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Extended Bell inequality and maximum violation*

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The original formula of Bell inequality (BI) in terms of two-spin singlet has to be modified for the entangled-state with parallel spin polarization. Based on classical statistics of the particle-number correlation, we prove in this paper an extended BI, which is valid for two-spin entangled states with both parallel and antiparallel polarizations. The BI and its violation can be formulated in a unified formalism based on the spin coherent-state quantum probability statistics with the state-density operator, which is separated to the local and non-local parts. The local part gives rise to the BI, while the violation is a direct result of the non-local quantum interference between two components of entangled state. The Bell measuring outcome correlation denoted by $P_{\rm B}$ is always less than or at most equal to one for the local realistic model ($P_{\rm B}^{lc} \leq 1$) regardless of the specific superposition coefficients of entangled state. Including the non-local quantum interference the maximum violation of BI is found as $P_{\rm B}^{\max} = 2$, which, however depends on state parameters and three measuring directions as well. Our result is suitable for entangled photon pairs.

Keywords: Bell inequality, quantum entanglement, non-locality, spin coherent state

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1. Introduction

The non-locality is the most striking characteristic of quantum mechanics beyond our intuition of space and time in the classical field theory.[1–3] It has no classical counterpart and therefore has been receiving continuously theoretical attention ever since the birth of quantum mechanics. The two-particle entangled-state as a typical example of non-locality was originally considered by Einstein–Podolsky–Rosen to question the complicity of quantum mechanics. It has become the essential ingredients in quantum information and computation.[4–6] The quantum nonlocal correlation by local measurements on distant parts of a quantum system is a consequence of entanglement, which is incompatible with local hidden variable models.[7] This was discovered by Bell, who further established a theorem known as Bell inequality (BI)[7] to provide a possibility of quantitative test for non-local correlations, which lead necessarily to the violation of BI. The overwhelming experimental evidence[8–13] for the violation of BI in some entangled-states invalidates local realistic interpretations of quantum mechanics. Various extensions of the original BI have proposed from both theoretical and experimental viewpoints.[14–21] The nonlocality has been also justified undoubtedly in various aspects.[14,15] Soon after the pioneer work of Bell, Clauser–Horne–Shimony–Holt (CHSH) formulated a modified form of the inequality,[22] which is more suitable for the quantitative test and therefore attracts most attentions of experiments. An alternative inequality for the local realistic model was formulated by Wigner[23–25] known as Wigner inequality (WI), which needs measurements of particle number probabilities only along one direction of spin-polarization. It is assumed that the joint probability distributions for measuring outcomes satisfy the locality condition[26] in the underlying stochastic hidden variable space. The experimental evidence strongly supports the quantum non-locality, however the underlying physical-principle is obscure.[27] Various aspects relating to the initial debate remain to be fully understood.

In order to have a better understanding of the underlying physics we in previous publications[28,29] formulated the BI and its violation in a unified formalism by means of the spin coherent-state quantum probability statistics along with the assumption of measurement-outcome-independence. The density operator of a bipartite entangled-state can be separated into the local and non-local parts,[29] with which the measuring outcome correlation is then evaluated by the quantum probability statistics in the spin coherent-state base vectors. The local part of density operator gives rise to the BI, while its violation is a direct result of non-local correlations of entangled states.[29] We predicted a spin parity effect[28] in the violation of BI, which is violated by the entangled-states of

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half-integer but not the integer spins. It was moreover demonstrated that the violation is seen to be an effect of Berry phase induced by relative-reversal measurements of two spins.

The original formula of BI is actually valid for arbitrary two-spin entangled-state with antiparallel polarization[30] beyond the singlet state. However, it has to be modified by the change of a sign for the parallel spin polarization.[29] It is an interesting question whether or not a unified inequality exists for both antiparallel and parallel spin-polarizations. It is the main goal of the present paper to establish an extended BI valid for two kinds of entangled states. Following the recent work for the maximum violation[31] of WI, the maximum violation bound of the BI is also obtained to demonstrate a fact that the BI with three-direction measurements is equally convenient as CHSH inequality for the experimental test. A loophole-free experimental verification of the violation of CHSH inequality was reported recently by means of electronic spin associated with a single nitrogen-vacancy defect center in a diamond chip[32,33] and also for the two-photon entangled states with mutually perpendicular polarizations.[34] The formalism and results in the present paper are also suitable for the entangled photon pairs.

