ISSN 1674-1056
http://cpb.iphy.ac.cn
http://iopscience.iop.org/cpb
中国物理 B
Chinese Physics
中国物理
Chinese Physics B

Non-Gaussianity dynamics of two-mode squeezed number states subject to different types of noise based on cumulant theory

Shaohua Xiang(向少华), Xixiang Zhu(朱喜香), Kehui Song(宋克慧)

Citation: Chin. Phys. B . 2018, 27(10): 100305. **doi:** 10.1088/1674-1056/27/10/100305

Journal homepage: http://cpb.iphy.ac.cn; http://iopscience.iop.org/cpb

What follows is a list of articles you may be interested in

Decoherence of macroscopic objects from relativistic effect

Guo-Hui Dong(董国慧), Yu-Han Ma(马宇翰), Jing-Fu Chen(陈劲夫), Xin Wang(王欣), Chang-Pu Sun(孙昌璞)
Chin. Phys. B . 2018, 27(10): 100301. **doi:** 10.1088/1674-1056/27/10/100301

Decoherence for a two-qubit system in a spin-chain environment

Yang Yang(杨阳), An-Min Wang(王安民), Lian-Zhen Cao(曹连振), Jia-Qiang Zhao(赵加强), Huai-Xin Lu(逯怀新)
Chin. Phys. B . 2018, 27(9): 090302. **doi:** 10.1088/1674-1056/27/9/090302

Classical-driving-assisted coherence dynamics and its conservation

De-Ying Gao(高德营), Qiang Gao(高强), Yun-Jie Xia(夏云杰)
Chin. Phys. B . 2018, 27(6): 060304. **doi:** 10.1088/1674-1056/27/6/060304

Superconducting phase qubits with shadow-evaporated Josephson junctions

Fei-Fan Su(宿非凡), Wei-Yang Liu(刘伟洋), Hui-Kai Xu(徐晖凯), Hui Deng(邓辉), Zhi-Yuan Li(李志远), Ye Tian(田野), Xiao-Bo Zhu(朱晓波), Dong-Ning Zheng(郑东宁), Li Lv(吕力), Shi-Ping Zhao(赵士平)
Chin. Phys. B . 2017, 26(6): 060308. **doi:** 10.1088/1674-1056/26/6/060308

Bidirectional multi-qubit quantum teleportation in noisy channel aided with weak measurement

Guang Yang(杨光), Bao-Wang Lian(廉保旺), Min Nie(聂敏), Jiao Jin(金娇)
Chin. Phys. B . 2017, 26(4): 040305. **doi:** 10.1088/1674-1056/26/4/040305

Non-Gaussianity dynamics of two-mode squeezed number states subject to different types of noise based on cumulant theory*

Shaohua Xiang(向少华)†, Xixiang Zhu(朱喜香), and Kehui Song(宋克慧)

School of Mechanical, Optoelectronics and Physics, Huaihua University, Huaihua 418008, *China*

(Received 23 May 2018; revised manuscript received 23 July 2018; published online 27 September 2018)

We provide a measure to characterize the non-Gaussianity of phase-space function of bosonic quantum states based on the cumulant theory. We study the non-Gaussianity dynamics of two-mode squeezed number states by analyzing the phase-averaged kurtosis for two different models of decoherence: amplitude damping model and phase damping model. For the amplitude damping model, the non-Gaussianity is very fragile and completely vanishes at a finite time. For the phase damping model, such states exhibit rich non-Gaussian characters. In particular, we obtain a transition time that such states can transform from sub-Gaussianity into super-Gaussianity during the evolution. Finally, we compare our measure with the existing measures of non-Gaussianity under the independent dephasing environment.

Keywords: non-Gaussianity, cumulants, squeezed number states, decoherence

PACS: 03.65.Yz, 03.67.Mn, 42.50.Dv, 42.50.Lc **DOI:** 10.1088/1674-1056/27/10/100305

1. Introduction

The non-Gaussianity of a quantum state has been widely acknowledged to be a useful resource in implementing relevant quantum-information tasks and in understanding quantum mechanics since it can enhance the fidelity in continuous-variable quantum teleportation,[1,2] construct a test platform for optical links suitable for quantum key distribution,[3] reveal the non-classicality of quantum states,[4] and measure the robustness of real single photons against optical losses.[5] To characterize this property in a quantitative manner, different measures have been proposed in recent years. Genoni *et al.* first used the Hilbert–Schmidt distance to quantify the non-Gaussian character of a bosonic quantum state and evaluated the non-Gaussianity of some relevant states.[6] Subsequently, they developed the entropic measure of non-Gaussianity based on the quantum relative entropy,[7] by which they investigated the performance of conditional Gaussification toward twin-beam and de-Gaussification processes driven by Kerr interaction. These measures have recently been used in the studies of the non-Gaussianity of other quantum states. For instance, Tang *et al.* have investigated the non-Gaussianity of the single-photon-added and -subtracted coherent (SPASC) superposition states. It was found that squeezing the input field can enhance the robustness of non-Gaussianity.[8] Taghiabadi *et al.* have analyzed the non-Gaussianity of two two-mode continuous-variable separable states with the same marginal states and shown that the difference in behavior of their non-Gaussianity is the same as the difference between negativity of their Wigner functions.[9] Park *et al.* have examined the non-Gaussianity for the correlation of a bipartite quantum states and found that Gaussian extremality holds for none of these measures.[10] However, we emphasize that computing those measures is a formidable task for multimode continuous-variable (CV) non-Gaussian entangled states because one requires to look for the corresponding reference Gaussian counterparts.

The present work aims at overcoming this roadblock by introducing the cumulants. As is well known, the cumulants were first introduced in the late 1800s by Thiele under the name of semi-invariants,[11] but entered the wider scene of statistics only with Fisher's fundamental paper under the name of cumulative moment functions.[12] Up to now it has been widely accepted that the cumulants are useful quantities for measuring the statistical properties of probability distributions. In particular, the cumulants of order greater than two serves as a quantitative statement of the departure of the shape of probability distribution from Gaussian.[13,14] For a univariate case, the third-order (or called skewness) is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean, while the fourth-order cumulant (or called kurtosis) provides a measure of the distance to Gaussianity. Positive kurtosis indicates heavy tails and peakedness relative to the Gaussian distribution, whereas negative kurtosis indicates light tails and flatness. Therefore, the larger the kurtosis is different from zero, the more highly the distribution is non-Gaussian. The multivariate skewness and kurtosis measures were also developed to assess departure from multivariate normality.[15–18] On the other hand, the characteristic function to any quantum operator including the density matrix operator can be

*Project supported by the Natural Science Foundation of Hunan Province, China (Grant No. 2017JJ2214) and the Key Project Foundation of the Education Department of Hunan Province, China (Grant No. 14A114.

