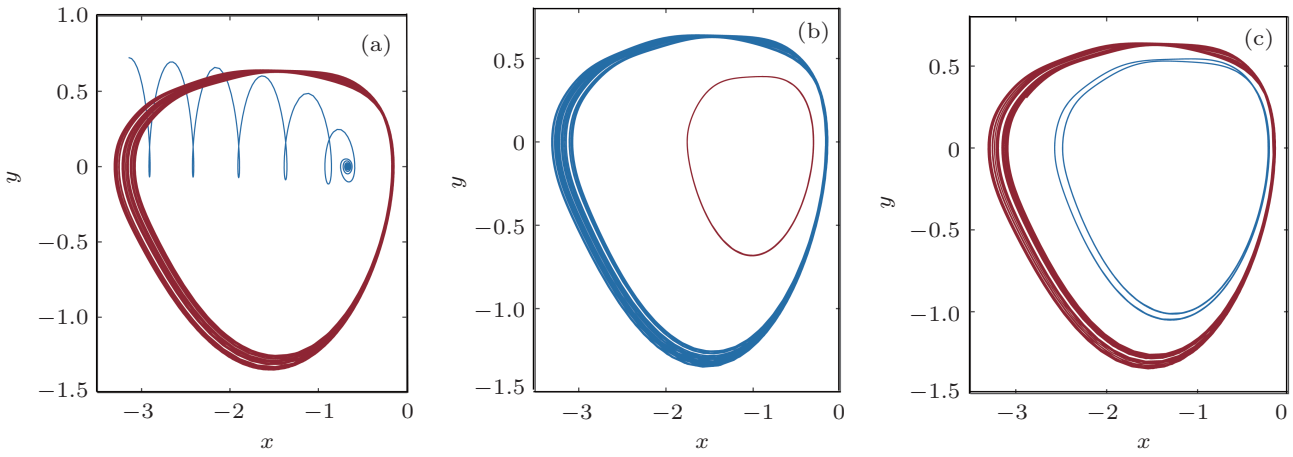


Fig. 4. (color online) (a) Coexistence of chaotic and point attractors with $(x(0), y(0), z(0), w(0)) = (0, 0, \pm 1, 0))$ (red, blue), (b) coexistence of period-1 and chaotic attractors with $(x(0), y(0), z(0), w(0)) = (0, 0, \pm 0.2, 0)$ (red, blue), (c) coexistence of period-2 and chaotic attractors with $(x(0), y(0), z(0), w(0)) = (0, 0, \pm 0.5, 0)$ (blue, red) of system (1) $a = 2.60$, $b = 1.43$, $c = 2.18$, $d = 6.11$, $f = 2.65$, and $g = 1.74$.