2. Spin coherent-state quantum probability statistics and BI

In our formalism the Bell-type inequalities and their violation are formulated in a unified manner by means of the spin coherent-state quantum probability statistics.[28,29] We begin with an arbitrary two-spin entangled state with antiparallel polarization

$$|\psi\rangle = c_1|+,-\rangle + c_2|-,+\rangle, \tag{1}$$

where $|\pm\rangle$ are considered as the usual spin-1/2 eigenstates ($\hat{\sigma}_z|\pm\rangle = \pm|\pm\rangle$). The normalized coefficients can be parameterized as $c_1 = \mathrm{e}^{\mathrm{i}\eta}\sin\xi$, $c_2 = \mathrm{e}^{-\mathrm{i}\eta}\cos\xi$ in terms of the arbitrary real parameters η, ξ. The density operator of an entangled state can be separated to the local (or classical) and non-local (or quantum coherent) parts such that

$$\hat{\rho} = \hat{\rho}_{lc} + \hat{\rho}_{nlc}. \tag{2}$$

The local part

$$\hat{\rho}_{\mathrm{lc}} = \sin^2\xi|+,-\rangle\langle+,-| + \cos^2\xi|-,+\rangle\langle-,+|, \tag{3}$$

which is the classical two-particle probability-density operator, gives rise to the local realistic bound of measuring outcome correlation, namely the BIs. While the non-local part

$$\hat{\rho}_{\mathrm{nlc}} = \sin\xi\cos\xi\left(\mathrm{e}^{2\mathrm{i}\eta}|+,-\rangle\langle-,+| + \mathrm{e}^{-2\mathrm{i}\eta}|-,+\rangle\langle+,-|\right)$$

describing the quantum coherence between two remote spins results in the violation of the BIs.[28,29] For the entangled state of parallel polarization

$$|\psi\rangle = c_1|+,+\rangle + c_2|-,-\rangle,$$

the local and non-local parts of density operator become[29]

$$\hat{\rho}_{\mathrm{lc}} = \sin^2\xi|+,+\rangle\langle+,+| + \cos^2\xi|-,-\rangle\langle-,-|, \tag{4}$$

and

$$\hat{\rho}_{\mathrm{nlc}} = \sin\xi\cos\xi\left(\mathrm{e}^{2\mathrm{i}\eta}|+,+\rangle\langle-,-| + \mathrm{e}^{-2\mathrm{i}\eta}|-,-\rangle\langle+,+|\right)$$

respectively.

2.1. Spin measuring outcome correlation and BI

We assume to measure two spins independently along two arbitrary directions, say $\boldsymbol{a}$ and $\boldsymbol{b}$. Each measuring outcome falls necessarily into the eigenvalues of projection spin-operators $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{a}$ and $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}$, i.e.,

$$\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{a}|\pm\boldsymbol{a}\rangle = \pm|\pm\boldsymbol{a}\rangle, \quad \hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|\pm\boldsymbol{b}\rangle = \pm|\pm\boldsymbol{b}\rangle,$$

according to the quantum measurement theory. Solving the above eigenvalue equations for each direction $\boldsymbol{r}$ ($\boldsymbol{r} = \boldsymbol{a}, \boldsymbol{b}$) we obtain two orthogonal eigenstates given by

$$\begin{aligned} |+\boldsymbol{r}\rangle &= \cos\frac{\theta_r}{2}|+\rangle + \sin\frac{\theta_r}{2}\mathrm{e}^{\mathrm{i}\phi_r}|-\rangle, \\ |-\boldsymbol{r}\rangle &= \sin\frac{\theta_r}{2}|+\rangle - \cos\frac{\theta_r}{2}\mathrm{e}^{\mathrm{i}\phi_r}|-\rangle, \end{aligned} \tag{5}$$