†Corresponding author. E-mail: shxiang97@163.com

© 2018 Chinese Physical Society and IOP Publishing Ltd

expanded by its cumulants.[19] A quantum operator is Gaussian if all the cumulants of greater two orders of its operator's characteristic function vanish identically; otherwise, it is non-Gaussian. Recently, an ongoing research effort has been devoted to the study of the non-Gaussianity of quantum operators on quantum Bosonic systems. Dubost *et al.* have studied theoretically and experimentally the quantification of non-Gaussian spin distributions using theory of cumulants and demonstrated that cumulant-based estimation is very efficient.[20] In the same spirit, Moreno–Cardoner *et al.* have analyzed the behavior of the full distribution of collective observables in quantum spin chains.[21] It has been shown that the non-Gaussianity in the critical phase can be evident from the analysis of the Binder cumulant and quantum fluctuations at criticality leads to highly non-Gaussian distribution. Olsen *et al.*[22] investigated the non-Gaussian statistic of the Kerr-squeezed state by calculating higher-order cumulants of quadrature variables. It was found that the nonlinear interaction can skew the distribution of the quadrature variables, giving rise to large third- and fourth-order cumulants for sufficiently long interaction times. However, unlike the proposed measures in Refs. [20] and [22], we use the phase-averaged technique to estimate the non-Gaussianity of quadrature distribution of a collective quadrature operator of bosonic quantum system, which belongs to a single-variable estimation of non-Gaussianity, thus greatly removing the complexity of cumulant computation. Additionally, we note that resource theories of quantum non-Gaussianity have been developed very recently for non-Gaussian operations[23] and for non-Gaussian states.[24,25] It is of great help to guide the deeply understanding of non-Gaussian resources and the development of quantum information processing thereof.

Two-mode squeezed number states are a broad and meaningful class of continuous-variable quantum bipartite states. The set of them contains mostly non-Gaussian entangled states but includes the known two-mode squeezed vacuum state. Such states were originally introduced by Chizhow where the photon number statistics and the phase properties of these states are investigated.[26] It is shown that the photon number distribution and the Pegg–Barnett phase distribution for such states have a similar $(N+1)$-peak structure for a nonzero value of the difference in the numbers of photons between the modes. The analysis of their inseparabilities has been carried out via the Neumann entropy and the Shchukin-Vogel positive partial transposition (PPT) criterion.[27] However, to the best of our knowledge, non-Gaussianity dynamics of such states was never studied as yet.

In the present paper, we mainly study the non-Gaussianity of two-mode squeezed Bell states, which can be generated by operating with two-mode squeezer on Bell states, thus allowing us to discuss the influence of Gaussian operation on non-Gaussian entangled states. The remainder of this paper is organized as follows. In Section 2, we define the cumulants of a quantum state in the characteristic function representation and provide an explicit formula for cumulants of any order. We illustrate in Section 3 how two-mode squeezer effects the non-Gaussianity of quantum state using two examples of two-mode squeezed Bell states. In Section 4, we compare our measure with the Hilbert–Schmidt degree of non-Gaussianity for a case of the two modes subjected to their own phase-diffusive noise. We conclude with a summary and outlook in Section 5.

2. Higher-order cumulants of quantum non-Gaussian states

As is well known, it is convenient to describe a continuous-variable quantum state by the s-order characteristic function, which is defined as[28,29]

$$\chi[\rho](\boldsymbol{\xi}) = \mathrm{Tr}\left[\rho \exp\left(\sum_{l=1}^{N} \xi_l \hat{a}_l^{\dagger} - \xi_l^{*}\hat{a}_l + \frac{1}{2}s|\xi_l|^2\right)\right], \quad (1)$$

where $\boldsymbol{\xi} = (\xi_1, \xi_1^{*}, \ldots, \xi_N, \xi_N^{*})$, ρ is the density operator, and ξ_l is the complex variable corresponding to the operator $\hat{a}_l$. The values $s = -1, 0, 1$ correspond, respectively, to the Q function,[30] the Wigner function,[31] and the P function.[32] Here we make use of the Wigner characteristic function, i.e., $s = 0$. Therefore, once these functions are known, the expectation values of any products of creation operators $\hat{a}^{\dagger}$ and annihilations $\hat{a}$ quantum system can be computed.

Without loss of generality, the characteristic function of any continuous-variable quantum state can be expressed as in the following form:

$$\chi[\rho](\boldsymbol{\xi}) = f(\boldsymbol{\xi})\chi_{\mathrm{G}}(\boldsymbol{\xi}), \quad (2)$$

where $\chi_{\mathrm{G}}(\boldsymbol{\xi})$ is equivalently given by an $N-$mode Gaussian state as follows:

$$\begin{aligned}\chi[\hat{\rho}_{\mathrm{G}}](\boldsymbol{\xi}) &= \mathrm{Tr}\left[\hat{\rho}_{\mathrm{G}} \bigotimes_{j=1}^{N} \hat{D}(\xi_j)\right] \\ &= \exp\left[-\boldsymbol{\xi}\boldsymbol{V}\boldsymbol{\xi}^{\mathrm{T}} - \mathrm{i}\langle \boldsymbol{d}\rangle_{\hat{\rho}_{\mathrm{G}}}\boldsymbol{\xi}^{\mathrm{T}}\right], \end{aligned} \quad (3)$$

where the average value is given by $\langle \boldsymbol{d}\rangle_j = \mathrm{Tr}[\hat{\rho}_{\mathrm{G}}\hat{\boldsymbol{A}}_j]$ and a matrix element of $2N \times 2N$ covariance matrix $\boldsymbol{V}$ is given by $V_{lm} = \mathrm{Tr}\left[\hat{\rho}_{\mathrm{G}}\{\Delta\hat{\boldsymbol{A}}_l, \Delta\hat{\boldsymbol{A}}_m^{\dagger}\}\right]$, in which for convenience to write, we collect the N-pair quadrature operators into a vector of operators $\hat{\boldsymbol{A}} \equiv (\hat{a}_1^{\dagger}, \hat{a}_1, \ldots, \hat{a}_N^{\dagger}, \hat{a}_N)$, $\Delta\hat{\boldsymbol{A}} \equiv \hat{\boldsymbol{A}} - \boldsymbol{d}$, and $\{\cdot\,,\,\cdot\}$ denotes the anti-commutator. We see from Eq. (3) that if the function $f(\boldsymbol{\xi})$ is a constant, then a state described by Eq. (2) is a Gaussian one; otherwise, it is a non-Gaussian state.