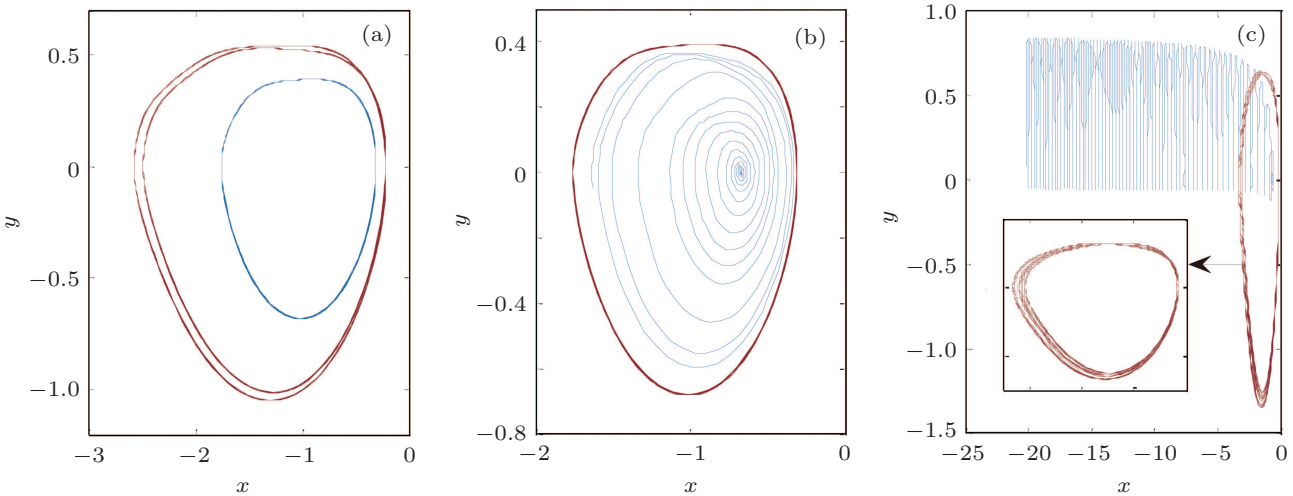


Fig. 5. (color online) (a) Coexistence of period-1 attractor and period-2 attractor with $(x(0), y(0), z(0), w(0)) = (0, \pm 0.1, \pm 0.1, \pm 0.1)$ (blue, red), (b) coexistence of period-1 attractor and stable equilibrium with $(x(0), y(0), z(0), w(0) = (0, 0, \pm 0.8, 0)$ (red, blue), (c) coexistence of stable equilibrium and strange attractor with $(x(0), y(0), z(0), w(0) = (0, 0.8, \pm 0.8, -0.8)$ (red, blue) of system (2), and $a = 2.60$, $b = 1.43$, $c = 2.18$, $d = 6.11$, $f = 2.65$, and $g = 1.74$.

Table 2. Lyapunov exponents for the coexistence of attractors for Figs. 4 and 5.

SL. No.	Initial conditions	Lyapunov exponents
1	$(x(0),y(0),z(0),w(0)=(0,0,1,0)$	$L_i=(-0.0123,-0.0543,-0.585,-0.7882)$
2	$(x(0),y(0),z(0),w(0)=(0,0,-1,0)$	$L_i=(0.1171,0,-0.485,-1.0510)$
3	$(x(0),y(0),z(0),w(0)=(0,0,0.2,0)$	$L_i=(0.119,0,-0.493,-1.0501)$
4	$(x(0),y(0),z(0),w(0)=(0,0,-0.2,0)$	$L_i=(0,-0.0593,-0.493,-1.0501)$
5	$(x(0),y(0),z(0),w(0)=(0,0,0.5,0)$	$L_i=(0,-0.0693,-0.593,-1.0703)$
6	$(x(0),y(0),z(0),w(0)=(0,0,-0.5,0)$	$L_i=(0.129,0,-0.482,-1.0710)$
7	$(x(0),y(0),z(0),w(0)=(0,0.1,0.1,0.1)$	$L_i=(0,0.0712,-0.4821,-1.0520)$
8	$(x(0),y(0),z(0),w(0)=(0,-0.1,-0.1,-0.1)$	$L_i=(0,0.0612,-0.5031,-1.0621)$
9	$(x(0),y(0),z(0),w(0)=(0,0,0.8,0)$	$L_i=(-0.110,-0.340,-0.4821,-1.0583)$
10	$(x(0),y(0),z(0),w(0)=(0,0,-0.8,0)$	$L_i=(0,-0.2457,-0.486,-0.7066)$
11	$(x(0),y(0),z(0),w(0)=(0,0.8,0.8,-0.8)$	$L_i=(-0.1201,-0.341,-0.5012,-0.802)$
12	$(x(0),y(0),z(0),w(0)=(0,0.8,-0.8,-0.8)$	$L_i=(0.139,0,-0.4931,-1.08201)$

