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The stabilizer for n-qubit symmetric states*

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The stabilizer group for an n-qubit state $|\phi\rangle$ is the set of all invertible local operators (ILO) $g = g_1 \otimes g_2 \otimes \cdots \otimes g_n$, $g_i \in \mathscr{GL}(2,\mathbb{C})$ such that $|\phi\rangle = g|\phi\rangle$. Recently, Gour *et al.* [Gour G, Kraus B and Wallach N R 2017 *J. Math. Phys.* **58** 092204] presented that almost all n-qubit states $|\psi\rangle$ own a trivial stabilizer group when $n \geq 5$. In this article, we consider the case when the stabilizer group of an n-qubit symmetric pure state $|\psi\rangle$ is trivial. First we show that the stabilizer group for an n-qubit symmetric pure state $|\phi\rangle$ is nontrivial when $n \leq 4$. Then we present a class of n-qubit symmetric states $|\phi\rangle$ with a trivial stabilizer group when $n \geq 5$. Finally, we propose a conjecture and prove that an n-qubit symmetric pure state owns a trivial stabilizer group when its diversity number is bigger than 5 under the conjecture we make, which confirms the main result of Gour *et al.* partly.

Keywords: symmetric states, stabilizer group

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1. Introduction

Quantum entanglement[1] is a valuable resource for a variety of tasks that cannot be finished by classical resource. Among the most popular tasks are quantum teleportation[2] and quantum superdense coding.[3] Due to the importance of quantum entanglement, the classification of quantum entanglement states is a big issue for the quantum information theory.

Entanglement theory is a resource theory with its free transformation is local operations and classical communication (LOCC). As LOCC is hard to deal with mathematically, and with the number of the parties of the quantum systems growing, the classification of all entanglement states under the LOCC restriction becomes very hard. A conventional way is to consider other operations, such as stochastic LOCC (SLOCC), local unitary operations (LU), and separable operations (SEP).

Two n-partite states $|\phi\rangle$ and $|\psi\rangle$ are SLOCC equivalent[4] if and only if there exists n invertible local operations (ILO) $A_i, i = 1,2,\ldots,n$ such that

$$|\phi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\psi\rangle. \tag{1}$$

The classification for multi-qubit pure states under SLOCC is attracting a great deal of attention.[5–10] However, there are an uncountable number of SLOCC inequivalent classes in n-qubit systems when $n \geq 4$, so it is a formidable task to classify multipartite states under SLOCC.

SEP is simple to describe mathematically and contains LOCC strictly, as pure state transformations exist belonging to SEP, but they cannot be achieved by LOCC. The authors in Ref. [11] presented that the existence of transformations under separable operations between two pure states depends largely on the stabilizer of the state. Recently, Gour *et al.* showed that almost all of the stabilizer group for 5 or more qubits pure states contains only the identity.[12] The authors in Ref. [13] generalized this result to n-qudit systems when $n > 3, d > 2$.

Symmetric states belong to the space that is spanned by the pure states invariant under particle exchange, and there are some results done on the classifications under SLOCC limited to symmetric states.[14–18] The authors in Ref. [15] proved if $|\psi\rangle$ and $|\phi\rangle$ are n-qubit symmetric pure states, and n invertible operations $A_i, i = 1,2,\ldots,n$ exist such that $|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\phi\rangle$, then there exists an invertible matrix A such that

$$|\psi\rangle = A^{\otimes n} |\phi\rangle. \tag{2}$$

Moreover, Migdal *et al.*[16] generalized the results from qubit systems to qudit systems.

In this article, we consider the problem on the stabilizer groups for n-qubit symmetric states. This article is organized as follows. In Section 2, we present preliminary knowledge on n-qubit symmetric pure states. In Section 3, we present our main results. First, we present the stabilizer group for an n-qubit symmetric state is nontrivial when $n \leq 4$, then we present

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a class of n-qubit symmetric states whose stabilizer group is trivial $n \geq 5$, at last, based on the Majorana representation for symmetric states, we apply the Möbius transformation to present when the diversity number m of a pure symmetric state $|\psi\rangle$ is 5 or 6, it owns a trivial stabilizer group, when $m = 2$, the pure state $|\psi\rangle$ owns a nontrivial stabilizer group, 5 when $m = 4$, there exists only one case when $|\psi\rangle$ owns a nontrivial stabilizer group, when $m \geq 7$, under a conjecture we make, a symmetric pure state $|\psi\rangle$ owns a trivial stabilizer group. In Section 4, we will end with a summary.

2. Preliminary knowledge

In this section, we will first recall the definition of symmetric states, and then we present the Majorana representation for an n-qubit symmetric pure state.

A pure state $|\psi\rangle \in \mathscr{H}_2$ can be represented by a point on a Bloch sphere geometrically as $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, here two parameters $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$.

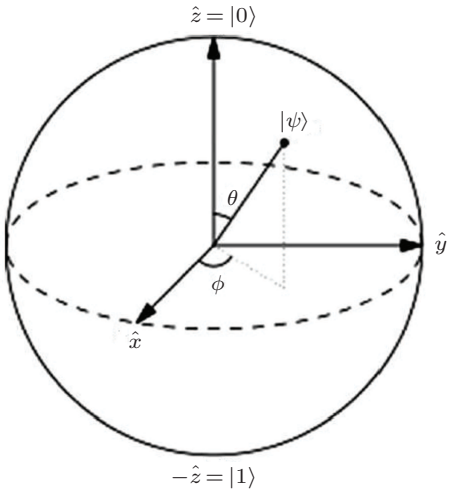


Fig. 1. Bloch sphere.