where the unit vector $\boldsymbol{r} = (\sin\theta_r\cos\phi_r, \sin\theta_r\sin\phi_r, \cos\theta_r)$ is parameterized with the polar and azimuthal angles θ_r, ϕ_r. The two orthogonal states $|\pm\boldsymbol{r}\rangle$ are known as spin coherent states of north- and south-pole gauges.[35–37] The eigenstates of projection spin-operators $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{a}$ and $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}$ form a measuring-outcome independent vector base for two spins measured respectively along the $\boldsymbol{a}$, $\boldsymbol{b}$ directions. The four base vectors are labeled as

$$\begin{aligned} |1\rangle &= |+\boldsymbol{a},+\boldsymbol{b}\rangle, \quad |2\rangle = |+\boldsymbol{a},-\boldsymbol{b}\rangle, \\ |3\rangle &= |-\boldsymbol{a},+\boldsymbol{b}\rangle, \quad |4\rangle = |-\boldsymbol{a},-\boldsymbol{b}\rangle \end{aligned} \tag{6}$$

for the sake of simplicity. The measuring outcome correlation[29,30] is obviously

$$P(a,b) = \mathrm{Tr}[\hat{\Omega}(a,b)\hat{\rho}] = \rho_{11} - \rho_{22} - \rho_{33} + \rho_{44}, \tag{7}$$

where

$$\hat{\Omega}(ab) = (\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{a})(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b})$$

is the spin correlation operator and $\rho_{ii} = \langle i|\hat{\rho}|i\rangle$ ($i = 1,2,3,4$) denote matrix elements of the density operator. The measuring outcome correlation can be also separated to local and non-local parts

$$P(a,b) = P_{\mathrm{lc}}(a,b) + P_{\mathrm{nlc}}(a,b),$$

with

$$P_{\rm lc}(a,b) = {\rm Tr}[\hat{\Omega}(a,b)\hat{\rho}_{\rm lc}],$$

and

$$P_{\rm nlc}(a,b) = {\rm Tr}[\hat{\Omega}(a,b)\hat{\rho}_{\rm nlc}].$$

Submitting the local parts of density operators Eqs. (3) and (4) into the local measuring-outcome correlation $P_{\rm lc}(a,b)$ we have

$$P_{\rm lc}(a,b) = \mp\cos\theta_a\cos\theta_b,$$

respectively for the antiparallel and parallel spin polarizations. The BI becomes correspondingly[28,29]

$$1 \pm P_{\rm lc}(b,c) \geq |P_{\rm lc}(a,b) - P_{\rm lc}(a,c)|$$

for the antiparallel and parallel entangled states.

2.2. Particle-number correlation probability

In the Wigner formalism[31] the particle-number correlation probability is considered instead of the spin measuring-outcome correlation. The quantity defined by

$$\begin{aligned} N(+a,+b) &= |\langle +a,+b|\psi\rangle|^2 \\ &= \langle +a,+b|\hat{\rho}|+a,+b\rangle = \rho_{11} \end{aligned} \tag{8}$$

describes the particle-number correlation probability for two positive-spin particles measured respectively along $\boldsymbol{a}$, $\boldsymbol{b}$ directions. Correspondingly we have

$$\begin{aligned} &N(+a,-b) = \rho_{22}, \quad N(-a,+b) = \rho_{33}, \\ &N(-a,-b) = \rho_{44}, \end{aligned} \tag{9}$$

which are all positive quantities different from the spin measuring-outcome correlations. The spin measuring-outcome correlation $P(a,b)$ in the BI are related to the four particle-number correlation probabilities by

$$P(a,b) = N(+a,+b) - N(+a,-b) - N(-a,+b) + N(-a,-b), \tag{10}$$

which is the difference between the particle number probabilities of same direction measurement and that of opposite directions.

3. Extended BI and maximum violation

The extended BI for both parallel and antiparallel polarizations is obviously

$$1 + |P_{\rm lc}(b,c)| \geq |P_{\rm lc}(a,b) - P_{\rm lc}(a,c)|,$$

for the local model, since

$$1 + |P_{\rm lc}(b,c)| \geq 1 \pm P_{\rm lc}(b,c).$$

We define a quantum Bell correlation probability (QBCP) that

$$P_B = |P(a,b) - P(a,c)| - |P(b,c)|.$$

The extended BI is then

$$P_B^{\rm lc} \leq 1,$$

which is violated once $P_B > 1$.

