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Two-qubit pure state tomography by five product orthonormal bases*

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In this paper, we focus on two-qubit pure state tomography. For an arbitrary unknown two-qubit pure state, separable or entangled, it has been found that the measurement probabilities of 16 projections onto the tensor products of Pauli eigenstates are enough to uniquely determine the state. Moreover, these corresponding product states are arranged into five orthonormal bases. We design five quantum circuits, which are decomposed into the common gates in universal quantum computation, to simulate the five projective measurements onto these bases. At the end of each circuit, we measure each qubit with the projective measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. Then, we consider the open problem whether three orthonormal bases are enough to distinguish all two-qubit pure states. A necessary condition is given. Suppose that there are three orthonormal bases $\{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}$. Denote the unitary transition matrices from $\mathscr{B}_1$ to $\{\mathscr{B}_2, \mathscr{B}_3\}$ as U_1 and U_2. All 32 elements of matrices U_1 and U_2 should not be zero. If not, these three bases cannot distinguish all two-qubit pure states.

Keywords: two-qubit pure state, tomography, projective measurement, quantum circuit

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1. Introduction

A quantum state is the carrier of quantum information. One of the central topics in quantum science is the ability to characterize and manipulate quantum states, which would be applied to quantum communications, quantum cryptography, and quantum computation. Suppose that there is an unknown quantum state. It is impossible to create an identical copy of this state by the no-cloning theorem.[1] However, the information of the unknown quantum state can be transmitted to a remote location by the process of quantum teleportation,[2] with the help of classical communication and previously shared quantum entanglement. Now consider the following problem. There is a machine which can produce a large number of identical unknown quantum states. We are permitted to design any device to repeatedly measure this state and read out the probability distributions of the measurement outcomes. Can we infer all the information of the unknown state?

This is the problem of quantum state tomography. Two aspects are essential for this problem: the measurements and the method to reconstruct the unknown state via the measurement probabilities. Various measurements are considered for achieving quantum state tomography, such as Pauli measurements,[3,4] projective measurement onto mutually unbiased bases,[5–8] or onto general orthogonal bases,[9–12] and symmetric informationally complete measurements.[13–15] The data of measurement probabilities is collected after we take measurements on the ensemble of identical unknown quantum states. There are different methods applied on the data in order to calculate all the coefficients of the unknown state, such as linear inversion,[16,17] maximum-likelihood estimation,[9,18,19] Bayesian mean estimation,[20] and the convex optimization program.[12,21]

In this paper, we focus on the two-qubit pure state tomography. From the result of papers,[9,22] the density matrix of an unknown two-qubit state can be determined with the expectation values of 16 Pauli measurements. It was proved that a two-qubit pure state $|\phi\rangle$ is uniquely determined among all states (UDA) with 11 Pauli measurements.[4] That is to say, there does not exist any other state, pure or mixed, which has the same measurement results as $|\phi\rangle$. The projective measurements of five mutually unbiased bases have the same property.[6] In Ref. [12], another set of five orthonormal bases are constructed by a sequence of orthogonal polynomials. The method of the convex optimization program can be used to search for a density matrix, whose measurement probabilities match the data obtained experimentally. It has been found that several projections onto the tensor products of Pauli eigenstates are enough for two-qubit pure states tomography. These tensor product states are arranged into five orthonormal bases. Furthermore, the projective measurements onto these

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bases is simulated by five quantum circuits. Each circuit is decomposed by the common quantum gates used in universal quantum computing,[22] which is the footstone in building a universal quantum computer and has been researched on various materials. The measurement parts are the projective measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on each qubit. We can directly calculate the unknown two-qubit pure state with partial measurement probabilities of these bases.