We call an n-partite pure state $|\psi\rangle$ symmetric state if it is invariant under permuting the particles. That is, for any permutation operator $P_\pi, P_\pi|\psi\rangle = |\psi\rangle$. Generally, there are two main characterizations for an n-qubit symmetric pure state $|\psi\rangle$, Majorana representation[19] and Dicke representation. The Majorana representation for an n-qubit symmetric pure state is that single particles $|\phi_i\rangle, i = 1,2,\ldots,n$, exist such that

$$|\psi\rangle = \frac{e^{i\theta}}{\sqrt{K}} \sum_{\sigma\in \text{perm}} |\phi_{\sigma(1)}\rangle|\phi_{\sigma(2)}\rangle\cdots|\phi_{\sigma(n)}\rangle, \tag{3}$$

where the sum runs over all distinct permutations σ, K is a normalization prefactor, and the $|\phi_i\rangle$ are single qubit states $|\phi_i\rangle = \cos(\theta_i/2)|0\rangle + e^{i\phi_i}\sin(\theta_i/2)|1\rangle, i = 1,2,\ldots n$, we would denote $|\phi_i\rangle$ as $|\phi_i\rangle = a_i|0\rangle + b_i|1\rangle$ below. An n-qubit symmetric pure state $|\psi\rangle$ can also be characterized as the sum of the Dicke states $|D(n,k)\rangle$, that is,

$$|\psi\rangle = \sum x_k |D(n,k)\rangle. \tag{4}$$

Here the Dicke states $|D(n,k)\rangle$ are defined as

$$|D(n,k)\rangle = \frac{1}{\sqrt{C_n^k}} \sum |\underbrace{0\cdots0}_{n-k}\underbrace{1\cdots1}_{k}\rangle, \tag{5}$$

where the sum runs over all the permutations of the qubits. Up to a global phase factor, the parameters a_i/b_i in a pure state $|\phi_i\rangle = a_i|0\rangle + b_i|1\rangle$ are the roots of the polynomial $P(z) = \sum(-1)^k x_k\sqrt{C_n^k}z^k$, here C_n^k denotes the binomial coefficient of n and k.

Next we introduce an isometric linear map

$$\mathscr{M} : \mathscr{L}(\mathscr{H}_1,\mathscr{H}_2) \longrightarrow \mathscr{H}_1 \otimes \mathscr{H}_2, \tag{6}$$

$$|b\rangle\langle a| \longrightarrow |b\rangle|a\rangle, \tag{7}$$

here we denote that $\mathscr{L}(\mathscr{H}_1,\mathscr{H}_2)$ is the set of all linear maps from the Hilbert space $\mathscr{H}_1$ to the Hilbert space $\mathscr{H}_2$. This map is useful for considering the stabilizer group for a 2-qubit symmetric state. Now we introduce some properties of this map:

(i) Assume $A \in \mathscr{L}(\mathscr{H}_1), B \in \mathscr{L}(\mathscr{H}_1)$, we have that

$$A \otimes B\mathscr{M}(X) = \mathscr{M}(AXB^{\mathrm{T}}). \tag{8}$$

(ii) Assume $|\psi\rangle \in \mathscr{H}_1, |\phi\rangle \in \mathscr{H}_2$, then

$$\mathscr{M}(|\phi\rangle\langle\psi|) = |\phi\rangle|\bar{\psi}\rangle. \tag{9}$$

Then we recall the definition and some important properties of the Möbius transformation, which is useful for the last part of this article. The Möbius transformation is defined on the extended complex plane onto itself,[20] it can be represented as

$$f(z) = \frac{az+b}{cz+d}, \tag{10}$$

with $a,b,c,d \in \mathbb{C}, ad-bc \neq 0$. From the above equality, we see that when $c \neq 0$, this function $f : \mathbb{C} - \{-d/c\} \to \mathbb{C} - \{a/c\}$, $f(-d/c) = \infty$, $f(\infty) = a/c$, when $c = 0$, this function $f : \mathbb{C} \to \mathbb{C}$, $f(\infty) = \infty$. The Möbius transformation owns the following properties:

i) Möbius transformation maps circles to circles.

ii) Möbius transformations are conformal.

iii) If two points are symmetric with respect to a circle, then their images under a Möbius transformation are symmetric with respect to the image circle. This is called the "Symmetry Principle".

iv) With the exception of the identity mapping, a Möbius transformation has at most two fixed points.

v) There exists a unique Möbius transformation sending any three points to any other three points.

vi) The unique Möbius transformation $z \to M(z)$ sending three points q,r,s to any other three points q',r',s' is given by

$$\frac{(M(z)-q')(r'-s')}{(M(z)-s')(r'-q')} = \frac{(z-q)(r-s)}{(z-s)(r-q)}.$$

vii) The Möbius transformation forms a group, Möbius transformation is isotropic to the projective linear group $PSL(2,\mathbb{C}) \cong SL(2,\mathbb{C})/\{I,-I\}$.

As we know, the stereographic projection is a mapping that projects a sphere onto a plane. This projection is defined on the whole sphere except a point, and this map is smooth and bijective. It is conformal, i.e., it preserves the angles at where curves meet. By transforming the majorana points of a pure state $|\psi\rangle$ to an extended complex plane, we may get the following proposition.[21]

Lemma 1 Assume $|\psi_1\rangle, |\psi_2\rangle$ are two pure symmetric states, if $|\psi_1\rangle$ and $|\psi_2\rangle$ are SLOCC-equivalent if there exists a Möbius transformation (10) between their Majorana points.

At last, we recall two parameters defined in Ref. [14], the diversity degree and degeneracy configuration of an n-qubit symmetric pure state. Both parameters can be used to identify the SLOCC entanglement classes of all n-qubit symmetric pure states. Assume $|\psi\rangle$ is an n-qubit symmetric pure state,

$$|\psi\rangle = \frac{e^{i\theta}}{\sqrt{K}} \sum_{\sigma\in perm} |\phi_{\sigma(1)}\rangle|\phi_{\sigma(2)}\rangle\cdots|\phi_{\sigma(n)}\rangle,$$

up to a global phase factor, two states $|\phi_i\rangle$ and $|\phi_j\rangle$ are identical if and only if $a_i b_j - a_j b_i = 0$, and we define their number as the degeneracy number. Then we define the degeneracy configuration $\{n_i\}$ of a symmetric state $|\psi\rangle$ as the list of its degeneracy numbers n_i ordered in decreasing order. We denote the number of the elements in the set $\{n_i\}$ as the diversity degree m of the symmetric state, it stands for the number of distinct $|\phi_i\rangle$ in Eq. (3). For example, for a 3-qubit GHZ state $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$, we have $|\phi_1\rangle = (|0\rangle + w|1\rangle)/2$, $|\phi_2\rangle = (|0\rangle + w^2|1\rangle)/2$, $|\phi_3\rangle = (|0\rangle + |1\rangle)/2$, w is roots of $P(z) = 1 - z^3$, the degeneracy number of $|GHZ\rangle$ is 3, and the degeneracy configuration of $|GHZ\rangle$ is $\{1,1,1\}$.