Another equivalent description of a probability distribution is through the cumulant generating function, which is the logarithm of the moment-generation function

$$\Gamma(\boldsymbol{\xi}) = \ln\chi[\rho](\boldsymbol{\xi}). \quad (4)$$

Once $\chi[\rho](\boldsymbol{\xi})$ is known, the k-th order cumulant can be generated by evaluating the l-th derivative of $\Gamma(\boldsymbol{\xi})$ at $\boldsymbol{\xi}=0$ and is given by

$$\begin{aligned}\kappa_l &= \left.\frac{\partial^l \Gamma(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^l}\right|_{\boldsymbol{\xi}=0} = \left.\frac{\partial^l \ln f(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^l}\right|_{\boldsymbol{\xi}=0} + \left.\frac{\partial^l \ln \chi_{\mathrm{G}}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^l}\right|_{\boldsymbol{\xi}=0} \\ &= \kappa_l^{\mathrm{NG}} + \kappa_l^{\mathrm{G}}, \end{aligned} \tag{5}$$

where the superscript NG (G) denotes the non-Gaussian (Gaussian) component, $\partial \boldsymbol{\xi}^l = \partial(\xi_1^{l_1}, \xi_1^{*l_2}, \ldots, \xi_N^{l_{2N-1}}, \xi_N^{*l_{2N}})$, and $l = l_1 + l_2 + \cdots + l_{2N}$. So we see that the cumulant of a CV quantum state is a sum of the cumulants of its Gaussian and the resulting non-Gaussian parts.

Any multi-mode Gaussian state is fully determined in the phase space by the first and second moments of the quadrature operators.[33,34] Thus, in terms of the definition of cumulant, one can easily show that the first-order cumulants of Gaussian states are given by

$$\kappa_1(\xi_j) = \langle d_j \rangle_{\hat{\rho}_{\mathrm{G}}}, \tag{6}$$

and the corresponding second-order cumulants are the vectors of the covariance matrix, i.e.,

$$\kappa_2(\xi_l \xi_m) = \mathrm{vec}(\boldsymbol{V}_{l,m}), \quad (l,m = 1,2,\ldots,2N), \tag{7}$$

where ξ_l denotes the l-th component of vector $\boldsymbol{\xi}$.

We see that the function $f(\boldsymbol{\xi})$ is of high relevance in deciding whether a quantum state is Gaussian or not. So we return to consider the non-Gaussianity of such a function, namely, its higher order cumulants. In order to calculate these cumulants, we can write the function $f(\boldsymbol{\xi})$ as a formal Taylor expansion

$$\begin{aligned} f(\boldsymbol{\xi}) = 1 &+ \sum_l a_l \xi_l + \frac{1}{2!}\sum_{l,m} a_{lm}\xi_l\xi_m \\ &+ \frac{1}{3!}\sum_{l,m,k} a_{lmk}\xi_l\xi_m\xi_k \\ &+ \frac{1}{4!}\sum_{l,m,k,h} a_{lmkh}\xi_l\xi_m\xi_k\xi_h + \cdots, \end{aligned} \tag{8}$$

where $a_{lmk\ldots}$ are the expansion coefficients of Eq. (8) and are in fact given by

$$a_l \equiv \mathbb{E}[\xi_l] = \left.\frac{\partial f(\boldsymbol{\xi})}{\partial \xi_l}\right|_{\boldsymbol{\xi}=0}, \tag{9a}$$

$$a_{lm} \equiv \mathbb{E}[\xi_l \xi_m] = \left.\frac{\partial^2 f(\boldsymbol{\xi})}{\partial \xi_l \partial \xi_m}\right|_{\boldsymbol{\xi}=0}, \tag{9b}$$

$$a_{lmk} \equiv \mathbb{E}[\xi_l \xi_m \xi_k] = \left.\frac{\partial^3 f(\boldsymbol{\xi})}{\partial \xi_l \partial \xi_m \partial \xi_k}\right|_{\boldsymbol{\xi}=0}, \tag{9c}$$

$$\cdots, \tag{9d}$$

where $\mathbb{E}[\,\cdot\,]$ denotes the expectation value.

Among these higher-order cumulants, the fourth-order cumulant or the kurtosis can be regarded as the classical measure of non-Gaussianity. The kurtosis of a zero-mean random variable y is classically defined by

$$\kappa_4(y) = \mathbb{E}\left[y^4\right] - 3\left(\mathbb{E}\left[y^2\right]\right)^2. \tag{10}$$

Thus, kurtosis is zero for a Gaussian random variable. For non-Gaussian random variables, kurtosis is nonzero. Kurtosis can be both positive or negative. Random variables that have a negative kurtosis are called sub-Gaussian, and those with positive kurtosis are called super-Gaussian. On the other hand, it is possible to determinate quasi-probability distribution in terms of probability distribution for the rotated quadrature phase.[35,36] Such a quadrature operator is defined as

$$\hat{X}(\theta) = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger \mathrm{e}^{\mathrm{i}\theta} + \hat{a}\,\mathrm{e}^{-\mathrm{i}\theta}\right). \tag{11}$$

For two-mode case, we propose a scheme for measuring quadrature distribution via two-mode homodyne tomography with a single local oscillator, as shown in Fig. 1.

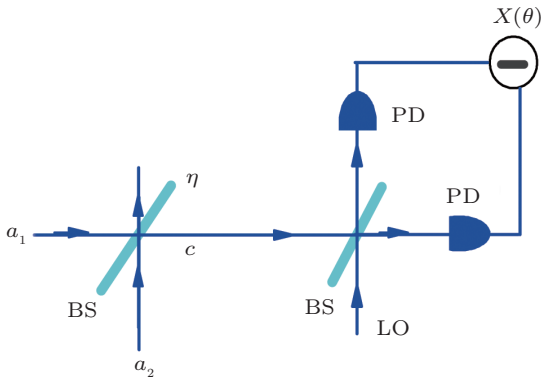


Fig. 1. (color online) Scheme of two-mode quadrature measurement. Mode $\hat{a}_1$ interacts with mode $\hat{a}_2$ through a beam splitter with reflectivity $\eta \in [0,1]$. After such a beam splitter, a probability distribution of a linear superposition of two single-mode quadratures is measured by means of standard single-mode homodyne detection. BS: beam splitter; LO: local oscillator; PD: photon-detectors.

Two light modes are expressed by the annihilations a_1 and a_2 and are mixed on a beam splitter with reflectivity η. After beam splitter, we have

$$\hat{c} = \sqrt{1-\eta}\hat{a}_1 + \sqrt{\eta}\hat{a}_2. \tag{12}$$

Mode c and a local-oscillator (LO) pulse are mixed at a beam splitter reflectivity η. Subsequently, the distribution of quadrature $X(\theta)$ is measured by means of standard single-mode homodyne detection. If we choose $\eta = \frac{1}{2}$, we have

$$\begin{aligned}\hat{X}(\theta) &= \frac{1}{\sqrt{2}}\left[\hat{c}^\dagger \mathrm{e}^{\mathrm{i}\theta} + \hat{c}\,\mathrm{e}^{-\mathrm{i}\theta}\right] \\ &= \frac{1}{\sqrt{2}}\left[(\hat{a}_1^\dagger + \hat{a}_2^\dagger)\,\mathrm{e}^{\mathrm{i}\theta} + (\hat{a}_1 + \hat{a}_2)\,\mathrm{e}^{-\mathrm{i}\theta}\right], \end{aligned} \tag{13}$$

where θ is the phase determined by a local oscillator. Thus a complete set of homodyne measurements over all phase angles $\theta \in [0, 2\pi]$ can be used to construct the density matrix