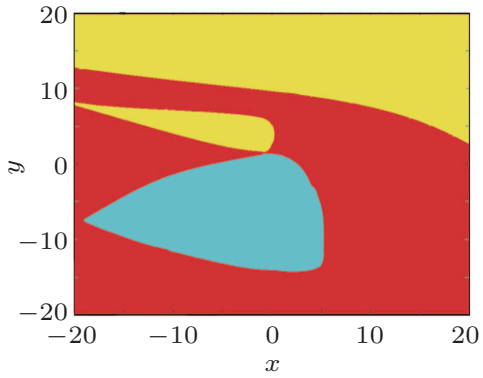


Fig. 6. (color online) Basin of attraction with $z=w=0$ and $a=2.73$, $b=1.43$, $c=2.18$, $d=6.11$, $f=2.65$, and $g=1.74$ of the new system, where points with blue, red, and yellow colors in the basin of attraction lead to strange attractor, stable fixed point, and unbounded solutions, respectively.

The Lyapunov exponents corresponding to the change in the initial condition for the coexistence of attractors for Figs. 4 and 5 are given in Table 2. It is seen from Table 2 that the new system has various responses.

4. Designed circuit for the hyperjerk chaotic system

This section presents a circuit which emulates the hyperjerk system in Eq. (2). By using the operational-amplifier approach,[81,82] the circuit is designed as shown in Fig. 7.

The main part of the circuit is based on four voltage integrators, which are constructed by operational amplifiers U_1, U_2, U_3, and U_4. Other operational amplifiers work as inverting amplifiers (U_5,U_6,U_7). For modelling the circuit, TL084 operational amplifiers have been used. The circuit equations are described by the following Eq. (7).

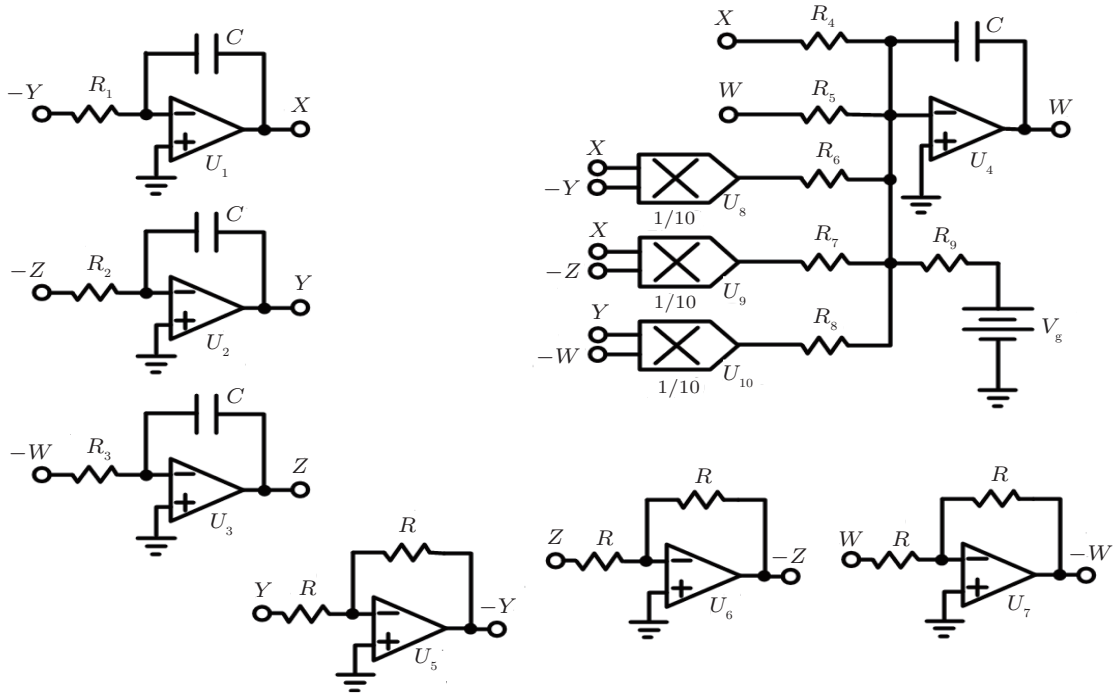


Fig. 7. The designed circuit of hyperjerk system (2) including electronic components.