3. Main Results

First we present that the stabilizer group for a two-qubit symmetric pure state $|\psi\rangle \in \mathscr{H}_2 \otimes \mathscr{H}_2$ is nontrivial.

Theorem 1 Assume that $|\psi\rangle$ is a 2-qubit symmetric pure state, $|\psi\rangle = \sum_{i=0}^{2} c_i |D(2,i)\rangle$, then the stabilizer group for the state $|\psi\rangle$ is nontrivial.

Proof Assume $A \in \mathscr{L}(\mathscr{H}_2)$ and $B \in \mathscr{L}(\mathscr{H}_2)$ and $A \otimes B|\psi\rangle = |\psi\rangle$, then we have $\mathscr{M}^{-1}(|\psi\rangle) = \mathscr{M}^{-1}(A \otimes B|\psi\rangle)$, according to the equality (8), we have that

$$AX = X(B^{\mathrm{T}})^{-1}. \tag{11}$$

here we assume that

$$X = \begin{pmatrix} c_0 & c_1 \\ c_1 & c_2 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B^{\mathrm{T}} = \begin{pmatrix} h & -f \\ -g & e \end{pmatrix}$$

and $\det(B) = 1$, then we can obtain the following equation set:

$$\begin{cases} c_0(e-a) = c_1(b-g), \\ c_0 f - c_2 b = c_1(a-h), \\ c_0 c - c_2 g = c_1(d-e), \\ c_1(f-c) = c_2(d-h). \end{cases} \tag{12}$$

As there are four equations and eight variables, then we know that the rank of the solution vectors is more than 1, that is, the stabilizer group for the state $|\psi\rangle$ contains more than the identity. □

Now we present a lemma to show an n-qubit symmetric pure state owns a nontrivial stabilizer group when $n = 3,4$.

Lemma 2 Assume $|\psi_1\rangle$ and $|\psi_2\rangle$ are symmetric pure states, and $|\psi_1\rangle$ is SLOCC equivalent to $|\psi_2\rangle$, if there exists a nontrivial ILO g such that $g^{\otimes n}|\psi_1\rangle = |\psi_1\rangle$, then the stabilizer group for $|\psi_2\rangle$ is nontrivial.

Proof As $|\psi_1\rangle$ and $|\psi_2\rangle$ are symmetric states, then there exists an ILO h such that $|\psi_2\rangle = h^{\otimes n}|\psi_1\rangle$. $|\psi_2\rangle = h^{\otimes n} g^{\otimes n} (h^{-1})^{\otimes n} |\psi_2\rangle$, that is, the stabilizer group for $|\psi_2\rangle$ is nontrivial. □

In Ref. [14], the authors presented that a three-qubit symmetric pure state is SLOCC equivalent to $|W\rangle$ or $|GHZ\rangle$. As we know, when we choose

$$g = \begin{pmatrix} \exp(\mathrm{i}\pi/4) & 0 \\ 0 & \exp(-\mathrm{i}\pi/2) \end{pmatrix},$$

$g^{\otimes 3}|W\rangle = |W\rangle$, $\sigma_z^{\otimes 3}|GHZ\rangle = |GHZ\rangle$. From Lemma 2, we see that the stabilizer group for all three-qubit pure symmetric states is nontrivial. A four-qubit symmetric state is SLOCC equivalent to one of the elements in

$$S = \left\{ |D(4,0)\rangle, |D(4,1)\rangle, |D(4,2)\rangle, \frac{|D(4,2)\rangle + |D(4,0)\rangle}{2}, \frac{|GHZ\rangle + \mu|D(4,2)\rangle}{\sqrt{1+\mu^2}} \right\},$$

then we present a nontrivial stabilizer for the elements in the set S, for the state $|D(4,0)\rangle$, we have $\sigma_z^{\otimes 4}|D(4,0)\rangle = |D(4,0)\rangle$, for the state $|D(4,1)\rangle$, we choose an ILO

$$g = \begin{pmatrix} \exp(\pi\mathrm{i}/6) & 0 \\ 0 & \exp(3\pi\mathrm{i}/2) \end{pmatrix},$$

for the state $|D(4,2)\rangle$, we can choose an ILO $g = \sigma_z$, and for the last element in the set S, we can also choose an ILO $g = \sigma_z$, then due to the Lemma 2, we have the stabilizer group for all four-qubit symmetric pure states is nontrivial. Note that for four-qubit pure states, the authors in Ref. [18] also proposed similar results.

Here we denote

$$\begin{aligned} G &\equiv SL(2,\mathbb{C})\otimes SL(2,\mathbb{C})\otimes\cdots\otimes SL(2,\mathbb{C})\\ K &\equiv SU(2)\otimes SU(2)\otimes\cdots\otimes SU(2)\\ \tilde{G} &\equiv GL(2,\mathbb{C})\otimes GL(2,\mathbb{C})\otimes\cdots\otimes GL(2,\mathbb{C})\\ \tilde{K} &\equiv U(2)\otimes U(2)\otimes\cdots\otimes U(2). \end{aligned} \tag{13}$$