Moreover, a necessary condition is given for the unsolved problem of whether three orthonormal bases can distinguish all two-qubit states. This problem dates back to the famous Pauli problem, whether or not the position and the momentum distributions can uniquely determine the wave function of a particle (up to a global phase).[23] The answer to the Pauli problem is negative and many related topics have been studied. The finite version of Pauli problem is considered by Asher Peres.[24] He noted that the probabilities of two orthonormal bases should be sufficient to determine a finite dimensional pure state up to a set of ambiguities. The findings in Ref. [25] suggest that the set is not a set of measure zero. A general question is the following. How many different orthonormal bases are needed at least in order to uniquely determine a pure quantum state in d-dimensional Hilbert space $\mathscr{H}_d$? The minimal number of orthonormal bases is gradually clear except for $d = 4$.[10,26–29] There exist four orthonormal bases which can distinguish any pair of different pure states in $\mathscr{H}_4$, but it is not clear whether three bases can do this. There is a partial answer for this problem. It is proved that three product orthonormal bases cannot distinguish all two-qubit states, not even four. The product basis here is restricted to the form of $\{\phi_i \otimes \psi_j : i, j = 1, 2\}$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ are two orthonormal bases of $\mathbb{C}^2$. Given any three orthonormal bases $\{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}$ of $\mathscr{H}_4$, the unitary transition matrices from $\mathscr{B}_1$ to $\{\mathscr{B}_2, \mathscr{B}_3\}$ can be written as U_1 and U_2. We show that all 32 elements of U_1 and U_2 should not be zero. If not, these three bases are not enough. For example, if we choose the first basis as the standard basis and choose the second basis as the basis of Bell states, no matter how we choose a third basis, these three bases cannot distinguish all two-qubit pure states.

The rest of this paper is organized as follows. In Section 2, we describe the five product bases and construct five quantum circuits to simulate the projective measurements onto these bases. Then, we show the method to directly calculate the unknown pure state with partial measurement probabilities. In Section 3, we give a necessary condition that three orthonormal bases must satisfy in order to distinguish all two-qubit pure states. We will end up with a conclusion in Section 4.

2. Five product bases for two-qubit pure state tomography

As a warm up, let us consider the one-qubit state tomography. It is known in many textbooks that three orthonormal bases uniquely determine an unknown qubit state,[22] which can be the eignenbases of Pauli operators σ_x, σ_y, and σ_z. Denote the eignenbases as $\{|+\rangle, |-\rangle\}$, $\{|L\rangle, |R\rangle\}$, and $\{|0\rangle, |1\rangle\}$ correspondingly. Here

$$|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, \quad |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}; \tag{1}$$

$$|L\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}, \quad |R\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}. \tag{2}$$

The Hadamard gate H can be described as $|+\rangle\langle 0| + |-\rangle\langle 1|$. Denote gate $\tilde{H}$ as $|+\rangle\langle 0| - i|-\rangle\langle 1|$.

Let U be a $d \times d$ unitary operation. If we want to measure a d-dimensional pure state $|\phi\rangle$ with the projective measurements onto the orthonormal bases $\{U|0\rangle, \ldots, U|d-1\rangle\}$, we can take the unitary operation $U^\dagger$ on the state $|\phi\rangle$, then measure $U^\dagger|\phi\rangle$ with the projective measurements onto the orthonormal bases $\{|0\rangle, \ldots, |d-1\rangle\}$. As the probabilities of the j-th outcome for both sides are equal, $|\langle\psi|U|j\rangle|^2 = |\langle j|U^\dagger|\psi\rangle|^2$, where $j = 0, \ldots, d-1$.

Let V_1 and V_2 be the unitary transition from basis $\{|0\rangle, |1\rangle\}$ to basis $\{|+\rangle, |-\rangle\}$ and $\{|L\rangle, |R\rangle\}$. Then $V_1 = H$, $V_1^\dagger = H$; $V_2 = |L\rangle\langle 0| + |R\rangle\langle 1|$, $V_2^\dagger = |0\rangle\langle L| + |1\rangle\langle R| = \tilde{H}$. We can simulate the projective measurements onto these Pauli eigenbases with the circuits in Fig. 1.

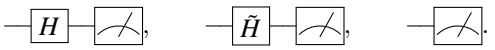


Fig. 1. Three circuits for one qubit tomography. The measurement parts are the projective measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. Before measurement, we take the unitary operation H, $\tilde{H}$, and I on the unknown qubit state respectively.