In Appendix A we specifically present a simple proof of the validity of the extended BI, in terms of the classical statistics with the particle-number correlation probabilities in the Wigner formulation.[23–25] An interesting question is to find the maximum violation bound, which is useful for the experimental verification.

3.1. Two-spin entangled state with antiparallel polarization

By means of the spin coherent-state quantum probability statistics we can obtain quantum correlation probability $P(a,b)$ for the two-spin entangled state with antiparallel polarization. The entire (quantum) correlation-probability including the non-local parts becomes

$$P(a,b) = -\cos\theta_a\cos\theta_b + \sin 2\xi\,\sin\theta_a\sin\theta_b\cos(\phi_a - \phi_b + 2\eta).$$

The QBCP for the three-direction measurement is found as

$$\begin{aligned} P_{\rm B} = &|-\cos\theta_a\cos\theta_b + \sin 2\xi\,\sin\theta_a\sin\theta_b\cos(\phi_a - \phi_b + 2\eta) \\ &+ \cos\theta_a\cos\theta_c - \sin 2\xi\,\sin\theta_a\sin\theta_c\cos(\phi_a - \phi_c + 2\eta)| - |-\cos\theta_b\cos\theta_c + \sin 2\xi\,\sin\theta_b\sin\theta_c\cos(\phi_b - \phi_c + 2\eta)|. \end{aligned} \tag{11}$$

Since the polar angle θ is restricted between 0 and π, the quantity $\sin\theta_a\sin\theta_b$ is larger than or equal to zero. We then obtain after a simple algebra the inequality of QBCP

$$P_{\rm B} \leq |-\cos(\theta_a \pm \theta_b) + \cos(\theta_a \mp \theta_c)|. \tag{12}$$

Thus we have the maximum violation bound

$$P_{\rm B}^{\rm max} = 2.$$

As a matter of fact the QBCP is bounded by $2 \geq P_{\rm B} \geq -1$.

For the measuring directions with polar and azimuthal angles $\theta_a = \theta_b = \theta_c = \pi/2$ and $\phi_a = \pi/2, \phi_b = 0, \phi_c = \pi$, the QBCP equation (11) becomes

$$P_B = 2\,|\sin 2\xi \sin(2\eta)| - |\sin 2\xi \cos(2\eta)|\,.$$

The three directions of measurements $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are set up with $\boldsymbol{a}$ along positive y axis, $\boldsymbol{b}$, $\boldsymbol{c}$ along positive and negative x axis respectively. The maximum violation $P_{\rm B}^{\max} = 2$ is approached with the state parameters, for example, $\xi = (\pi/4) \bmod 2\pi, \eta = (\pi/4) \bmod 2\pi$. The entangled state in this case is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\pi/4}|+,-\rangle + \mathrm{e}^{-\mathrm{i}\pi/4}|-,+\rangle\right).$$

The violation value depends not only on the entangled-state parameters ξ, η in our parametrization but also on the three directions of measurements.

Particularly for the two-spin singlet state

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)$$

with the state parameters $\xi = 3\pi/4$, $\eta = 0$, the QBCP value is found from Eq. (11) as

$$P_{\rm B} = |-\cos(\phi_a - \phi_b) + \cos(\phi_a - \phi_c)\,| - |\cos(\phi_b - \phi_c)|$$

for the polar angles of three-direction measurements $\theta_a = \theta_b = \theta_c = \pi/2$. The maximum QBCP value of two-spin singlet is obtained as

$$P_{\rm B} = \sqrt{2},$$

for azimuthal angles $\phi_a = 3\pi/4$, $\phi_b = \pi/2$, $\phi_c = 0$. It is less than the maximum violation bound $P_{\rm B}^{\max} = 2$ different from the common believe that the spin-singlet gives rise to the maximum violation bound. To obtain the violation value $\sqrt{2}$ for the spin singlet state the vector $\boldsymbol{b}$ is perpendicular to $\boldsymbol{c}$ and the vector $\boldsymbol{a}$ is parallel to the vector difference $(\boldsymbol{b} - \boldsymbol{c})$.