Next we will use the method proposed in Ref. [12] to give a class of symmetric states with its stabilizer group containing only the identity. First we introduce the definition of *SL*-invariant polynomials. A polynomial $f:\mathscr{H}\longrightarrow\mathbb{C}$ is *SL*-invariant if $f(g|\psi\rangle)=f(|\psi\rangle)$, $\forall g\in SL(2,\mathbb{C})\otimes SL(2,\mathbb{C})\cdots\otimes SL(2,\mathbb{C})$, here we denote $\mathscr{H}=\mathscr{H}_2\otimes\mathscr{H}_2\otimes\cdots\otimes\mathscr{H}_2$ and

$$SL(2,\mathbb{C})=\left\{A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}|a_{ij}\in\mathscr{C},i,j=1,2,\det(A)=1\right\}.$$

f_2 is an *SL*-invariant polynomial with degree 2, which is defined as $f_2(|\psi\rangle)=(|\psi\rangle,|\psi\rangle)=\langle\psi^*|\sigma_y^{\otimes n}|\psi\rangle$, here σ_y is the Pauli operator with its matrix representation

$$\begin{pmatrix} 0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}.$$

Due to the property of σ_y, we have that when n is odd, $f_2(\cdot)=0$. Another polynomial, $f_4(|\psi\rangle)$, is a polynomial with degree 4, it is defined as

$$f_4(|\psi\rangle)=\det\begin{pmatrix}(|\psi_0\rangle,|\psi_0\rangle) & (|\psi_0\rangle,|\psi_1\rangle)\\ (|\psi_1\rangle,|\psi_0\rangle) & (|\psi_1\rangle,|\psi_1\rangle)\end{pmatrix},$$

here we assume that $|\psi\rangle=|0\rangle|\psi_0\rangle+|1\rangle|\psi_1\rangle$. Below we denote the stabilizer group for a pure state $|\psi\rangle$ as

$$\tilde{G}_\psi=\{g_1\otimes g_2\otimes\cdots\otimes g_n\in\tilde{G}|g_1\otimes g_2\otimes\cdots\otimes g_n|\psi\rangle=|\psi\rangle\}.$$

Example 1 Assume $|\psi\rangle$ is an n-partite symmetric pure state $(n\geq 5)$, $|\psi\rangle=x_k|D(n,k)\rangle+x_l|D(n,l)\rangle+x_n|D(n,n)\rangle$, $k+l=n+1,k,\ l\neq n/2$, $gcd(k,l)=1$, here we denote that $gcd(k,l)$ is the greatest common divisor of k and l, then $\tilde{G}_\psi=\{I\}$.

Proof First we denote $g_2\otimes g_3\otimes\cdots\otimes g_n$ as h and let

$$g_1=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix},$$

then

$$\begin{aligned} g_1\otimes h|\psi\rangle &= |0\rangle h\big[a_1(x_k|D(n-1,k)\rangle+x_l|D(n-1,l)\rangle)+b_1(x_k|D(n-1,k-1)\rangle+x_l|D(n-1,l-1)\rangle\\ &\quad +x_n|D(n-1,n-1)\rangle)\big]+|1\rangle h\big[c_1(x_k|D(n-1,k)\rangle+x_l|D(n-1,l)\rangle)\\ &\quad +d_1(x_k|D(n-1,k-1)\rangle+x_l|D(n-1,l-1)\rangle+x_n|D(n-1,n-1)\rangle)\big]\\ &= |0\rangle(x_k|D(n-1,k)\rangle+x_l|D(n-1,l)\rangle)+|1\rangle(x_k|D(n-1,k-1)\rangle+x_l|D(n-1,l-1)\rangle+x_n|D(n-1,n-1)\rangle). \end{aligned} \tag{14}$$

Apply $\langle 0|\otimes I^{\otimes(n-1)}$ to the left-hand side (LHS) and the right-hand side (RHS) of the above equality, and we denote that $|\zeta_1\rangle=x_k|D(n-1,k)\rangle+x_l|D(n-1,l)\rangle$, $|\zeta_2\rangle=x_k|D(n-1,k-1)\rangle+x_l|D(n-1,l-1)\rangle+x_n|D(n-1,n-1)\rangle$, then

$$h[a_1|\zeta_1\rangle+b_1|\zeta_2\rangle]=x_k|D(n-1,k)\rangle+x_l|D(n-1,l)\rangle. \tag{15}$$

Assume n is odd, then by using $f_2(\cdot)$ to the equality (15), we have $2b_1^2x_kx_l=0$, $b_1=0$. As $|\psi\rangle$ is a symmetric state, we may assume

$$g_i=\begin{pmatrix}a_i & 0\\ c_i & d_i\end{pmatrix},\ \ i=2,\ldots,n,$$

due to $g_1\otimes g_2\otimes\cdots\otimes g_n|\psi\rangle=|\psi\rangle$ and through simple computation, we have $c_i=0$. Applying

$$g_i=\begin{pmatrix}a_i & 0\\ 0 & d_i\end{pmatrix}$$

to the equality (14), we have $d_i=\lambda a_i$, $\prod d_i=1$, $\lambda^k=\lambda^l=1$, as $gcd(k,l)=1$, we have $\lambda=1$, that is $\tilde{G}_\psi=\{I\}$.

When n is even, we use $f_4(\cdot)$ to the equality (15), we have $2b_1^4x_k^2x_l^2=0$, $b_1=0$. From the same method as the case when n is odd, we have that $\tilde{G}_\psi=\{I\}$. □

Then we present a class of symmetric critical states $|\psi\rangle$ with $G_\psi=\{I\}$. First we present the definition of critical states and a meaningful characterization of critical states. The set of critical states is defined as:

$$\mathrm{Crit}(\mathscr{H}_n)=\{\psi|\langle\psi|X|\psi\rangle=0,X\in\mathrm{Lie}(G)\}. \tag{16}$$

Here Lie(G) is the Lie algebra of G. The critical set is valuable, as many important states in quantum information theory, such as the Bell states, GHZ states, cluster states, and graph states, belong to the set of critical states. Then we present a fundamental characterization of critical states.

Lemma 3 (The Kempf–Ness theorem)

(I) $\psi\in\mathrm{Crit}(\mathscr{H}_n)$ if and only if $||g\psi||\geq||\psi||$ for all $g\in G$, $||\cdot||$ denotes the Euclidean norm of $|\psi\rangle\in\mathscr{H}_n$.

(II) If $\psi\in\mathrm{Crit}(\mathscr{H}_n)$, then $||g\psi||\geq||\psi||$ with equality if and only if $g\psi\in K\psi$. Moreover, if g is positive-semidefinite then the equality condition holds if and only if $g\psi=\psi$.

(III) If $\psi\in\mathscr{H}_n$, then $G\psi$ is closed in $\mathscr{H}_n$ if and only if $G\psi\cap\mathrm{Crit}(\mathscr{H}_n)\neq\emptyset$.