Now, we construct five product orthonormal bases to directly determine an unknown two-qubit pure state, which are as follows:

$$\begin{aligned}
\mathscr{B}_0 &= \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\},\\
\mathscr{B}_1 &= \{|0+\rangle, |L-\rangle, |1+\rangle, |R-\rangle\},\\
\mathscr{B}_2 &= \{|0L\rangle, |LR\rangle, |1L\rangle, |RR\rangle\},\\
\mathscr{B}_3 &= \{|+0\rangle, |-L\rangle, |+1\rangle, |-R\rangle\},\\
\mathscr{B}_4 &= \{|L0\rangle, |RL\rangle, |L1\rangle, |RR\rangle\}.
\end{aligned}$$

It is easy to check that each basis above is orthonormal. The first basis $\mathscr{B}_0$ is the standard basis in 4-dimensional Hilbert space $\mathscr{H}_4$. The projective measurement onto the states of $\mathscr{B}_0$ is simulated by the circuit in Fig. 2.

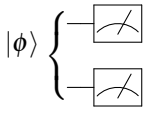


Fig. 2. Circuit for the projective measurement onto $\mathscr{B}_0$. Let the unknown two-qubit pure state be $|\phi\rangle$. We directly measure each qubit with the projective measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. The state would collapse to $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$.

Next we use the circuits to simulate the projective measurements onto the bases $\{\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4\}$, which are depicted in Fig. 3. Let U_j be the unitary transition from the basis $\mathscr{B}_0$ to the basis $\mathscr{B}_j$, $j=1,2,3,4$. The task is to decompose the gates $U_j^\dagger$ by the common gates in universal quantum computation. For example, $U_1^\dagger=(I\otimes|0\rangle|\langle 0|+\tilde{H}\otimes|1\rangle\langle 1|)\cdot(I\otimes H)$. If we take the operation $U_1^\dagger$ on the states of the basis $\mathscr{B}_1$, we will obtain the states of the basis $\mathscr{B}_0$.

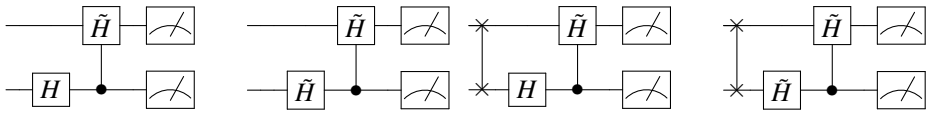


Fig. 3. Circuits for the projective measurements onto bases $\{\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3,\mathscr{B}_4\}$. The circuits at top left, top right, bottom left, and bottom right simulate the projective measurements onto $\mathscr{B}_1,\mathscr{B}_2,\mathscr{B}_3$, and $\mathscr{B}_4$ respectively. There are two two-qubit operations in these circuits. The first one is the swap operation, which is depicted in Fig. 4. The other one is a controlled unitary operation, $I\otimes|0\rangle|\langle 0|+\tilde{H}\otimes|1\rangle\langle 1|$. The one-qubit operations are the Hadamard gate H and the gate $\tilde{H}$ that appeared in the circuit for one qubit tomography. At the end of each circuit, we measure each qubit with the projective measurement $\{|0\rangle\langle 0|,|1\rangle\langle 1|\}$.

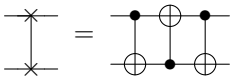


Fig. 4. (color online) Simulation of swap operation. The swap operation is: $|00\rangle\langle 00|+|10\rangle\langle 01|+|01\rangle\langle 10|+|11\rangle\langle 11|$. If we take the swap operation on input state $|jk\rangle$, the output state will be $|kj\rangle$, $j,k=0,1$. It can be decomposed into three control-not operations. The bases states of $\mathscr{B}_1$ and $\mathscr{B}_2$ would be transformed into the bases states of $\mathscr{B}_3$ and $\mathscr{B}_4$ by this operation.

3. Method to calculate the unknown two-qubit pure state

An arbitrary unknown two-qubit pure state, separable or entangled, can be expressed with the vector form $a|00\rangle+b|01\rangle+c|10\rangle+d|11\rangle$, where $|a|^2+|b^2|+|c|^2+|d|^2=1$. Any complex number g can be expressed as $|g|e^{i\theta_g}$. So we can write the unknown state with a new form

$$|\phi\rangle=|a|\,\mathrm{e}^{\mathrm{i}\theta_a}|00\rangle+|b|\,\mathrm{e}^{\mathrm{i}\theta_b}|01\rangle+|c|\,\mathrm{e}^{\mathrm{i}\theta_c}|10\rangle+|d|\,\mathrm{e}^{\mathrm{i}\theta_d}|11\rangle, \tag{3}$$

where θ_a, θ_b, θ_c, $\theta_d\in[0,2\pi)$. In this section, we demonstrate that all the unknown coefficients can be calculated by partial measurement probabilities of the five bases.