$$\dot{X} = \frac{1}{R_1 C} Y, \tag{7a}$$

$$\dot{Y} = \frac{1}{R_2 C} Z, \tag{7b}$$

$$\dot{Z} = \frac{1}{R_3 C} W, \tag{7c}$$

$$\dot{W} = -\frac{1}{R_4 C} X - \frac{1}{R_5 C} W + \frac{1}{10 R_6 C} XY + \frac{1}{10 R_7 C} XZ + \frac{1}{10 R_8 C} YW - \frac{1}{R_9 C} V_{\mathrm{g}}, \tag{7d}$$

where X, Y, Z, and W are the voltages at the output of the operational amplifiers U_1, U_2, U_3, and U_4 respectively.

Obviously, the derived circuit equations correspond to the hyperjerk system in Eq. (2). As a result, by comparing the circuit equations in Eq. (7) and the hyperjerk system in Eq. (2), we have selected the values of the electronic components as follows: $R_1 = R_2 = R_3 = R_9 = R = 100$ kΩ, $R_4 = 36.63$ kΩ, $R_5 = 69.93$ kΩ, $R_6 = 4.587$ kΩ, $R_7 = 1.637$ kΩ, $R_8 = 3.774$ kΩ, $C = 2.2$ nF, and $V_{\mathrm{g}} = 1.74 V_{\mathrm{DC}}$. The chaotic attractors of the circuit by using PSpice are shown in the following Fig. 8. These results match with the results provided in Fig. 3(a).

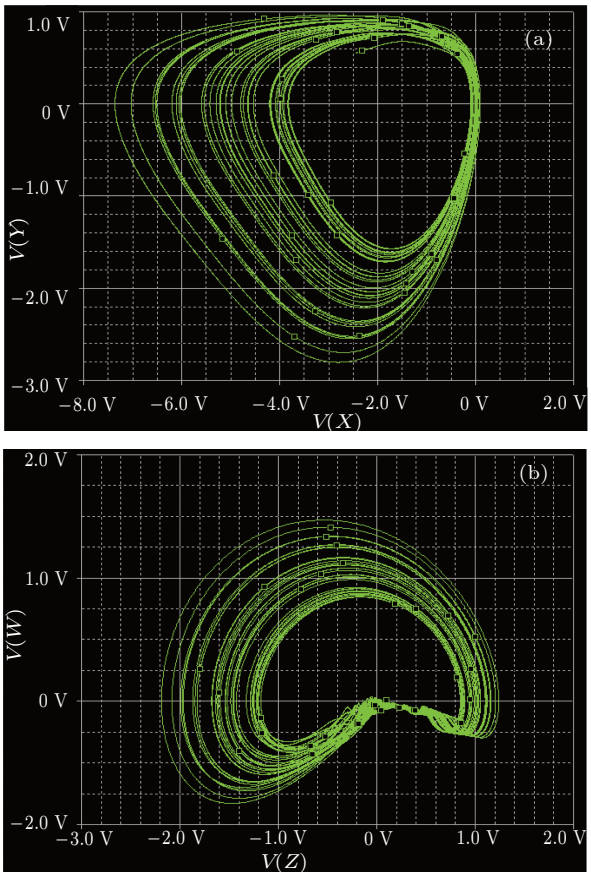


Fig. 8. (color online) Chaotic attractors of the circuit by PSpice: (a) X–Y plane and (b) Z–W plane.

5. The synchronization of two identical new hyperjerk systems by an adaptive integrator backstepping controller

This section describes the complete synchronization between two new identical hyperjerk chaotic systems by an adaptive integrator backstepping controller.

Using the system in Eq. (2), the dynamics of the master and slave systems are described in Eqs. (8) and (9), respectively.