Due to this lemma, Gour and Wallach in Ref. [22] showed that $\psi\in\mathscr{H}_n$ is critical if and only if all the local density matrices of $|\psi\rangle$ are proportional to the identity. By the lemma

in Ref. [12], for a class of pure states $|\psi\rangle \in \mathrm{Crit}(\mathscr{H}_n)$, if we could show the set $K_\psi = \{U_1 \otimes U_2 \otimes \cdots \otimes U_n \in K | U_1 \otimes U_2 \otimes \cdots \otimes U_n |\psi\rangle = |\psi\rangle\}$ contains only the identity, then $G_\psi = \{I\}$.

Example 2 Assume

$$|\psi\rangle = a_0|D(n,0)\rangle + a_{n-1}|D(n,n-1)\rangle$$

or

$$|\psi\rangle = a_1|D(n,1)\rangle + a_n|D(n,n)\rangle$$

is an n-qubit symmetric critical pure state with $n > 4$, then $G_\psi = \{I\}$.

Proof At first we will prove $K_\psi = \{I\}$. When $|\psi\rangle = a_1|D(n,1)\rangle + a_n|D(n,n)\rangle$, assume $U^{\otimes n} \in K$ satisfies $U^{\otimes n}|\psi\rangle = |\psi\rangle$, then

$$(U \otimes U)\rho_{1,2}(U \otimes U)^+ = \rho_{1,2}, \tag{17}$$

the equality (17) can be changed as

$$\begin{aligned}&(U\otimes U)\left[\frac{(n-2)|a_1|^2}{n}|00\rangle\langle 00| + \frac{|a_1|^2}{n}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|) + |a_n|^2|11\rangle\langle 11|\right](U\otimes U)^+\\ &= \frac{|a_1|^2(n-2)}{n}|00\rangle\langle 00| + \frac{|a_1|^2}{n}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|) + |a_n|^2|11\rangle\langle 11|. \end{aligned} \tag{18}$$

As the formula on the RHS of the equality above can be seen as a Schmidt decomposition, and $U \otimes U$ cannot increase the Schmidt rank, then $U|1\rangle = u_1|1\rangle$, as U is a unitary operator, then $U|0\rangle = u_0|0\rangle$. At last, due to $U^{\otimes n}|\psi\rangle = |\psi\rangle$, then we have $u_0^{n-1}u_1 = 1$, $u_1^n = 1$, i.e., $u_0 = u_1$. The other case is similar. □

Here we present another proof on $G_{L_n} = \{I\}$, $|L_n\rangle$ is defined in the equality (8) of the article.[12] Note that the examples above tell us $G_\psi = \{I\}$, however, $\tilde{G}_\psi \supset \{I\}$. It seems that this result is simple, however, this method is very useful to present nontrivial examples of states in n-qudit systems with nontrivial stabilizer groups.[13]

Finally, I would like to apply Möbius transformation to show when the diversity number m of an n-qubit symmetric pure state $|\psi\rangle$ is 5 or 6, the stabilizer group of $|\psi\rangle$ is trivial, when $m \geq 7$, under a conjecture we make, the stabilizer group of $|\psi\rangle$ is trivial.

Assume a pure symmetric state $|\psi\rangle$ can be represented in terms of Majorana representation:

$$|\psi\rangle = \frac{e^{i\alpha}}{\sqrt{K}} \sum_{\sigma\in \mathrm{perm}} |\phi_{\sigma(1)}\rangle|\phi_{\sigma(2)}\rangle \cdots |\phi_{\sigma(n)}\rangle, \tag{19}$$

where the sum takes over all the permutations and K is the normalization for the state $|\psi\rangle$. Due to the main results proposed by Mathonet *et al.*,[15] we see that if $|\psi\rangle = g_1 \otimes g_2 \otimes \cdots \otimes g_n|\psi\rangle$, then there exists an ILO g such that $|\psi\rangle = g^{\otimes n}|\psi\rangle$, i.e., if we could prove $\{g|g^{\otimes n}|\psi\rangle = |\psi\rangle\} = \{I\}$, then the stabilizer group of $|\psi\rangle$ is trivial. From Eq. (19),

$$g^{\otimes n}|\psi\rangle = |\psi\rangle, \tag{20}$$

$$\sum_{\sigma\in \mathrm{perm}} \bigotimes_i g|\phi_{\sigma(i)}\rangle = \sum_{\sigma\in \mathrm{perm}} \bigotimes_i |\phi_{\sigma(i)}\rangle. \tag{21}$$

Due to the uniqueness of the Majorana representation for a symmetric state and according to the equality (21), we see that there is a permutation σ such that

$$g|\phi_i\rangle = \lambda_i|\phi_{\sigma(i)}\rangle. \tag{22}$$

Assume the diversity number of a symmetric state $|\psi\rangle$ is m, the divergence configuration for the state $|\psi\rangle$ is $\{k_1, k_2, \ldots, k_m\}$ with $k_1 \geq k_2 \cdots \geq k_m$ and $\sum_i k_i = n$. Then from the Lemma 1 and the property iv) of Möbius transformation, we have:

Corollary 1 Assume $|\psi\rangle$ is an n-qubit symmetric state, its degeneracy configuration is $\{k_1, k_2, \ldots, k_m\}, m \geq 3$, if there exists k_i, k_j, and k_l such that these three values are unequal to the residual elements in the set, then the stabilizer group for the state $|\psi\rangle$ contains only the identity.

Finally, we talk about the stabilizer group for an n-qubit symmetric pure state $|\psi\rangle$ with the increase of the diversity number m of $|\psi\rangle$, when $m = 2$ the stabilizer group for $|\psi\rangle$ is nontrivial, when $m \geq 5$, under a conjecture we make, the stabilizer group for $|\psi\rangle$ is trivial.

$m = 1$ (separable states): When a symmetric state $|\psi\rangle$ is separable, then it can be represented as

$$|\psi\rangle = |\varepsilon\varepsilon\cdots\varepsilon\rangle, \tag{23}$$

it is easy to see when an ILO g satisfies that $|\varepsilon\rangle$ is an eignvector of g, $g^{\otimes n}$ is the stabilizer for the state $|\psi\rangle$.