If we measure $|\phi\rangle$ with the projective measurement onto the basis $\mathscr{B}_0$, the state will collapse to $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$. Denote the measurement probabilities as p_0^1, p_0^2, p_0^3, and p_0^4 respectively. The coefficients of $\{|a|,|b|,|c|,|d|\}$ can be calculated by the probabilities of the basis $\mathscr{B}_0$. The following equations hold:

$$p_0^1=|a|^2,\ \ p_0^2=|b|^2,\ \ p_0^3=|c|^2,\ \ p_0^4=|d|^2. \tag{4}$$

So, $|a|=\sqrt{p_0^1}$, $|b|=\sqrt{p_0^2}$, $|c|=\sqrt{p_0^3}$, $|d|=\sqrt{p_0^4}$. The unknown state will be clear until the phases $\{\theta_a,\theta_b,\theta_c,\theta_d\}$ are calculated. We can let one of these phases be zero. Two pure states $|\phi_1\rangle$ and $|\phi_2\rangle$ correspond to the same one if there is a nonzero complex number g such that $|\phi_1\rangle=g|\phi_2\rangle$, where $|g|=1$. As $|\phi_1\rangle\langle\phi_1|=gg^\dagger|\phi_2\rangle\langle\phi_2|=|\phi_2\rangle\langle\phi_2|$, their density matrices are the same. So the global phase can be extracted and neglected.

At least one probability in $\{p_0^1,p_0^2,p_0^3,p_0^4\}$ is not equal to zero. Next we divide it into four cases and calculate the remaining phases $\{\theta_a,\theta_b,\theta_c,\theta_d\}$ at each case. Record the probability as p_j^k if the state collapses to the k-th element of basis $\mathscr{B}_j$.

Case 1 Only one probability in $\{p_0^1,p_0^2,p_0^3,p_0^4\}$ is not zero. In this case, just the probabilities of basis $\mathscr{B}_0$ are sufficient to reconstruct the unknown two-qubit pure state.

For example, the probabilities of the basis $\mathscr{B}_0$ are the following: $p_0^1=1$, $p_0^2=0$, $p_0^3=0$, and $p_0^4=0$. Then $|a|=1$, $b=c=d=0$. We do not need to calculate the phases $\{\theta_b,\theta_c,\theta_d\}$ and the unknown state is $|00\rangle$ for the freedom choice of the global phase. The nonzero probability may be another one. We list the nonzero probability and the corresponding state in the following table (Table 1).

Table 1. The nonzero probability and the corresponding state.

Nonzero probability	p_0^1	p_0^2	p_0^3	p_0^4
Unknown state	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$

Case 2 Two probabilities in $\{p_0^1,p_0^2,p_0^3,p_0^4\}$ are not zero. There are $\binom{4}{2}=6$ subcases totally. For example, $p_0^1,p_0^2\neq 0;p_0^3,p_0^4=0$. The unknown state can be written as $|a||00\rangle+|b|e^{i\theta_b}|01\rangle$. We only need to calculate the phase θ_b. This task can be fulfilled by considering the probabilities that the state collapses to $|0+\rangle$ and $|0L\rangle$. The following equations hold:

$$\begin{cases} 2p_1^1=2|\langle\phi|0+\rangle|^2=p_0^1+p_0^2+2\sqrt{p_0^1p_0^2}\cos\theta_b. \\ 2p_2^1=2|\langle\phi|0L\rangle|^2=p_0^1+p_0^2+2\sqrt{p_0^1p_0^2}\sin\theta_b. \end{cases} \tag{5}$$

So the values of $\cos\theta_b$ and $\sin\theta_b$ can be determined by the probabilities $\{p_1^1,p_2^1,p_0^1,p_0^2\}$. Then we assign a unique value of θ_b at the range of $[0,2\pi)$.