ρ or equivalently its phase-space function. That is, the measured quadrature distribution can contain the complete statistics information on a quantum state. Therefore, we introduce a phase-averaged kurtosis of two-mode quadrature $\hat{X}(\theta)$ as the non-Gaussianity measure of quantum state, which is defined as

$$\kappa_4(X) = \frac{1}{2\pi}\int_0^{2\pi}\left[\langle\hat{X}^4(\theta)\rangle - 3\langle\hat{X}^2(\theta)\rangle^2\right]\mathrm{d}\theta, \tag{14}$$

where the nth moment of operator $\hat{X}(\theta)$ is given by

$$\langle\hat{X}^n(\theta)\rangle = \sum_{j=0}^{n}\sum_{m=0}^{j}\sum_{l=0}^{n-j}\left(\frac{1}{\sqrt{2}}\right)^n \mathrm{e}^{\mathrm{i}(n-2j)\theta}\binom{n}{j}\binom{j}{l}\binom{n-j}{m}\left\langle :\hat{a}_1^{+l}\hat{a}_1^m\hat{a}_2^{+(j-l)}\hat{a}_2^{n-j-m}:\right\rangle, \tag{15}$$

where : : denotes the normal ordering and the terms $\langle : : \rangle$ can be obtained from Eq. (9).

3. Non-Gaussianity evolution of two-mode squeezed number states

3.1. Independent amplitude damping

It is well known that any quantum system is unavoidably influenced by the surrounding environment, thus giving rise to deteriorate the degree of non-classicality of quantum states. Let us consider a simple case where two modes are independently coupled to their own thermal environment with the same coupling constant γ, so the dynamics of the system is described by the following master equation in the interaction picture

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \sum_{j=1,2}\frac{\gamma}{2}\left\{\bar{n}_\mathrm{R}L[\hat{a}_j^\dagger] + (\bar{n}_\mathrm{R}+1)L[\hat{a}_j]\right\}\rho, \tag{16}$$

where $\bar{n}_\mathrm{R}$ stands for the average photon number of the reservoir and the Lindblad superoperator is defined as

$$L[\hat{O}]\rho = 2\hat{O}\rho\hat{O}^+ - \hat{O}^+\hat{O}\rho - \rho\hat{O}^+\hat{O}.$$

This model refers as the amplitude damping (AD).

Using standard operator correspondences we can transform the mater equation (16) into the Fokker–Planck equation:

$$\frac{\partial}{\mathrm{d}t}\chi(\xi_1,\xi_2;t) = -\frac{\gamma}{2}\sum_{j=1,2}\left[(2\bar{n}_\mathrm{R}+1)|\xi_j|^2 + \xi_j\frac{\partial}{\partial\xi_j} + \xi_j^*\frac{\partial}{\partial\xi_j^*}\right]\chi(\xi_1,\xi_2;t). \tag{17}$$

The interaction with reservoir leads to the time development

$$\chi(\xi_1,\xi_2;t) = \chi_{12}\left[\xi_1\exp\left(-\frac{1}{2}\gamma t\right),\xi_2\exp\left(-\frac{1}{2}\gamma t\right);0\right]\chi_\Gamma(\xi_1,\xi_2), \tag{18}$$

where $\chi_{12}(\xi_1,\xi_2;0)$ is the characteristic function at time $t=0$ and $\chi_\Gamma(\xi_1,\xi_2)$ is given by

$$\chi_\Gamma(\xi_1,\xi_2) = \prod_{j=1,2}\exp\left[-\left(\bar{n}_\mathrm{R}+\frac{1}{2}\right)\left(1-\mathrm{e}^{-\gamma t}\right)|\xi_j|^2\right]. \tag{19}$$

Assume that the system is initially in a two-mode squeezed number state, which is given by

$$\rho_\mathrm{sn} = S(r)|n,n\rangle\langle n,n|S^+(r), \tag{20}$$

where the two-mode squeezing operator is given by $S(r) = \exp\left[r(\hat{a}_1\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger)\right]$ with r being the squeezing degree and where $|n,m\rangle$ denotes the Fock state of two modes.

The characteristic function of the state (20) can be written as

$$\chi(\xi_1,\xi_2;0) = \exp\left[-\frac{1}{2}\cosh(2r)\left(|\xi_1|^2+|\xi_2|^2\right) - \frac{1}{2}\sinh(2r)(\xi_1\xi_2+\xi_1^*\xi_2^*)\right]L_n\left(|z_1|^2\right)L_n\left(|z_2|^2\right), \tag{21}$$

where $L_n(z)$ denotes a Laguerre polynomial, $z_l = \xi_l\cosh r + \xi_m^*\sinh r$, $l,m = 1,2$, $l\neq m$, and we have used the formula

$$\langle k|D(\xi)|l\rangle = \left(\frac{l!}{k!}\right)^{1/2}\xi^{k-l}\exp\left(-\frac{|\xi|^2}{2}\right)L_l^{(k-l)}(|\xi|^2).$$

From Eq. (18), we find that the characteristic function of two modes at time t becomes

$$\chi(\xi_1,\xi_2;t) = f_I(\xi_1,\xi_2;t)\exp\left\{\left[\left(\bar{n}_\mathrm{R}+\frac{1}{2}\right)(1-\mathrm{e}^{-\gamma t}) - \frac{1}{2}\mathrm{e}^{-\gamma t}\cosh^2(2r)\right]\left(|\xi_1|^2+|\xi_2|^2\right)\right.$$

$$-\frac{1}{2}\mathrm{e}^{-\gamma t}\sinh^2(2r)\left(\xi_1\xi_2+\xi_1^*\xi_2^*\right)\Big\}, \tag{22}$$

where the function $f(\xi_1,\xi_2;t)$ is given by

$$f_I(\xi_1,\xi_2;t)=\sum_{m_1,m_2=0}^{n}\sum_{k_1=0}^{m_1-l_1}\sum_{k_2=0}^{l_1}\sum_{q_1=0}^{m_2-l_2}\sum_{q_2=0}^{l_2}\frac{(-1)^{m_1+m_2}}{m_1!m_2!}\binom{n}{m_1}\binom{n}{m_2}\binom{m_1-l_1}{k_1}\binom{m_2-l_2}{q_1}\binom{l_1}{k_2}\binom{l_2}{q_2}$$
$$\cdot(\cosh r)^{l_1+l_2+2k_1+2q_1}(\sinh r)^{2m_1+2m_2-l_1-l_2-2k_1-2q_1}\,\xi_1^{A_{11}}(\xi_1^*)^{A_{12}}\,\xi_2^{A_{21}}(\xi_2^*)^{A_{22}}\,\mathrm{e}^{-(m_1+m_2)\gamma t}, \tag{23}$$

where

$$A_{11}=m_2-l_2-q_1+q_2+k_1+k_2, \tag{24a}$$
$$A_{12}=m_2+l_1-q_1-q_2+k_1-k_2, \tag{24b}$$
$$A_{21}=m_1-l_1+q_1+q_2-k_1+k_2, \tag{24c}$$
$$A_{22}=m_1+l_2+q_1-q_2-k_1-k_2, \tag{24d}$$

and where the Newton binomial

$$\binom{n}{m}=\frac{n!}{m!(n-m)!}.$$

In the light of the discussion above and using Eqs. (8), (9), and (15), we can obtain the phase-averaged kurtosis for operator $\hat{X}(\theta)$ at time t as

$$\kappa_4(X)=-\frac{3}{4}n\left[(9n+1)\cosh^2(2r)+(n+5)\sinh^2(2r)-4n\cosh(2r)-n+3\right]\mathrm{e}^{-2\gamma t}. \tag{25}$$