$m = 2$: In this case, the state $|\psi\rangle$ can be represented as

$$|\psi\rangle = \frac{e^{i\theta}}{\sqrt{K}} \sum |\underbrace{\varepsilon_1\cdots\varepsilon_1}_{k_1}\underbrace{\varepsilon_2\cdots\varepsilon_2}_{n-k_1}\rangle, \tag{24}$$

where the sum takes over all the permutation of k_1 $|\varepsilon_1\rangle s$ and $n-k_1$ $|\varepsilon_2\rangle s$. In this case, when $k_1 \neq n-k_1$, first we denote a local operator $h = [|\varepsilon_1\rangle, |\varepsilon_2\rangle]^{-1}$, this means if $|\varepsilon_1\rangle = a_1|0\rangle + b_1|1\rangle$, $|\varepsilon_2\rangle = a_2|0\rangle + b_2|1\rangle$,

$$h = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}^{-1},$$

then g can be written as

$$g = h^{-1}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} h.$$

When $k_1 = n - k_1$, the ILO g can also be written as

$$g = h^{-1} \begin{pmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{pmatrix} h, \quad h = [|\varepsilon_1\rangle, |\varepsilon_2\rangle]^{-1}.$$

When $m \geq 3$, due to Lemma 3, by searching for a nontrivial Möbius transformation between the Majorana points of the pure state $|\psi\rangle$, we can see whether a pure state $|\psi\rangle$ owns a nontrivial stabilizer group.

When $m = 3$, assume the degeneracy configuration of a pure state $|\psi\rangle$ is $\{k_1, k_2, k_3\}$, due to the property v), we see that except when $k_1 \neq k_2 \neq k_3$, $|\psi\rangle$ owns a nontrivial stabilizer group.

Here we assume that each z_i is not ∞, as if there exists k such that $z_k = \infty$, we can always make z_k not be ∞ by Möbius transformation. When $m = 4$, assume the degeneracy configuration of the pure state $|\psi\rangle$ is $\{k_1, k_2, k_3, k_4\}$, when all the four numbers are different from each other or only two of them are equal, the stabilizer group for $|\psi\rangle$ is trivial. First we show when $k_1 = k_2$, $k_3 = k_4$, the stabilizer group is nontrivial, i.e., there exists a nontrivial Möbius transformation f that can permute $z_i, i = 1, 2, 3, 4$. Let $f(z_1) = z_2$, $f(z_2) = z_1$, $f(z_3) = z_4$, $f(z_4) = z_3$, from the property vi) of the Möbius transformation, we see f exists. When three of them are equal, we can always assume $k_1 = k_2 = k_3 \neq k_4$, then the Möbius transformation f should satisfy one of the following cases,

I) $f(z_1) = z_2$, $f(z_2) = z_3$, $f(z_3) = z_1$, $f(z_4) = z_4$;
II) $f(z_1) = z_2$, $f(z_2) = z_1$, $f(z_3) = z_3$, $f(z_4) = z_4$;
III) $f(z_1) = z_1$, $f(z_2) = z_3$, $f(z_3) = z_2$, $f(z_4) = z_4$;
IV) $f(z_1) = z_3$, $f(z_3) = z_1$, $f(z_2) = z_2$, $f(z_4) = z_4$.

Here we only consider the first two cases, the other two are similar to the second one. In case I), from property vi) of Möbius transformation f, we have

$$\begin{aligned} \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} &= \frac{(z_2 - z_3)(z_1 - z_4)}{(z_2 - z_4)(z_1 - z_3)} \\ &= \frac{(z_3 - z_1)(z_2 - z_4)}{(z_3 - z_4)(z_2 - z_1)}, \\ \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} &= \frac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)} \\ &= \frac{(z_3 - z_2)(z_1 - z_4)}{(z_3 - z_4)(z_1 - z_2)}, \end{aligned} \tag{25}$$

from the above equalities, we have

$$(z_1 - z_4)^2 (z_2 - z_3)^2 + (z_1 - z_2)(z_3 - z_4)(z_2 - z_4)(z_1 - z_3) = 0, \tag{26}$$

$$(z_1 - z_2)^2 (z_3 - z_4)^2 + (z_1 - z_4)(z_2 - z_3)(z_3 - z_1)(z_2 - z_4) = 0, \tag{27}$$

then from Eqs. (26) and (27), we have

$$(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_1 - z_3) = 0, \tag{28}$$

i.e., $z_1 = z_4$, $z_2 = z_3$, $z_4 = z_2$, or $z_3 = z_1$, however, each of these four conditions is invalid, that is, case I) is invalid. For case II), we may also from property vi) of Möbius transformation f have

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)}, \tag{29}$$

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)}, \tag{30}$$

from the above equalities, we have

$$(z_1 - z_3)(z_2 - z_4) - (z_2 - z_3)(z_1 - z_4) = 0, \tag{31}$$

$$(z_1 - z_2)(z_3 - z_4) = 0, \tag{32}$$

this is invalid. Then we see the above four conditions are invalid. When $k_1 = k_2 \neq k_3 \neq k_4$, similar to the above analyses, the stabilizer group for the pure states is trivial. Then we see that only when $k_1 = k_2$, $k_3 = k_4$ the stabilizer group is nontrivial.

Here we note that when $|\psi\rangle$ is a four-qubit symmetric pure state, $|\psi\rangle$ always owns a nontrivial stabilizer group. As the diversity configuration can only be $\{1,1,1,1\}$, $\{2,1,1\}$, $\{3,1\}$, or $\{4\}$, from the above analyses, we see that the stabilizer group is nontrivial.