3.2. Parallel polarization

For the two-spin entangled state with parallel polarization[29]

$$|\psi\rangle = c_1\,|+,+\rangle + c_2\,|-,-\rangle$$

the entire correlation–probability including the non-local parts becomes

$$P(a,b) = \cos\theta_a \cos\theta_b + \sin 2\xi \sin\theta_a \sin\theta_b \cos(\phi_a + \phi_b + 2\eta)\,.$$

The QBCP then is

$$\begin{aligned} P_{\rm B} = |&\cos\theta_a \cos\theta_b + \sin 2\xi \sin\theta_a \sin\theta_b \cos(\phi_a + \phi_b + 2\eta) \\ &- \cos\theta_a \cos\theta_c - \sin 2\xi \sin\theta_a \sin\theta_c \cos(\phi_a + \phi_c + 2\eta)\,| - |\cos\theta_b \cos\theta_c + \sin 2\xi \sin\theta_b \sin\theta_c \cos(\phi_b + \phi_c + 2\eta)|\,, \end{aligned} \quad (13)$$

from which we have the same inequality of QBCP as Eq. (12). Thus, the maximum value of QBCP is still $P_{\rm B}^{\max} = 2$.

The maximum violation of BI can be realized from Eq. (13) for the parallel polarization state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\pi/4}|+,+\rangle + \mathrm{e}^{-\mathrm{i}\pi/4}|-,-\rangle\right)$$

with the state parameters $\xi = (\pi/4) \bmod 2\pi, \eta = (\pi/4) \bmod 2\pi$. The three-direction measurements should be arranged respectively with the polar and azimuthal angles $\theta_a = \theta_b = \theta_c = \pi/2$, $\phi_a = \pi/2, \phi_b = 0, \phi_c = \pi$. Namely $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are perpendicular to the original spin polarization (z axis) with $\boldsymbol{a}$ along y direction; $\boldsymbol{b}$, $\boldsymbol{c}$ along $\pm x$ directions.

In conclusion the value of QBCP is restricted by $-1 \le P_{\rm B} \le 2$ for two-spin entangled states with both parallel and antiparallel polarizations. The extended BI is violated if $1 < P_{\rm B}$.

4. Polarization-entangled photon pairs

Polarization-entangled photon pairs play an important role in various quantum information experiments instead of the two-spin entangled state. We reformulate the extended BI and maximum violation in terms of our formalism.

4.1. Entangled photon pairs with mutually perpendicular polarizations

Two perpendicular polarization states of a single photon may be denoted by $|e_x\rangle$ and $|e_y\rangle$ in our framework, where we have assumed that the polarization plane is perpendicular to the z axis. The entangled state of a photon pair with mutually perpendicular polarizations can be represented as

$$|\psi\rangle = c_1|e_x, e_y\rangle + c_2|e_y, e_x\rangle,$$

which corresponds to the two-spin entangled state with antiparallel spin-polarizations. The normalized coefficients c_1 and c_2 are parameterized as in the spin case. The local and non-local parts of density operator are the same, however, with the spin states $|\pm\rangle$ replaced respectively by the photon polarization states $|e_x\rangle$, $|e_y\rangle$. The entangled photon pairs are measured in three arbitrary directions, say $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$, in the plane also perpendicular to the z axis. With respect to a measuring direction, say $\boldsymbol{r} = (\cos\phi_r, \sin\phi_r, 0)$ ($\boldsymbol{r} = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$), the horizontal ($h$) and vertical ($v$) polarization states are represented as

$$\left|\boldsymbol{r}_h\right\rangle = \cos\phi_r\left|e_x\right\rangle + \sin\phi_r\left|e_y\right\rangle,$$

$$|\boldsymbol{r}_v\rangle = -\sin\phi_r|e_x\rangle + \cos\phi_r|e_y\rangle,$$

where ϕ_r is the azimuthal angle of the measuring direction $\boldsymbol{r}$. In the measuring-outcome independent vector base denoted similarly by

$$|1\rangle = |\boldsymbol{a}_h, \boldsymbol{b}_h\rangle, |2\rangle = |\boldsymbol{a}_h, \boldsymbol{b}_v\rangle, |3\rangle = |\boldsymbol{a}_v, \boldsymbol{b}_h\rangle, |4\rangle = |\boldsymbol{a}_v, \boldsymbol{b}_v\rangle,$$