Let us first study a case where the system does not interact with the environment. As seen from Eq. (25) that if the system is initially in a two-mode number state, we can calculate the phase-averaged kurtosis of operator $\hat{X}(\theta)$ as $\kappa_4(X)=-3n(n+1)$, that is, it is a sum of kurtosis for two single-mode Fock states obtained in Ref. [35]. When two modes are initially in two-mode squeezed vacuum state, i.e., $n=0$, this phase-averaged kurtosis naturally vanishes, as expected. We now focus on the dynamics of this phase-averaged kurtosis of two-mode squeezed number state evolving in the independent amplitude damping, as shown in Fig. 2. We see that under the amplitude damping, the phase-averaged kurtosis for two-mode operator $\hat{X}(\theta)$ shows an exponential law decay behavior and ultimately vanishes completely in a finite interaction time, meaning that the non-Gaussian correlations in two-mode squeezed number state disappear. To further see this point, we can calculate all other higher-order cumulants of such an operator at any time as $\kappa_m(X;t)=\mathrm{e}^{-m\gamma t/2}\kappa_m(X)$, in which $\kappa_m(X)$ is the m-th order cumulant of the two-mode phase-averaged quadrature operator at the initial time. It tells again that the cumulant-based non-Gaussianity is very fragile against the local amplitude damping decoherence, which can drive non-Gaussian states into Gaussian ones and makes the states more Gaussian. This process is called a Gaussification one.[37] Additionally, we observe from Fig. 2 that increasing the squeezing degree and the photon numbers of two modes have the same dynamical behavior of the $\kappa_4(X)$ and can lead to higher non-Gaussian correlations of quantum states. For instance, for a case of $r=0.8$ and $n=4$, one gets $\kappa_4(X)|_{\tau=0.5}=-372.53$, while for $r=1.2$ and $n=10$, $\kappa_4(X)|_{\tau=0.5}=-1.134\times10^4$ where $\tau=1-\mathrm{e}^{-2\gamma t}$, which is 0 for $t=0$ and 1 for $t=\infty$. That is to say, they can improve the robustness of non-Gaussianity.

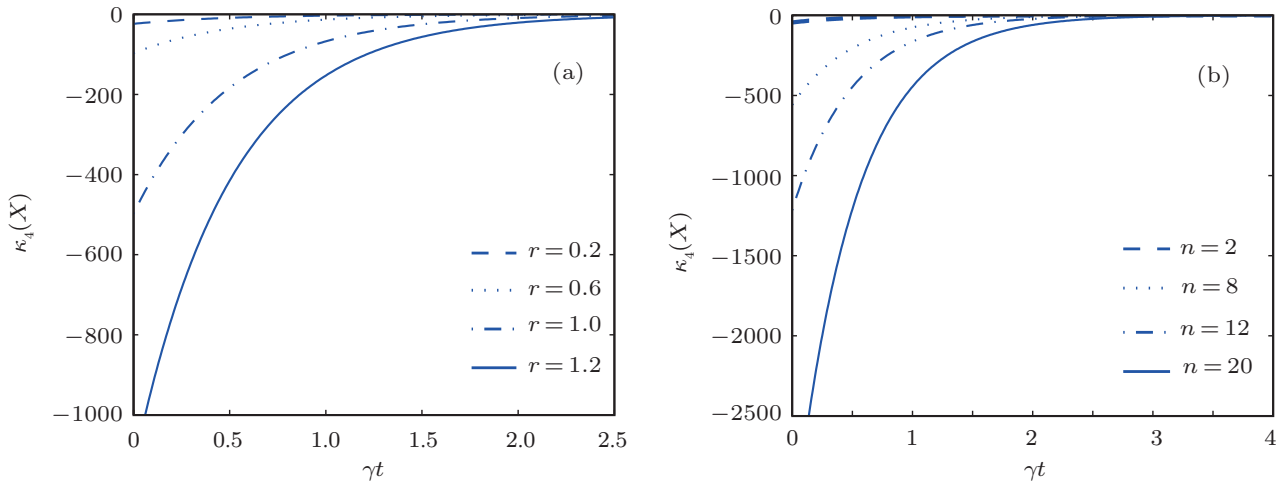


Fig. 2. (color online) The phase-averaged kurtosis of $\hat{X}(\theta)$ as a function of interaction time γt for (a) various squeezing parameters r at $n=2$ and for (b) different photon numbers n at $r=0.4$ when two modes are initially in a squeezed number state under the independent amplitude damping model.

3.2. Independent phase damping

Let us return to another decoherence model: phase damping (PD), whose mater equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \sum_{j=1,2}\frac{\gamma}{2}\left[2\hat{a}_j^{\dagger}\hat{a}_j\rho\hat{a}_j^{\dagger}\hat{a}_j - (\hat{a}_j^{\dagger}\hat{a}_j)^2\rho - \rho(\hat{a}_j^{\dagger}\hat{a}_j)^2\right]. \tag{26}$$

Repeating the same procedure as earlier, we can obtain the characteristic function of the evolved state as[38]

$$\chi(\xi_1,\xi_2,t) = \int \frac{1}{2\pi\gamma t}\exp\left(-\sum_{j=1,2}\frac{x_j^2}{2\gamma t}\right)\chi(\{\xi_j\,\mathrm{e}^{\mathrm{i}x_j}\},0)\,\mathrm{d}\{x\}, \tag{27}$$

where $\{\xi_j\,\mathrm{e}^{\mathrm{i}x_j}\} \equiv (\xi_1\,\mathrm{e}^{\mathrm{i}x_1}, \xi_1^*\,\mathrm{e}^{-\mathrm{i}x_1}, \xi_2\,\mathrm{e}^{\mathrm{i}x_2}, \xi_2^*\,\mathrm{e}^{-\mathrm{i}x_2})$ and $\mathrm{d}\{x\} \equiv \mathrm{d}x_1\,\mathrm{d}x_2$.