When $p_0^1,p_0^3\neq 0$, the unknown state is $|a||00\rangle+|c|\,\mathrm{e}^{\mathrm{i}\theta_c}|10\rangle$. In order to calculate the phase θ_c, we can consider the probabilities that the state collapses to $|+0\rangle$ and $|L0\rangle$.

$$\begin{cases} 2p_3^1=2|\langle\phi|+0\rangle|^2=p_0^1+p_0^3+2\sqrt{p_0^1p_0^3}\cos\theta_c. \\ 2p_4^1=2|\langle\phi|L0\rangle|^2=p_0^1+p_0^3+2\sqrt{p_0^1p_0^3}\sin\theta_c. \end{cases} \tag{6}$$

When $p_0^1, p_0^4 \neq 0$, the unknown state is $|a||00\rangle + |d|e^{i\theta_d}|11\rangle$. In order to calculate the phase θ_d, we can consider the probabilities that the state collapses to $|L-\rangle$ and $|LR\rangle$.

$$\begin{cases} 4p_1^2 = 2|\langle\phi|L-\rangle|^2 = p_0^1 + p_0^4 - 2\sqrt{p_0^1 p_0^4}\sin\theta_d. \\ 4p_2^2 = 2|\langle\phi|LR\rangle|^2 = p_0^1 + p_0^4 + 2\sqrt{p_0^1 p_0^4}\cos\theta_d. \end{cases} \tag{7}$$

When $p_0^2, p_0^3 \neq 0$, the unknown state is $|b||01\rangle + |c|\mathrm{e}^{\mathrm{i}\theta_c}|10\rangle$. In order to calculate the phase θ_d, we can consider the probabilities that the state collapses to $|-L\rangle$ and $|RL\rangle$.

$$\begin{cases} 4p_3^2 = 2|\langle\phi|-L\rangle|^2 = p_0^2 + p_0^3 + 2\sqrt{p_0^2 p_0^3}\sin\theta_c. \\ 4p_4^2 = 2|\langle\phi|RL\rangle|^2 = p_0^2 + p_0^3 - 2\sqrt{p_0^2 p_0^3}\cos\theta_c. \end{cases} \tag{8}$$

When $p_0^2, p_0^4 \neq 0$, the unknown state is $|b||01\rangle + |d|\mathrm{e}^{\mathrm{i}\theta_d}|11\rangle$. In order to calculate the phase θ_d, we can consider the probabilities that the state collapses to $|+1\rangle$ and $|L1\rangle$.

$$\begin{cases} 2p_3^3 = 2|\langle\phi|+1\rangle|^2 = p_0^2 + p_0^4 + 2\sqrt{p_0^2 p_0^4}\cos\theta_d. \\ 2p_4^3 = 2|\langle\phi|L1\rangle|^2 = p_0^4 + p_0^4 + 2\sqrt{p_0^2 p_0^4}\sin\theta_d. \end{cases} \tag{9}$$

When $p_0^3, p_0^4 \neq 0$, the unknown state is $|c||10\rangle + |d|\mathrm{e}^{\mathrm{i}\theta_d}|11\rangle$. In order to calculate the phase θ_d, we can consider the probabilities that the state collapses to $|1+\rangle$ and $|1L\rangle$.

$$\begin{cases} 2p_1^3 = 2|\langle\phi|1+\rangle|^2 = p_0^3 + p_0^4 + 2\sqrt{p_0^3 p_0^4}\cos\theta_d. \\ 2p_2^3 = 2|\langle\phi|1L\rangle|^2 = p_0^4 + p_0^4 + 2\sqrt{p_0^4 p_0^4}\sin\theta_d. \end{cases} \tag{10}$$

Case 3 Three probabilities in $\{p_0^1, p_0^2, p_0^3, p_0^4\}$ are not zero.

There are four subcases in total. For example, $p_0^1, p_0^2, p_0^3 \neq 0; p_0^4 = 0$. The unknown state can be written as $|a||00\rangle + |b|e^{i\theta_b}|01\rangle + |c|e^{i\theta_c}|10\rangle$. The phases θ_b and θ_c can be determined by Eqs. (5) and (6).