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_3 = x_3, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -a x_1 - 1.43 x_4 + c x_1 x_2 + d x_1 x_3 + 2.65 x_2 x_4 - g, \end{cases} \tag{8}$$

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_3 = y_3, \\ \dot{y}_3 = y_4, \\ \dot{y}_4 = -a y_1 - 1.43 y_4 + c y_1 y_2 + d y_1 y_3 \\ \qquad + 2.65 y_2 y_4 - g + u, \end{cases} \tag{9}$$

where u is the control input added in the slave system. The complete synchronization error dynamics between the master and slave systems is given in Eq. (10).

$$\begin{cases} \dot{e}_1 = \dot{y}_1 - \dot{x}_1 = e_2, \\ \dot{e}_2 = \dot{y}_2 - \dot{x}_2 = e_3, \\ \dot{e}_3 = \dot{y}_3 - \dot{x}_3 = e_4, \\ \dot{e}_4 = \dot{y}_4 - \dot{x}_4 = -a e_1 - 1.43 e_4 + c e_1 y_2 + c x_1 e_2 \\ \qquad + d e_1 y_3 + d x_1 e_3 + 2.65 e_2 y_4 + 2.65 x_2 e_4 + u. \end{cases} \tag{10}$$

The design of an adaptive integrator backstepping for the complete synchronization of the systems is achieved by using the following steps.

Step 1 Suppose the desired error for the second state error (e_2) is chosen as

$$e_{2\mathrm{des}} = \alpha_1 = -k_1 e_1, \tag{11}$$

where $k_1 > 0$. Then, the error dynamics $\dot{e}_1$ contracts and stabilizes to the origin. Therefore, the difference between the actual and desired errors for e_2 can be written as

$$z_1 = e_2 - \alpha_1 = e_2 + k_1 e_1. \tag{12}$$

Using Eq. (10) we can write as

$$\dot{e}_1 = z_1 - k_1 e_1. \tag{13}$$

And also the dynamics for z_1 can be written as

$$\dot{z}_1 = \dot{e}_2 - \dot{\alpha}_1 = \dot{e}_2 - \frac{\partial \alpha_1}{\partial e_1} \dot{e}_1 = e_3 + k_1 \dot{e}_1. \tag{14}$$

Step 2 The stabilization of error dynamics $\dot{e}_1$ and $\dot{z}_1$ is achieved by considering a desired variable $\alpha_2 = e_{3\mathrm{des}}$ as

$$\alpha_2 = -k_1 z_1 + k_1^2 e_1 - e_1 - k_2 z_1, \tag{15}$$

where $k_2 > 0$. Thus, a difference of actual and desired for the error e_3 can be written as

$$z_2 = e_3 - \alpha_2 = e_3 - (-k_2 z_1 + k_1^2 e_1 - e_1 - k_2 z_1). \tag{16}$$

Using Eq. (16), the dynamics for z_1 is rewritten as

$$\dot{z}_1 = z_2 - k_2 z_1 - e_1. \tag{17}$$

Next is the computation of $\dot{z}_2$

$$\dot{z}_2 = \dot{e}_3 - \frac{\partial \alpha_2}{\partial e_1}\dot{e}_1 - \frac{\partial \alpha_2}{\partial z_1}\dot{z}_1. \tag{18}$$

Step 3 Here the stabilization of error dynamics $\dot{e}_1$, $\dot{z}_1$, and $\dot{z}_2$ is achieved by considering a desired variable $\alpha_3 = e_{4\text{des}}$ as

$$\alpha_3 = -k_3 z_2 - z_1 + \left(k_1^2 - 1\right)\dot{e}_1 - (-k_1 - k_2)\dot{z}_1, \tag{19}$$

where $k_3 > 0$. Thus, a difference of the actual and desired value for the error e_4 can be defined as

$$\begin{aligned} z_3 &= e_4 - \alpha_3 \\ &= e_4 - (-k_3 z_2 - z_1 + \left(k_1^2 - 1\right)\dot{e}_1 - (-k_1 - k_2)\dot{z}_1). \end{aligned} \tag{20}$$