Using the integration

$$\int_{-\infty}^{\infty}\mathrm{e}^{-q^2x^2}\sin[p(x+\lambda)]\,\mathrm{d}x = \frac{\sqrt{\pi}}{q}\exp\left(-\frac{p^2}{4q^2}\right)\sin(p\lambda), \tag{28a}$$

$$\int_{-\infty}^{\infty}\mathrm{e}^{-q^2x^2}\cos[p(x+\lambda)]\,\mathrm{d}x = \frac{\sqrt{\pi}}{q}\exp\left(-\frac{p^2}{4q^2}\right)\cos(p\lambda), \tag{28b}$$

we can easily obtain the characteristic function of the state (16) subject to PD decoherence as

$$\chi(\xi_1,\xi_2;t) = f_{\mathrm{II}}(\xi_1,\xi_2;t)\exp\left\{-\frac{1}{2}\cosh(2r)\left[|\xi_1|^2+|\xi_2|^2\right]\right\}, \tag{29}$$

where the function $f_{II}(\xi_1,\xi_2;t)$ is given by

$$\begin{aligned} f_{\mathrm{II}}(\xi_1,\xi_2;t) = &\sum_{g_1,g_2=0}^{\infty}\sum_{m_1,m_2=0}^{n}\sum_{k_1=0}^{m_1-l_1}\sum_{k_2=0}^{l_1}\sum_{q_1=0}^{m_2-l_2}\sum_{q_2=0}^{l_2}\frac{(-1)^{m_1+m_2}}{m_1!m_2!}\binom{n}{m_1}\binom{n}{m_2}\binom{m_1-l_1}{k_1}\binom{m_2-l_2}{q_1}\binom{l_1}{k_2}\binom{l_2}{q_2}\\ &\times 2^{-g_1-g_2}(\cosh r)^{l_1+l_2+2k_1+2q_1}(\sinh r)^{2m_1+2m_2+g_1+g_2-l_1-l_2-2k_1-2q_1}\,\xi_1^{A_{11}+g_1}(\xi_1^*)^{A_{12}+g_2}\,\xi_2^{A_{21}+g_1}(\xi_2^*)^{A_{22}+g_2}\\ &\times\exp\left(-\frac{1}{2}h^2\gamma t\right), \end{aligned} \tag{30}$$

where $h = 2q_2+2k_2+g_1-g_2-l_1-l_2$.

Furthermore, we can obtain the phase-averaged kurtosis of two-mode quadrature operator $\hat{X}(\theta)$ as

$$\kappa_4(X) = \frac{1}{4}\left[\left(27n^2+3n+3\right)\cosh(4r) - 48n^2\cosh^2(2r) + 9n^2 - 15n - 3 - 6\left(1+4n+4n^2\right)\sinh^2(2r)\,\mathrm{e}^{-2\gamma t}\right]. \tag{31}$$

As seen from Eq. (31), when the two modes are initially in a Fock state, the averaged-phase kurtosis remains unchanged and equals to a sum of that of two individual Fock states, i.e., $\kappa_4(X) = -3n(n+1)$, implying that there exists the possibility of non trivial phenomenon of time-invariant non-Gaussianity for two-mode quantum system in initially a number state. For the case of the system in being squeezed number state, we see that in the long-time limit, the $\kappa_4(X)$ does not vanish and approaches asymptotically to a nonzero steady value, which is determined by the squeezing parameters and photon numbers of the initial quantum state. Especially we have $\kappa_4(X)|_{t\to\infty} = (3/2)\sinh^2(2r)$ in the long-time limit when the initial condition is the known two-mode squeezed vacuum state of the system. We plot the dynamics of the averaged-phase kurtosis of two-mode quadrature operator $\hat{X}(\theta)$ when the system is initially in two-mode squeezed number state evolving in the local dephasing environment, as shown in Fig. 3. It can be seen that if the mean initial intensity $\langle\hat{a}_1^{\dagger}\hat{a}_1+\hat{a}_2^{\dagger}\hat{a}_2\rangle = 2n\cosh(2r)$ is enough small, the presence of dephasing environment almost have no great influence to the evolution of the $\kappa_4(X)$; on the contrary, one sees that the bigger the mean initial intensity is, the more pronounced the variation of non-Gaussianity becomes. Or more accurately the $\kappa_4(X)$ has a linear-dynamics behavior in the short-time limit, while at other evolution times, it exhibits an exponential-law behavior. For example, with Eq. (31), we can obtain the time-dependent phase-averaged kurtosis as $\kappa_4(X) = (3/2)\sinh^2(2r)\left(1-\mathrm{e}^{-2\gamma t}\right)$ when taking into account two-mode squeezed vacuum state as initial state. Additionally, it is interesting to see that the independent phase damping can lead to flip the sign of the $\kappa_4(X)$ of the evolved state, namely, the transition from sub-Gaussian state to super-

Gaussian one. We can obtain easily this transition time

$$t_{\rm f}=\frac{1}{2\gamma}\ln\left[\frac{(1+4n+4n^2)\sinh^2(2r)}{(n^2+n+1)\sinh^2(2r)-2n(n+1)}\right]$$

in such a way that the $\kappa_4(X)$ will be positive if the evolution time $t>t_f$; otherwise, it is negative.

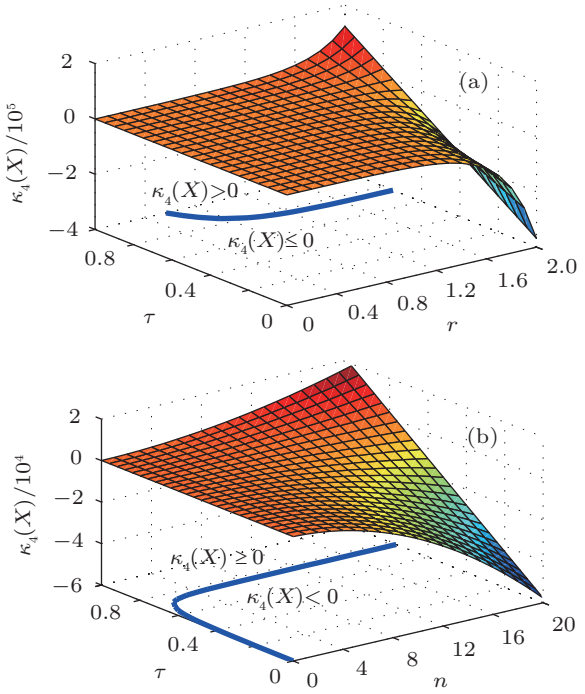


Fig. 3. (color online) The dynamics of $\kappa_4(X)$ for two-mode squeezed number state as a function of (a) the squeezing parameter at $n=10$ and (b) photon numbers at $r=1.2$ under the independent phase damping, where time $\tau=1-{\rm e}^{-2\gamma t}$. We also present the crossover lines for the negative and positive regimes of the phase-averaged kurtosis.