When $p_0^1, p_0^2, p_0^4 \neq 0$, the unknown state is $|a||00\rangle + |b|\mathrm{e}^{\mathrm{i}\theta_b}|01\rangle + |d|\mathrm{e}^{\mathrm{i}\theta_d}|11\rangle$. The phases θ_b can be determined by Eq. (5). The phase difference $\theta_d - \theta_b$ can be determined by considering the probabilities that the state collapses to $|+1\rangle$ and $|L1\rangle$.

$$\begin{cases} 2p_3^3 = 2|\langle\phi|+1\rangle|^2 = p_0^2 + p_0^4 + 2\sqrt{p_0^2 p_0^4}\cos(\theta_d - \theta_b). \\ 2p_4^3 = 2|\langle\phi|L1\rangle|^2 = p_0^4 + p_0^4 + 2\sqrt{p_0^2 p_0^4}\sin(\theta_d - \theta_b). \end{cases} \tag{11}$$

When $p_0^1, p_0^3, p_0^4 \neq 0$, the unknown state is $|a||00\rangle + |c|\mathrm{e}^{\mathrm{i}\theta_c}|10\rangle + |d|\mathrm{e}^{\mathrm{i}\theta_d}|11\rangle$. The phase θ_c can be determined by Eq. (6). The phase difference $\theta_d - \theta_c$ can be determined by considering the probabilities that the state collapses to $|1+\rangle$ and $|1L\rangle$.

When $p_0^2, p_0^3, p_0^4 \neq 0$, the unknown state is $|b||01\rangle + |c|\mathrm{e}^{\mathrm{i}\theta_c}|10\rangle + |d|\mathrm{e}^{\mathrm{i}\theta_d}|11\rangle$. The phase θ_d can be determined by Eq. (9). The phase difference $\theta_d - \theta_c$ can be determined by considering the probabilities that the state collapses to $|1+\rangle$ and $|1L\rangle$.

Case 4 All probabilities in $\{p_0^1, p_0^2, p_0^3, p_0^4\}$ are not zero.

We write the unknown two-qubit pure state as $|a||00\rangle + |b|\mathrm{e}^{\mathrm{i}\theta_b}|01\rangle + |c|\mathrm{e}^{\mathrm{i}\theta_c}|10\rangle + |d|\mathrm{e}^{\mathrm{i}\theta_d}|11\rangle$. We can use Eqs. (5) and (6) to determine the phases θ_b and θ_c. The phase difference $\theta_d - \theta_b$ can be determined by Eq. (11). Then all the phases are clear.

Now we draw a conclusion. The coefficients $\{|a|, |b|, |c|, |d|\}$ can be calculated by the probabilities $\{p_0^k : k = 1,2,3,4\}$, which are obtained from the projective measurement onto the basis $\mathscr{B}_0$. The remaining task is to calculate the phases for the nonzero amplitudes. The number of nonzero amplitudes may be k as listed in the case k, where $k \in \{1,2,3,4\}$. We can let one phase be zero for the freedom choice of the global phase. For the cases 2, 3, and 4, we use two, four, and six measurement probabilities respectively to determine the phases. These probabilities are some parts obtained from the projective measurements onto the bases $\{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3, \mathscr{B}_4\}$, where each basis state is the tensor product of Pauli eigenstates. Until we determine the amplitudes and the phases, the unknown two-qubit state will be clear.

4. A necessary condition

Let $\mathscr{B}_1, \ldots, \mathscr{B}_m$ be m orthonormal bases of the d-dimensional Hilbert space $\mathscr{H}_d$, where $\mathscr{B}_j = \{|\phi_j^k\rangle : k = 1, \ldots, d\}$. Given any pair of different pure states ρ_1 and ρ_2, these bases can distinguish them if $\mathrm{tr}(\rho_1|\phi_\mathrm{j}^\mathrm{k}\rangle\langle\phi_\mathrm{j}^\mathrm{k}|) \neq \mathrm{tr}(\rho_2|\phi_\mathrm{j}^\mathrm{k}\rangle\langle\phi_\mathrm{j}^\mathrm{k}|)$ for some basis states $|\phi_j^k\rangle$. The minimal number of orthonormal bases needed to distinguish all d-dimensional pure states is clear except for $d = 4$.[10] Four orthonormal bases can distinguish all two-qubit pure states; however, the problem remains unsolved whether three orthonormal bases are enough. Here we give a necessary condition that three orthonormal bases should satisfy. If not, three bases cannot distinguish all two-qubit pure states.