the local part of the density-operator elements becomes

$$\begin{aligned}
\rho_{11}^{\rm lc} &= \sin^2\xi\cos^2\phi_a\sin^2\phi_b + \cos^2\xi\sin^2\phi_a\cos^2\phi_b,\\
\rho_{22}^{\rm lc} &= \sin^2\xi\cos^2\phi_a\cos^2\phi_b + \cos^2\xi\sin^2\phi_a\sin^2\phi_b,\\
\rho_{33}^{\rm lc} &= \sin^2\xi\sin^2\phi_a\sin^2\phi_b + \cos^2\xi\cos^2\phi_a\cos^2\phi_b,\\
\rho_{44}^{\rm lc} &= \sin^2\xi\sin^2\phi_a\cos^2\phi_b + \cos^2\xi\cos^2\phi_a\sin^2\phi_b.
\end{aligned}$$

The non-local part is

$$\rho_{11}^{\rm nlc} = \rho_{44}^{\rm nlc} = -\rho_{22}^{\rm nlc} = -\rho_{33}^{\rm nlc} = \frac{1}{4}\sin 2\xi\cos 2\eta\sin 2\phi_a\sin 2\phi_b.$$

The local measuring-outcome correlation is

$$P_{\rm lc}(a,b) = \rho_{11}^{\rm lc} - \rho_{22}^{\rm lc} - \rho_{33}^{\rm lc} + \rho_{44}^{\rm lc} = -\cos 2\phi_a\cos 2\phi_b, \quad (14)$$

with which it is easy to verify the extended BI

$$P_{\rm B}^{\rm lc} \le |\cos 2\phi_b - \cos 2\phi_c| - |\cos 2\phi_b\cos 2\phi_c| \le 1. \quad (15)$$

Including the nonlocal part

$$P_{\rm nlc}(a,b) = \sin 2\xi\sin 2\phi_a\sin 2\phi_b\cos 2\eta,$$

the entire correlation-probability becomes

$$P(a,b) = -\cos 2\phi_a\cos 2\phi_b + \sin 2\xi\cos 2\eta\sin 2\phi_a\sin 2\phi_b.$$

The QBCP is

$$\begin{aligned}
P_{\rm B} = |&-\cos 2\phi_a\cos 2\phi_b + \sin 2\xi\cos 2\eta\sin 2\phi_a\sin 2\phi_b\\
&+ \cos 2\phi_a\cos 2\phi_c - \sin 2\xi\cos 2\eta\sin 2\phi_a\sin 2\phi_c| - |-\cos 2\phi_b\cos 2\phi_c + \sin 2\xi\cos 2\eta\sin 2\phi_b\sin 2\phi_c|, \quad (16)
\end{aligned}$$

which is then bounded by

$$\begin{aligned}
P_{\rm B} &\le |-\cos(2\phi_a + 2\phi_b) + \cos(2\phi_a + 2\phi_c)|\\
&\le P_{\rm B}^{\max} = 2.
\end{aligned}$$

From the general form of QBCP Eq. (16) it is easy to check that the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|e_x, e_y\rangle + |e_y, e_x\rangle) \quad (17)$$

results in the maximum violation bound $P_{\rm B}^{\max} = 2$ for the three-direction measurements with $\phi_a = \pi/8$, $\phi_b = 3\pi/8$, and $\phi_c = 15\pi/8$. Namely, $\boldsymbol{b}$ is perpendicular to $\boldsymbol{c}$, and the angle between $\boldsymbol{a}$ and $\boldsymbol{c}$ equals $\pi/4$. It is remarkably to find that the state

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}(|e_x, e_y\rangle - |e_y, e_x\rangle),$$

which may be regarded as the the counterpart of two-spin singlet, gives rise to the QBCP-value again $\sqrt{2}$ less than the maximum bound $P_{\rm B}^{\max}$. The three angles of measuring directions should be arranged as $\phi_a = 3\pi/8$, $\phi_b = \pi/4$, and $\phi_c = 0$ in this state.