When $m = 5$, assume the degeneracy configuration of the pure state $|\psi\rangle$ is $\{k_1, k_2, k_3, k_4, k_5\}$. First we analyze the case when $k_1 = k_2 = k_3 = k_4 = k_5$, if the stabilizer group is nontrivial, then there exists a nontrivial Möbius transformation f that can permute $z_i, i = 1, 2, 3, 4, 5$, and we need to consider the following cases,

Case a) $f(z_1) = z_2$, $f(z_2) = z_3$, $f(z_3) = z_4$, $f(z_4) = z_5$, and $f(z_5) = z_1$;
Case b) $f(z_1) = z_2$, $f(z_2) = z_3$, $f(z_3) = z_4$, $f(z_4) = z_1$, and $f(z_5) = z_5$;
Case c) $f(z_1) = z_2$, $f(z_2) = z_3$, $f(z_3) = z_1$, $f(z_4) = z_4$, and $f(z_5) = z_5$;
Case d) $f(z_1) = z_2$, $f(z_2) = z_3$, $f(z_3) = z_1$, $f(z_4) = z_5$, and $f(z_5) = z_4$;
Case e) $f(z_1) = z_2$, $f(z_2) = z_1$, $f(z_3) = z_4$, $f(z_4) = z_3$, and $f(z_5) = z_5$.

Similar to the analysis when $m = 4$, we have that the above case c) is invalid. For case d), we can consider the Möbius transformation $g = f \circ f$, as $g(z_1) = z_3$, $g(z_2) = z_1$, $g(z_3) = z_2$, $g(z_4) = z_4$, $g(z_5) = z_5$, so Case d) is invalid. For Case e), we can get

$$(z_1-z_4)(z_2-z_5)(z_3-z_5)+(z_1-z_5)(z_2-z_3)(z_4-z_5)=0, \tag{33}$$
$$(z_1-z_5)(z_2-z_4)(z_3-z_5)+(z_1-z_3)(z_2-z_5)(z_4-z_5)=0, \tag{34}$$
$$(z_1-z_5)(z_2-z_3)(z_4-z_5)=0. \tag{35}$$

Then Case e) is invalid. For Case b), similar to the analysis when $m=4$, we see this is invalid. For Case a), we have

$$\frac{(z_2-z_3)(z_4-z_5)}{(z_2-z_5)(z_4-z_3)}=\frac{(z_4-z_5)(z_1-z_2)}{(z_4-z_2)(z_1-z_5)}, \tag{36}$$
$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}=\frac{(z_3-z_4)(z_5-z_1)}{(z_3-z_1)(z_5-z_4)}, \tag{37}$$
$$\frac{(z_5-z_3)(z_4-z_2)}{(z_5-z_2)(z_4-z_3)}=\frac{(z_5-z_3)(z_1-z_4)}{(z_5-z_4)(z_1-z_3)}, \tag{38}$$
$$\frac{(z_5-z_2)(z_1-z_4)}{(z_5-z_4)(z_1-z_2)}=\frac{(z_5-z_2)(z_1-z_3)}{(z_5-z_3)(z_1-z_2)} \tag{39}$$
$$\frac{(z_5-z_4)(z_2-z_1)}{(z_5-z_1)(z_2-z_4)}=\frac{(z_2-z_3)(z_5-z_4)}{(z_2-z_4)(z_5-z_3)} \tag{40}$$

from the equalities (36) and (37), we have

$$(z_2-z_3)(z_5-z_1)=(z_5-z_2)(z_4-z_3)+(z_5-z_4)\cdot(z_1-z_3), \tag{41}$$

combining Eqs. (41) and (38), we have

$$(z_2-z_3)(z_5-z_1)(z_1-z_4)=(z_5-z_4)(z_1-z_3)\cdot(z_1-z_2), \tag{42}$$

combining Eqs. (42) and (39), we have

$$(z_1-z_4)(z_5-z_3)(z_1-z_2)=(z_1-z_4)(z_2-z_3)\cdot(z_5-z_1), \tag{43}$$

similarly, we have

$$(z_1-z_4)(z_2-z_3)=(z_3-z_4)(z_1-z_2), \tag{44}$$

combining Eqs. (43) and (44), we have $z_1-z_2=z_2-z_3$, then combining Eq. (40) and the above equality, we have $(z_2-z_4)(z_3-z_1)=0$, then we see this is invalid. Due to the above analyses, we see that when $m=5$, the pure state $|\psi\rangle$ owns a trivial stabilizer.

When $m=6$, assume the degeneracy configuration of $|\psi\rangle$ is $\{k_1,k_2,\ldots,k_6\}$ with $k_1=k_2=\cdots=k_6$, here we first present that there cannot exist a Möbius transformation f such that $f(z_1)=z_2$, $f(z_2)=z_3$, $\ldots$, $f(z_5)=z_6$, $f(z_6)=z_1$. Here we also apply the method *reductio ad absurdum*. If there exists a Möbius transformation f satisfying $f(z_1)=z_2$, $f(z_2)=z_3$, $\ldots$, $f(z_5)=z_6$, $f(z_6)=z_1$, due to the property vi) of Möbius transformation, we have

$$\frac{(z_1-z_4)(z_3-z_2)}{(z_1-z_2)(z_3-z_4)}=\frac{(z_1-z_4)(z_5-z_6)}{(z_1-z_6)(z_5-z_4)}, \tag{45}$$
$$\frac{(z_1-z_6)(z_5-z_4)}{(z_1-z_4)(z_5-z_6)}=\frac{(z_3-z_2)(z_1-z_6)}{(z_1-z_2)(z_3-z_6)}, \tag{46}$$

that is,

$$(z_1-z_2)(z_3-z_4)(z_5-z_6)-(z_1-z_6)(z_3-z_2)(z_5-z_4)=0, \tag{47}$$
$$(z_1-z_2)(z_3-z_6)(z_5-z_4)-(z_1-z_4)(z_3-z_2)(z_5-z_6)=0, \tag{48}$$
$$(z_6-z_4)[(z_1-z_2)(z_3-z_5)+(z_1-z_5)(z_3-z_2)]=0. \tag{49}$$

Due to Eq. (49) and $z_4\neq z_6$, if we could prove $(z_1-z_2)(z_3-z_5)+(z_1-z_5)(z_3-z_2)=0$ is invalid, then this case is invalid. Due to the Möbius transformation f, we have

$$\frac{(z_3-z_4)(z_2-z_6)}{(z_3-z_6)(z_2-z_4)}=\frac{(z_4-z_5)(z_3-z_1)}{(z_3-z_5)(z_4-z_1)}, \tag{50}$$
$$\frac{(z_1-z_5)(z_2-z_6)}{(z_1-z_6)(z_2-z_5)}=\frac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)} \tag{51}$$

from Eq. (45) and $(z_1-z_2)(z_3-z_5)+(z_1-z_5)(z_3-z_2)=0$, we have

$$\frac{z_2-z_6}{z_2-z_5}=\frac{(z_3-z_1)(z_4-z_6)}{(z_3-z_5)(z_4-z_1)}, \tag{52}$$

combining Eq. (50) and Eq. (52), we have $z_4-z_6=0$, which is invalid.