Denote three orthonormal bases as $\{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}$. We record the bases $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$ as $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$, $\{|\phi_1\rangle, \ldots, |\phi_4\rangle\}$, $\{|\psi_1\rangle, \ldots, |\psi_4\rangle\}$ respectively. The unitary transition matrices from $\mathscr{B}_1$ to $\{\mathscr{B}_2, \mathscr{B}_3\}$ are written as U_1 and U_2. $U_1 = (x_{i,j})$, $U_2 = (y_{i,j})$, $i, j = 1, \ldots, 4$. In other words, the bases of $\mathscr{B}_2$ and $\mathscr{B}_3$ are the following:

$$\mathscr{B}_2 = \{|\phi_k\rangle = \sum_{j=1}^{4} x_{kj}|j\rangle, k = 1, \ldots, 4\}, \tag{12}$$

$$\mathscr{B}_3 = \{|\psi_k\rangle = \sum_{j=1}^{4} y_{kj}|j\rangle, k = 1, \ldots, 4\}. \tag{13}$$

All 32 elements of matrices U_1 and U_2 should not be zero. If zero appears, we can always construct a pair of different pure states, whose measurement probabilities under the three orthonormal bases are the same.

Here we give the proof. Without loss of generality, the zero matrix element is x_{11}. We may as well let the first rows of matrix U_1 and U_2 be real numbers. Namely, x_{k1} and y_{k1} are all real, $k = 1,2,3,4$. This is because the probabilities that state $|\phi\rangle$ collapses to the basis states $|\phi_k\rangle$ or $c|\phi_k\rangle$ are equal, $|c| = 1$. So the bases $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle\}$, and $\{c_1|\phi_1\rangle, c_2|\phi_2\rangle, c_3|\phi_3\rangle, c_4|\phi_4\rangle\}$ have the same power of discrimination, where $|c_k| = 1$ for $k = 1,2,3,4$.

We can construct a pair of unnormalized vector states:

$$|\Phi_1\rangle = |1\rangle + \sum_{k=2}^{4} [\mathrm{Re}(c_k) + \mathrm{i}\cdot \mathrm{Im}(c_k)]|k\rangle, \tag{14}$$

$$|\Phi_2\rangle = |1\rangle - \sum_{k=2}^{4} [\mathrm{Re}(c_k) + i\cdot \mathrm{Im}(c_k)]|k\rangle. \tag{15}$$

where $\mathrm{Re}(c_k)$ and $\mathrm{Im}(c_k)$ are the real and imaginary parts of the complex number c_k. Now we prove that there is a nontrivial solution of $\{c_k : k = 2,3,4\}$, so that the probability distributions of $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are the same under the projective measurements onto the bases $\mathscr{B}_1, \mathscr{B}_2$, and $\mathscr{B}_3$.

It is easy to prove that their probability distributions onto the first basis $\mathscr{B}_1$ are the same.

$$|\langle\Phi_1|k\rangle|^2 = |\langle\Phi_2|k\rangle|^2, \ \ k = 1,2,3,4. \tag{16}$$

For the second basis $\mathscr{B}_2$,

$$|\langle\Phi_1|\phi_k\rangle|^2 = [x_{k1} + X_k]^2 + Y_k^2, \tag{17}$$

$$|\langle\Phi_2|\phi_k\rangle|^2 = [x_{k1} - X_k]^2 + Y_k^2, \tag{18}$$

where $X_k = \sum_{j=2}^{4}[\mathrm{Re}(c_j)\mathrm{Re}(x_{kj}) + \mathrm{Im}(c_j)\mathrm{Im}(x_{kj}]$, $Y_k = \sum_{j=2}^{4}[\mathrm{Re}(c_j)\mathrm{Im}(x_{kj}) - \mathrm{Im}(c_j)\mathrm{Re}(x_{kj})]$. If $X_k = 0$ for all k, we have $|\langle\Phi_1|\phi_k\rangle|^2 = |\langle\Phi_2|\phi_k\rangle|^2$. Their probability distributions onto the basis $\mathscr{B}_2$ are the same. From the assumption, $x_{11} = 0$. We have $|\langle\Phi_1|\phi_1\rangle|^2 = |\langle\Phi_2|\phi_1\rangle|^2$ even $X_1 \neq 0$.