4.2. Parallel polarization

The state of entangled photon pairs with mutually parallel polarizations is

$$|\psi\rangle = c_1|e_x, e_x\rangle + c_2|e_y, e_y\rangle.$$

With the same calculation procedure we obtain the measuring outcome correlation

$$P_{\rm lc}(a,b) = \cos 2\phi_a\cos 2\phi_b,$$

which has a sign difference with the perpendicular case of Eq. (14). The QBCP is invariant as Eq. (15) comparing with the entangled state with perpendicular polarizations, so is the extended BI. Including the nonlocal part $P_{\rm nlc}(a,b) = \sin 2\xi\sin 2\phi_a\sin 2\phi_b\cos 2\eta$ the entire correlation-probability is

$$P(a,b) = \cos 2\phi_a\cos 2\phi_b + \sin 2\xi\sin 2\phi_a\sin 2\phi_b\cos 2\eta.$$

The QBCP then is

$$\begin{aligned}
P_{\rm B} = |&\cos 2\phi_a\cos 2\phi_b + \sin 2\xi\sin 2\phi_a\sin 2\phi_b\cos 2\eta\\
&- \cos 2\phi_a\cos 2\phi_c - \sin 2\xi\sin 2\phi_a\sin 2\phi_c\cos 2\eta|\\
&- |\cos 2\phi_b\cos 2\phi_c + \sin 2\xi\sin 2\phi_b\sin 2\phi_c\cos 2\eta|, \quad (18)
\end{aligned}$$

which leads again to the maximum violation bound $P_{\rm B} \le P_{\rm B}^{\max} = 2$.

In the entangled state of equal polarizations

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|e_x, e_x\rangle - |e_y, e_y\rangle),$$

the maximum violation $P_{\rm B}^{\max} = 2$ can be achieved from Eq. (18) for the same measuring directions of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ as in the state of Eq. (17).

5. Conclusions and discussions

The BI and its violation are formulated in a unified way by the spin coherent-state quantum probability statistics, in which the state density operator is separated to the local and

nonlocal parts. The BI is a direct result of local model, while the nonlocal part from the coherent interference of two components of the entangled state leads to the violation. The original BI, which was derived from the two-spin singlet, is extended to a unified form valid for the general entangled state with both antiparallel and parallel polarizations. Up to date the experimental test[32–34] of the inequality violation are mainly focused on the CHSH form, which provides a qualitative bound of the violation. The maximum violation value $P_{\rm B}^{\max}=2$ of the extended BI is two times of BI bound unit one ($P_B^{\rm lc}\leq 1$), while it is $\sqrt{2}$ times in the CHSH inequality case. We thus conclude that the extended BI is at least equally convenient for the experimental verification of its violation, which is expected in the future experiments. We moreover demonstrate that the violation value depends not only on the measuring directions but also two superposition coefficients of the entangled states, namely the angle parameters ξ, η in our formalism. It is remarkably to find that the maximum violation for the spin singlet is only $\sqrt{2}$ less than the maximum violation bound $P_{\rm B}^{\max}$. Our observation is different from the common believe that the spin-singlet would give rise to the maximum violation. The extended BI and violation are also suitable to entangled photon pairs. The BI and WI were formulated respectively with the spin and particle-number-probability correlations in the literature. The two measuring outcome correlations are also unified in our formalism of quantum probability statistics.[28,29,31]

Appendix A

Extended BI from viewpoint of classical statistics with realistic model by means of the measuring outcome correlation of particle number probabilities.

A1: The two-spin entangled state with the antiparallel polarization

In this case there are eight independent (measuring-outcome) particle-number probabilities given by the following table.[24]

Table A1. Antiparallel.