Using Eq. (20), equation (18) can be written as

$$\dot{z}_2 = z_3 - k_3 z_2 - z_1. \tag{21}$$

Then the dynamics for z_3 can be written as

$$\dot{z}_3 = \dot{e}_4 - \frac{\partial \alpha_3}{\partial e_1}\dot{e}_1 - \frac{\partial \alpha_3}{\partial z_1}\dot{z}_1 - \frac{\partial \alpha_3}{\partial z_2}\dot{z}_2. \tag{22}$$

Using the fourth state variable of Eq. (10), equation (22) can be written as

$$\begin{aligned} \dot{z}_3 = &-ae_1 - 1.43e_4 + ce_1y_2 + cx_1e_2 + de_1y_3 + dx_1e_3 \\ &+ 2.65e_2y_4 + 2.65x_2e_4 + u - \frac{\partial \alpha_3}{\partial e_1}\dot{e}_1 - \frac{\partial \alpha_3}{\partial z_1}\dot{z}_1 \\ &- \frac{\partial \alpha_3}{\partial z_2}\dot{z}_2. \end{aligned} \tag{23}$$

Step 4 Now, the stabilization of $\dot{e}_1$, $\dot{z}_1$, $\dot{z}_2$, and $\dot{z}_3$ is achieved by considering the control input u as

$$\begin{aligned} u = &-(-ae_1 - 1.43e_4 + ce_1y_2 + cx_1e_2 + de_1y_3 + dx_1e_3 \\ &+ 2.65e_2y_4 + 2.65x_2e_4) - \left(-\frac{\partial \alpha_3}{\partial e_1}\dot{e}_1 - \frac{\partial \alpha_3}{\partial z_1}\dot{z}_1 - \frac{\partial \alpha_3}{\partial z_2}\dot{z}_2\right) \\ &- k_4 z_3 - z_2, \end{aligned} \tag{24}$$

where $k_4 > 0$. Using the control input in Eq. (24), the dynamics $\dot{z}_3$ in Eq. (23) can be written as

$$\dot{z}_3 = -k_4 z_3 - z_2. \tag{25}$$

The overall dynamics of the errors in Eq. (10) in the transformed domain is described as

$$\begin{cases} \dot{e}_1 = z_1 - k_1 e_1, \\ \dot{z}_1 = z_2 - k_2 z_1 - e_1, \\ \dot{z}_2 = z_3 - k_3 z_2 - z_1, \\ \dot{z}_3 = -k_4 z_3 - z_2. \end{cases} \tag{26}$$

The stability of Eq. (26) can be shown by considering a Lyapunov function candidate as

$$V_1(e_1, z_1, z_2, z_3) = \frac{1}{2}(e_1^2 + z_1^2 + z_2^2 + z_3^2).$$

The time derivative of V_1 can be written as

$$\dot{V}_1(e_1, z_1, z_2, z_3) = -k_1 e_1^2 - k_2 z_1^2 - k_3 z_2^2 - k_4 z_3^2, \tag{27}$$

where k_1, k_2, k_3, and k_4 are positive constants and are user-dependent. Hence, the time derivative of V_1 in Eq. (27) is a negative-definite function. Therefore, the error dynamics asymptotically converges to the origin and the states of the master system (8) follow the slaves system (9).