We end this section with the following remarks. First, quantum homodyne tomographies are an important topic in the field of quantum optics and quantum information science. Especially, it is a desirable research how to reconstruct quantum state using by a simple experimental arrangement. In our two-mode measurement scheme, it is consist of two balanced lossless beam splitters, two photo-detectors and a strong coherent local oscillator (LO), thus greatly simplifying the complexity of the experiment. Second, we propose an one-parameter measure to characterize the non-Gaussian characters of a multimode continuous-variable system: the phase-averaged kurtosis. Working with our method can have an advantage over the method in Ref. [39] where one needs a lots of the joint cumulants to completely characterize the non-Gaussianity of a bosonic quantum state. It was found that among these cumulants, some are equivalent and redundant. But we are able to circumvent this obstruction in our work and more importantly, one removes the terms with the phase of local oscillator in Eq. (15) thanks to the phase-averaged technique, i.e., greatly simplifying the complexity of the complication. Third, unlike existing measures of non-Gaussianity,[6,7] the cumulant-based non-Gaussianity is aim at quantify the degree of peak of phase-space distribution for quantum state, so avoiding the selection of the exact reference Gaussian state. Finally, we analysis the dynamical difference of the phase-averaged kurtosis in two different decoherence models. As is well known, the amplitude damping channel is a quantum operation that describes the energy dissipation effects due to the loss of energy in a quantum system.[40] Such a channel is viewed as a Gaussian noise that can drive a quantum state into Gaussian thermal state in the long-time limit. Thus, it is valid for the case of two-mode squeezed number input state. While the phase damping channel describes a quantum map with loss of quantum phase information without loss of energy. It is regarded as a non-Gaussian noise, whose statistics may be generally expected away from thermal equilibrium. As a result, the dephase damping makes the system evolving towards a non-Gaussian state in the long-time limit.

4. Comparison with other measures

As stated in Refs. [6], [7], and [37], the geometric and entropic measures of non-Gaussianity have the same basic properties and also share the same qualitative behavior under Gaussification and de-Gaussification processes. So in this section, we only compare our phase-averaged kurtosis with the Hilbert-Schmidt measure of non-Gaussianity introduced in Ref. [6], where the non-Gaussianity of a state ρ is defined as

$$\delta_{\rm HS}[\rho]=\frac{{\rm Tr}[(\rho-\rho_{\rm G})^2]}{2\,{\rm Tr}[\rho^2]}=\frac{\mu[\rho]+\mu[\rho_{\rm G}]-2\kappa[\rho,\rho_{\rm G}]}{2\mu[\rho]},\quad(32)$$

where $\mu[\rho]$ is the purity of the state ρ, $\kappa[\rho,\rho_{\rm G}]$ denotes the overlap of ρ and $\rho_{\rm G}$, and $\rho_{\rm G}$ is a reference Gaussian state with the same vector and same covariance matrix as ρ under investigation, namely, $\boldsymbol{X}[\rho]=\boldsymbol{X}[\rho_{\rm G}]$ and $\boldsymbol{\sigma}[\rho]=\boldsymbol{\sigma}[\rho_{\rm G}]$.

To simplify our analysis, we focus our attention on the second noise model. For the matrix elements of a two-mode state: $\rho_{nn;mm}=\langle m,m|\rho|n,n\rangle$, we can obtain the solution of Eq. (26) as follows:

$$\rho_{nn;mm}(t)={\rm e}^{-\gamma t(n-m)^2}\rho_{nn;mm}(0).\quad(33)$$

To evaluate the non-Gaussianity of the damped state, we can write the state (20) in the two-mode Fock basis as[26]

$$|n,n\rangle_r=S(r)|n,n\rangle=\sum_l b_l|l,l\rangle,\quad(34)$$

where $S(r)$ is a two-mode squeezing operator as before and the expansion coefficients b_l are given by

$$b_l=\frac{n!l!\,[\tanh r]^{n+l}}{\cosh r}\sum_{k=0}^{\min(n,l)}\frac{(-1)^{l-k}(\sinh r)^{-2k}}{(k!)^2(l-k)!(n-k)!}.\quad(35)$$

According to Eqs. (33) and (34), we can calculate the purity of the damped state

$$\mu[\rho](t)=\sum_{l=0}^{\infty}b_j^4+2\sum_{l,m=0}^{\infty}b_l^2b_m^2\,{\rm e}^{-2\gamma t(l-m)^2},\quad(36)$$

and the purity of the reference Gaussian state

$$\mu[\rho_{\rm G}](t)=\frac{1}{4\sqrt{[f^2-g^2(t)][f^2-h^2(t)]}}, \tag{37}$$

where

$$f=\sum_{l=0}^{\infty} lb_l^2+\frac{1}{2}, \tag{38a}$$

$$g(t)=\sum_{l=1}^{\infty} lb_l b_{l-1}\,{\rm e}^{-\gamma t}, \tag{38b}$$

$$h(t)=\sum_{l=0}^{\infty}(l+1)b_l b_{l+1}\,{\rm e}^{-\gamma t}. \tag{38c}$$

The overlap of the state under examination and the reference Gaussian state is evaluated in the phase space by means of the characteristic functions

$$\kappa[\rho,\rho_{\rm G}](t)=\frac{1}{\pi^2}\iint {\rm d}^2\xi_1\,{\rm d}^2\xi_2\chi[\rho](\xi_1,\xi_2;t)\chi[\rho_{\rm G}](\xi_1,\xi_2;t), \tag{39}$$

where the characteristic functions $\chi[\rho](\xi_1,\xi_2;t)$ is given by Eq. (29) and since the vector is zero, the $\chi[\rho_{\rm G}](\xi_1,\xi_2;t)$ of the reference Gaussian state can be written as

$$\chi[\rho_{\rm G}](\xi_1,\xi_2;t)=\exp\left\{-\frac{1}{2}\left[f\left(|\xi_1|^2+|\xi_2|^2\right)+g(t)\xi_1\xi_2+h(t)\xi_1^*\xi_2^*\right]\right\}. \tag{40}$$

We have used Eqs. (35)–(40) to evaluate numerically two measures of non-Gaussianity we are interested in $\delta_{\rm HS}$ and $\kappa_4(X)$ via the formulas (31) and (32) in the specific case of two-mode squeezed number subject to the independent phase damping evolution. The results are shown in Fig. 4.