Combining the results when $m=5,6$, we may conjecture that:

Conjecture 1 Assume $z_1,z_2,\ldots,z_n$ are n different points on the extended complex plane with $n\geq 5$, there does not exist a Möbius transformation f such that $f(z_1)=z_2$, $f(z_2)=z_3$, $f(z_3)=z_4$, $\ldots$, $f(z_n)=z_1$.

As it is easy to see that when $n=4k$, $k\geq 2$, the above conjecture is valid by the similar idea when $n=5,6$, we think the above conjecture can be proven by the similar idea as when $n=5,6$.

Next we assume the degeneracy configuration of $|\psi\rangle$ is $\{k_1,k_2,\ldots,k_n\}$, if the stabilizer group of $|\psi\rangle$ is nontrivial, we could assume there exists a Möbius transformation f such that $f(z_1)=z_2$, $f(z_2)=z_3$, $\ldots$, $f(z_{i_1})=z_1$, $f(z_{i_1+1})=z_{i_1+2}$, $f(z_{i_1+2})=z_{i_1+3}$, $\ldots$, $f(z_{i_2})=z_{i_1+1}$, $\ldots$, $f(z_{i_j+1})=z_{i_j+2}$, $\ldots$,

$f(z_{i_{j+1}}) = z_{i_j+1}$, when there exists k such that $i_{k+1} - i_k \geq 5$, $k \in \{0,1,2,\ldots,j\}$, the stabilizer group for the state $|\psi\rangle$ is trivial. From the above analyses, it seems only when $\forall k$, $i_{k+1} - i_k = 2$, $k \in \{0,1,2,\ldots,j\}$, or $i_{k+1} - i_k = 3$, $k \in \{0,1,2,\ldots,j\}$, the pure state $|\psi\rangle$ owns a nontrivial stabilizer group, for the first case, it is enough to prove when $n = 6$, if there exists a Möbius transformation f such that $f(z_1) = z_2$, $f(z_2) = z_1$, $\ldots$, $f(z_5) = z_6$, $f(z_6) = z_5$. Similar to the analyses mentioned above, we have

$$\frac{(z_1 - z_2)(z_3 - z_5)}{(z_1 - z_5)(z_3 - z_2)} = \frac{(z_2 - z_1)(z_4 - z_6)}{(z_2 - z_6)(z_4 - z_1)},$$
$$(z_3 - z_5)(z_2 - z_6)(z_1 - z_4) = (z_1 - z_5)(z_3 - z_2)(z_4 - z_6), \tag{53}$$
$$(z_3 - z_5)(z_2 - z_6)(z_1 - z_4)(z_1 - z_2) = (z_2 - z_1)(z_1 - z_5)(z_3 - z_2)(z_4 - z_6), \tag{54}$$
$$\frac{(z_1 - z_4)(z_2 - z_6)}{(z_1 - z_6)(z_2 - z_4)} = \frac{(z_2 - z_3)(z_1 - z_5)}{(z_2 - z_5)(z_1 - z_3)},$$
$$(z_1 - z_4)(z_2 - z_6)(z_2 - z_5)(z_1 - z_3) = (z_1 - z_6)(z_2 - z_4)(z_2 - z_3)(z_1 - z_5), \tag{55}$$

combining Eqs. (54) and Eq. (55), we have

$$(z_2 - z_5)(z_3 - z_1)(z_4 - z_6) = (z_3 - z_5)(z_1 - z_6)(z_2 - z_4), \tag{56}$$

from Eqs. (53)–(56), we have

$$(z_4 - z_6)(z_1 - z_2)(z_3 - z_5) = 0, \tag{57}$$

as these six points are not identical, this is invalid. Next we prove the case $i_{k+1} - i_k = 3$, $k \in \{0,1,2,\ldots,j\}$ is invalid, here we also consider when $m = 6$, if there exists a Möbius transformation f such that $f(z_1) = z_2$, $f(z_2) = z_3$, $f(z_3) = z_1$, $f(z_4) = z_5$, $f(z_5) = z_6$, $f(z_6) = z_4$, then we have

$$\frac{(z_1 - z_3)(z_4 - z_5)}{(z_1 - z_5)(z_4 - z_3)} = \frac{(z_3 - z_1)(z_5 - z_6)}{(z_3 - z_6)(z_5 - z_1)}, \tag{58}$$

as each point is not identical to the others, then we have

$$(z_4 - z_3)(z_5 - z_6) = (z_4 - z_5)(z_3 - z_6) \to (z_4 - z_6)(z_5 - z_3) = 0. \tag{59}$$

As each point is not identical to the others, this is invalid. So under the conjecture we make, we have when the diversity number m of an n-qubit symmetric pure state $|\psi\rangle$ is larger than 5, the stabilizer group of $|\psi\rangle$ is trivial.

4. Conclusion

In this article, we consider the stabilizer group for a symmetric state $|\psi\rangle$. First we present that the stabilizer group for an n-qubit symmetric state $|\psi\rangle$ contains more than the identity when $n = 2,3,4$; then similar to the method presented in Ref. [12], we give a class of states whose stabilizer group contains only the identity; we also propose a class of states $|\psi\rangle$ with $G_\psi = \{I\}$, $\tilde{G}_\psi \neq \{I\}$; finally, we apply the Möbius transformation to present when the diversity number m of a symmetric pure state $|\psi\rangle$ is 5 and 6, the pure state $|\psi\rangle$ owns a trivial stabilizer group, and when $m \geq 7$, it is valid under the conjecture we make. This may confirm the main result in Ref. [12] partly when considering the n-qubit symmetric pure states.

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