Define $\tilde{X}_k = \sum_{j=2}^{4}[\mathrm{Re}(c_j)\mathrm{Re}(y_{kj}) + \mathrm{Im}(c_j)\mathrm{Im}(y_{kj}]$. With a similar analysis, if $\tilde{X}_k = 0$ for $k = 1,2,3,4$, their probability distributions onto the basis $\mathscr{B}_3$ are the same one.

Now we consider the following linear equations:

$$X_k = 0, \ \text{for } k = 2,3;$$
$$\tilde{X}_k = 0, \ \text{for } k = 1,2,3.$$

There are five linear equations about six unknown real variables $\{\mathrm{Re}(c_j), \mathrm{Im}(c_j) : j = 2,3,4\}$. There must be a nontrivial solution. Then for such a pair of two different unnormalized vector states $|\Phi_1\rangle$ and $|\Phi_2\rangle$, we have $|\langle\Phi_1|\phi_k\rangle|^2 = |\langle\Phi_2|\phi_k\rangle|^2$, $|\langle\Phi_1|\psi_k\rangle|^2 = |\langle\Phi_2|\psi_k\rangle|^2$, $k = 1,2,3$. As the bases $\mathscr{B}_2$ and $\mathscr{B}_3$ are orthonormal, we have $\sum_{k=1}^{4}|\phi_k\rangle\langle\phi_k| = \sum_{k=1}^{4}|\psi_k\rangle\langle\psi_k| = I$. Then,

$$\sum_{k=1}^{4}|\langle\Phi_1|\phi_k\rangle|^2 = \sum_{k=1}^{4}|\langle\Phi_2|\phi_k\rangle|^2, \ \ \sum_{k=1}^{4}|\langle\Phi_j|\psi_k\rangle|^2 = \sum_{k=1}^{4}|\langle\Phi_2|\psi_k\rangle|^2.$$

So the following equations are satisfied:

$$|\langle\Phi_1|\phi_k\rangle|^2 = |\langle\Phi_2|\phi_k\rangle|^2, \ k = 1,2,3,4;$$
$$|\langle\Phi_1|\psi_k\rangle|^2 = |\langle\Phi_2|\psi_k\rangle|^2, \ k = 1,2,3,4.$$

Let N be the norm of $|\Phi_1\rangle$ and $|\Phi_2\rangle$. The probability distributions of pure states $|\Phi_1\rangle/N$ and $|\Phi_2\rangle/N$ are the same under the projective measurements onto the bases $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$.

The zero element may appear at the j-th row of the matrices U_1 or U_2. Then the two vector states can be constructed as: $|\Phi_1\rangle = |j\rangle + \sum_{k\neq j}[\mathrm{Re}(c_k) + \mathrm{i}\cdot\mathrm{Im}(c_k)]|k\rangle$, $|\Phi_2\rangle = |j\rangle - \sum_{k\neq j}[\mathrm{Re}(c_k) + \mathrm{i}\cdot\mathrm{Im}(c_k)]|k\rangle$. With a similar analysis, there exists a nontrivial solution so that the probability distributions onto the three orthonormal bases are equal. This completes the proof.

This necessary condition can give us some insights to the minimal number of orthonormal bases which can distinguish all two-qubit pure states. If the minimal number is actually three, we should find them with the constraint of the condition. For example, if we choose basis $\mathscr{B}_1$ as the standard basis and choose basis $\mathscr{B}_2$ as the basis of Bell states, no matter how we choose a third basis, these three bases cannot distinguish all two-qubit pure states. As there are zero elements in matrix U_1, if any three orthonormal bases which satisfy this condition cannot distinguish all two-qubit pure states, the minimal number is four.

5. Conclusion

In this paper, we show that any unknown two-qubit pure state can be directly reconstructed by the measurement probabilities of projections onto 16 tensor products of Pauli eigenstates. These projections are arranged into five orthonormal bases. Moreover, we construct five quantum circuits to simulate the projective measurements onto the states of these bases. In the end, we focus on the unsolved problem of whether three orthonormal bases can distinguish all two-qubit pure states. A necessary condition is given. If the condition is not satisfied, three orthonormal bases are not sufficient to distinguish all two-qubit pure states.

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