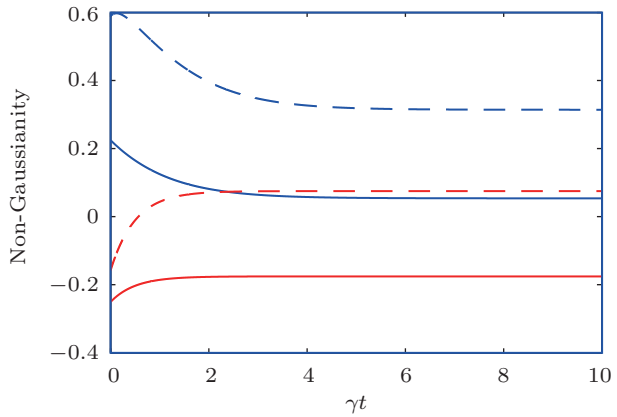


Fig. 4. (color online) Time evolution of the phase-averaged kurtosis (red line) and the Hilbert-Schmidt degree of non-Gaussianity (blue line) under the independent phase damping model. The input state is a two-mode squeezed number state. Solid line: $N=2$, $r=0.1$; dotted line: $N=1$, $r=0.6585$. Throughout, the phase-averaged kurtosis is divided by a hundred for clarity.

It is seen that these two measures display the similar asymptotic behaviors of of non-Gaussianity. Namely, there exists a nonzero asymptotic value of the non-Gaussianity in the long-time limit, which is in very good agreement with the results obtained in Ref. [7] in the phase-diffusion evolution. However, they have a striking difference. One sees that the Hilbert–Schmidt degree of non-Gaussianities monotonically decreases with the evolution time, implying that the phase-damping noise has a merely detrimental effect on the $\delta_{\rm HS}$, in accordance with the corresponding property in the presence of the thermal noise reported in Ref. [37]. But this is not the case with our introduced measure. It is a complicated problem. We see that the kurtosis begins to decrease until it reaches zero, then increases and eventually approaches to the asymptotic value (see the red dashed line). This difference is due to the fact that the non-Gaussianity measure of Genoni *et al.* is based on the distance between the state ρ and the reference Gaussian state. So the key step is to choose the correct reference Gaussian state; if not, it will lead to misleading results in assessing which the quantum state is the most non-Gaussian. It is challenging task. On the contrary, our measure rests on the shape of the phase-space distribution of quantum state itself and hence can quantify the non-Gaussianity in a safe manner. Of course, how to quantify non-Gaussianity for quantum state by using a justified measure is our work in the future research.

5. Conclusions

In conclusion, we have proposed a method to characterize non-Gaussianity of multi-mode continuous-variable quantum state based on the cumulants. Unlike the other geometric and entropic distances, our measure only need to focus on the non-Gaussian components of the characteristic function of quantum states, thus greatly simplifying the process of calculation. Using the proposed measure, we have study the non-Gaussianity evolution of two-mode squeezed number states subject to two sources of decoherence: i) the amplitude damping and ii) the phase damping due to the coupling to the thermal environment. Our analysis shows that in the amplitude damping model, the non-Gaussianity can rapidly vanish at a finite time, while in the phase damping model, such a quantum state can transform from sub-Gaussianity into super-Gaussianity during the evolution. We hope that our work will be helpful for deeply understanding the non-Gaussianity of other quantum states and may be have potential applications in quantum entanglement distillation and quantum informa-

tion processing.

References

[1] Yang Y and Li F L 2009 *Phys. Rev. A* **80** 022315
[2] Jiang K X 2013 *Opt. Commun.* **300** 286
[3] Lasota M, Filip R and Usenko V C 2016 *Phys. Rev. A* **95** 062312
[4] Bednorz A and Belzig W 2011 *Phys. Rev. B* **83** 125304
[5] Straka I, Predojevic A, Huber T, Lachman L, Butschek L, Mikova M, Micuda M, Solomon G S, Weihs G, Jezek M and Filip R 2014 *Phys. Rev. Lett.* **113** 223603
[6] Genoni M G, Paris M G A and Banaszek K 2007 *Phys. Rev. A* **76** 042327
[7] Genoni M G and Paris M G A 2010 *Phys. Rev. A* **82** 052341
[8] Tang X B, Gao F, Wang Y X, Kuang S and Shuang F 2015 *Chin. Phys. B* **24** 034208
[9] Taghiabadi R, Akhtarshenas S J and Sarbishaei M 2016 *Quantum Infor. Process.* **15** 1999
[10] Park J, Lee J, Ji S W and Nha H 2017 *Phys. Rev. A* **96** 052324
[11] Thiele T N 1903 *Theory of Observations* (London: Charles and Edwin Layton)
[12] Fisher R A and Wishart J 1932 *Proc. London Math. Soc.* **2** 199
[13] Kendall M G and Stuart A 1963 *The Advanced Theory of Statistics* (London: Charles Griffin and Company Limited)
[14] Huber P J 1985 *Projection pursuit, Annal. Stat.* **13** 435
[15] Mardia K V 1970 *Biometrika* **57** 519
[16] Oja H 1983 *Statis. Probab. Lett.* **1** 327
[17] Looney S W 1995 *American Statis.* **49** 64
[18] Mustafa H and Doroslovacki M 2008 *Eur. Trans. Telecomms.* **19** 23
[19] Wunsche A 2015 *Phys. Scr.* **90** 07400323
[20] Dubost B, Koschorreck M, Napolitano M, Behbood N, Sewell R J and Mitchell M W 2012 *Phys. Rev. Lett.* **108** 183602
[21] Moreno-Cardoner M, Sherson J F, and De Chiara G 2016 *New J. Phys.* **18** 103015
[22] Olsen M K and Corney J F 2013 *Phys. Rev. A* **87** 033839
[23] Zhuang Q T, Peter Shor P W and Shapiro J H 2018 *Phys. Rev. A* **97** 052317
[24] Takagi R and Zhuang Q T 2018 arXiv: 1804.04669v2 [quant-ph]
[25] Albarelli F, Genoni M G and Paris M G A 2018 arXiv: 1804.05763v1 [quant-ph]
[26] Chizhow A V and Murzakhmetov B K 1993 *Phys. Lett. A* **176** 33
[27] Dell'Anno F, Siena S De and Illuminati F 2006 *Open Sys. and Inform.* **13** 383
[28] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882
[29] Barnett S M and Radmore P M 1997 *Methods in Theoretical Quantum Optics* (Oxford: Oxford University Press)
[30] Husimi K 1940 *Proc. Phys. Math. Soc. Jpn.* **22** 264
[31] Wigner E P 1932 *Phys. Rev.* **40** 749
[32] Glauber R J 1963 *Phys. Rev.* **131** 2766
[33] Adesso G, Serafini A and Illuminati F 2006 *Phys. Rev. A* **73** 032345
[34] Adesso G 2011 *Phys. Rev. A* **83** 024301
[35] Vogel K and Risken H 1989 *Phys. Rev. A* **40** 2847
[36] Villas-Boas C J, Prataviera G A and Moussa M H Y 2001 *Phys. Rev. A* **64** 065801
[37] Marian P, Ghiu I and Marian T A 2013 *Phys. Rev. A* **88** 012316
[38] Chen X Y 2006 *Phys. Rev. A* **73** 022307
[39] Jiang L Z 2010 *Proc. SPIE* **7846** 784612
[40] Nielsen M and Chuang I 2000 *Quantum Computation and Quantum communication* (Cambridge: Cambridge University Press)