Now, suppose the parameters a, c, and d are unknown to the user and are estimated online, then the control input in Eq. (24) is modified as

$$\begin{aligned} u = &-(-\hat{a}e_1 - 1.43e_4 + \hat{c}(e_1y_2 + x_1e_2) + \hat{d}(e_1y_3 + x_1e_3) \\ &+ 2.65e_2y_4 + 2.65x_2e_4) - \left(-\frac{\partial \alpha_3}{\partial e_1}\dot{e}_1 - \frac{\partial \alpha_3}{\partial z_1}\dot{z}_1 - \frac{\partial \alpha_3}{\partial z_2}\dot{z}_2\right) \\ &- k_4 z_3 - z_2, \end{aligned} \tag{28}$$

where $\hat{a}$, $\hat{c}$, and $\hat{d}$ are the estimates of a, c, and d parameters, respectively. The adaptation of these parameters is achieved using the following rules:

$$\begin{cases} \dot{\hat{a}} = -e_1 + k_5 e_a, \\ \dot{\hat{c}} = -y_2 e_1 - x_1 e_2 + k_6 e_c, \\ \dot{\hat{d}} = y_1 e_1 + x_1 e_3 + k_7 e_d, \end{cases} \tag{29}$$

where k_5, k_6, and k_7 are positive constants and e_a, e_c, and e_d are defined as $e_a = a - \hat{a}$, $e_c = c - \hat{c}$, and $e_d = d - \hat{d}$, respectively.

Theorem 1 Suppose the slave system (9) is controlled by Eq. (28) along with the adaptation laws given in Eq. (29), then the synchronization error defined in dynamics (26) converges to zero asymptotically.

Proof Consider a Lyapunov function candidate as

$$V_2(e_1, z_1, z_2, z_3, e_a, e_c, e_d) = \frac{1}{2}(e_1^2 + z_1^2 + z_2^2 + z_3^2 + e_a^2 + e_c^2 + e_d^2),$$

whose time derivative can be written as

$$\begin{aligned} \dot{V}_2(e_1, z_1, z_2, z_3, e_a, e_c, e_d) = &\, e_1\dot{e}_1 + z_1\dot{z}_1 + z_2\dot{z}_2 + z_3\dot{z}_3 + e_a\dot{e}_a \\ &+ e_c\dot{e}_c + e_d\dot{e}_d. \end{aligned} \tag{30}$$

Now, using Eqs. (26) and (29), equation (30) can be written

$$\begin{aligned} \dot{V}_2(e_1, z_1, z_2, z_3, e_a, e_c, e_d) = &-k_1 e_1^2 - k_2 z_1^2 - k_3 z_2^2 - k_4 z_3^2 - k_5 e_a^2 \\ &- k_6 e_c^2 - k_7 e_d^2. \end{aligned} \tag{31}$$

It can be seen that the Lyapunov function V_2 whose time derivative is given in Eq. (31) is a negative-definite function. Thus, the state of the slave system (9) follows the master system.

6. Results and discussion for the complete synchronisation of the new hyperjerk systems

This section describes the complete synchronization results between the master and the slave systems. The initial conditions considered for the master and slave systems are $x(0) = (0,0,-2,0)^{\mathrm{T}}$ and $y(0) = (0.5,0.5,-2.1,0.5)^{\mathrm{T}}$, respectively, and $\hat{a}(0) = 0, \hat{c}(0) = 0$, and $\hat{d}(0) = 0$ for the adaptation laws. The original parameters of the system are $a = 2.73$, $b = 1.43$, $c = 2.18$, $d = 6.11$, $f = 2.65$, and $g = 1.74$. The designed parameters considered for the synchronization are $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$, $k_5 = 1$, $k_6 = 1$, and $k_7 = 1$.

The synchronization of the states of the master and the slave systems is shown in Fig. 9. The control input is shown in Fig. 10. The estimation of the parameters a, c, and d is shown in Fig. 11. It is apparent from Fig. 9 that the synchronization is achieved before $t < 5$ time unit for three states, and for the first state, it is approximately the $t < 10$ time unit. It is obvious from Fig. 11 that the estimation of the parameters is achieved properly to their respective values.

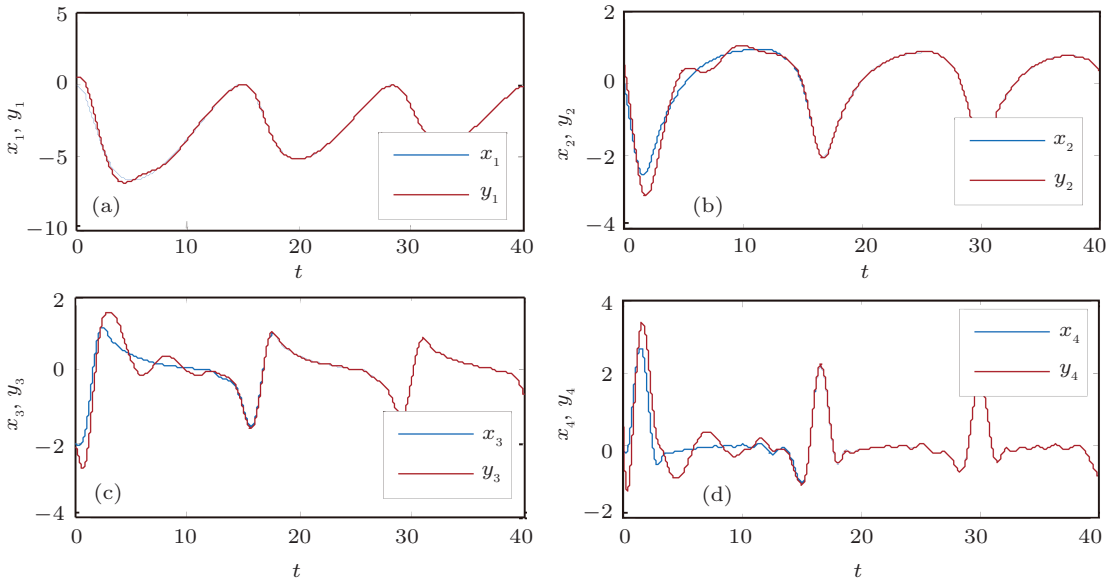


Fig. 9. (color online) Complete synchronized states of the master and the slave systems.

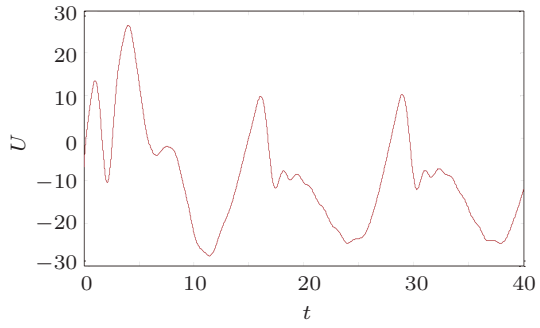


Fig. 10. (color online) Control input used for the synchronization between the master and the slave systems.

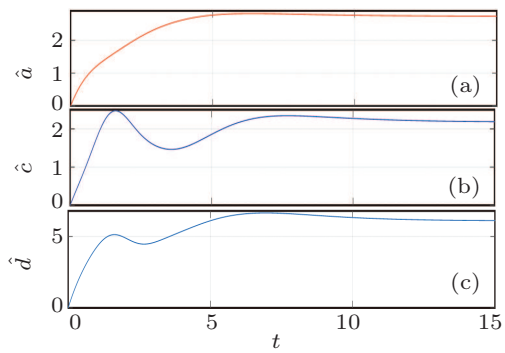


Fig. 11. (color online) The estimation of the parameters a, b, and c of the master and the slave systems.

7. Conclusions

This paper reports a new 4D hyperjerk chaotic system having a stable equilibrium point. Thus, the proposed system belongs to the category of hidden attractor chaotic systems. The proposed system shows the coexistence of different attractors and various dynamical behaviors like chaotic, periodic, and convergence to stable equilibrium. A circuit is designed to implement the dynamics of the new system and show its chaotic behavior. The application of the proposed system is shown by designing a complete synchronization strategy. An adaptive backstepping controller is designed for the synchronization of two identical proposed systems. The simulation results validate the complete synchronization strategy.

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