Population	Particle 1	Particle 2
N_1	$(+a,+b,+c)$	$(-a,-b,-c)$
N_2	$(+a,+b,-c)$	$(-a,-b,+c)$
N_3	$(+a,-b,+c)$	$(-a,+b,-c)$
N_4	$(+a,-b,-c)$	$(-a,+b,+c)$
N_5	$(-a,+b,+c)$	$(+a,-b,-c)$
N_6	$(-a,+b,-c)$	$(+a,-b,+c)$
N_7	$(-a,-b,+c)$	$(+a,+b,-c)$
N_8	$(-a,-b,-c)$	$(+a,+b,+c)$

Table a1 lists for the measuring outcomes of two spins respectively along three directions $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. For the local realistic model the measuring outcome correlation probabilities can be expressed in terms of the above population probabilities such that

$$N_{\rm lc}(+a,+b)=\frac{(N_3+N_4)}{\sum_i^8 N_i},$$
$$N_{\rm lc}(-a,-b)=\frac{(N_5+N_6)}{\sum_i^8 N_i},$$
$$N_{\rm lc}(+a,-b)=\frac{(N_1+N_2)}{\sum_i^8 N_i},$$
$$N_{\rm lc}(-a,+b)=\frac{(N_7+N_8)}{\sum_i^8 N_i}.$$

The spin measuring outcome correlation Eq. (10) in BI is related to the four particle-number probabilities as

$$\begin{aligned}P_{\rm lc}(a,b)&=N_{\rm lc}(+a,+b)+N_{\rm lc}(-a,-b)-N_{\rm lc}(+a,-b)\\&\quad-N_{\rm lc}(-a,+b)\\&=\frac{1}{\sum_i^8 N_i}(N_3+N_4+N_5+N_6-N_1-N_2-N_7-N_8).\end{aligned}$$

Correspondingly for particle-1 measured along a, particle-2 along c and particle-1 along b, particle-2 along c the measuring outcome correlations are similarly obtained as

$$\begin{aligned}P_{\rm lc}(a,c)&=N_{\rm lc}(+a,+c)+N_{\rm lc}(-a,-c)-N_{\rm lc}(+a,-c)\\&\quad-N_{\rm lc}(-a,+c)\\&=\frac{1}{\sum_i^8 N_i}(N_2+N_4+N_5+N_7-N_1-N_3-N_6-N_8).\end{aligned}$$

and

$$\begin{aligned}P_{\rm lc}(b,c)&=N_{\rm lc}(+b,+c)+N_{\rm lc}(-b,-c)-N_{\rm lc}(+b,-c)\\&\quad-N_{\rm lc}(-b,+c)\\&=\frac{1}{\sum_i^8 N_i}(N_2+N_6+N_3+N_7-N_1-N_5-N_4-N_8).\end{aligned}$$

We then have

$$P_{\rm lc}(a,b)-P_{\rm lc}(a,c)=\frac{2}{\sum_i^8 N_i}[N_3+N_6-N_2-N_7].$$

Thus Bell correlation becomes

$$\begin{aligned}P_{\rm B}^{\rm lc}&=|P_{\rm lc}(a,b)-P_{\rm lc}(a,c)|-|P_{\rm lc}(b,c)|\\&=\frac{1}{\sum_i^8 N_i}[2\,|N_3+N_6-N_2-N_7|\\&\quad-|N_2+N_6+N_3+N_7-N_1-N_5-N_4-N_8|].\end{aligned}\tag{A1}$$

It is easy to verify the inequality

$$P_{\rm B}^{\rm lc}\leq 1.$$

The equality holds only in the special cases when $N_2=N_7=0$ or $N_3=N_6=0$.

A2. Two-spin entangled state with parallel polarization

For the entangled state with parallel spin-polarization the eight independent particle-number probabilities become[31] those listed in the following table.

Table A2. Spin correlation measurements

Population	Particle 1	Particle 2
N_1	$(+a,+b,+c)$	$(+a,+b,+c)$
N_2	$(+a,+b,-c)$	$(+a,+b,-c)$
N_3	$(+a,-b,+c)$	$(+a,-b,+c)$
N_4	$(+a,-b,-c)$	$(+a,-b,-c)$
N_5	$(-a,+b,+c)$	$(-a,+b,+c)$
N_6	$(-a,+b,-c)$	$(-a,+b,-c)$
N_7	$(-a,-b,+c)$	$(-a,-b,+c)$
N_8	$(-a,-b,-c)$	$(-a,-b,-c)$

The measuring outcome correlation probabilities then are

$$N_{\rm lc}(+a,+b) = \frac{(N_1+N_2)}{\sum_i^8 N_i},$$
$$N_{\rm lc}(-a,-b) = \frac{(N_7+N_8)}{\sum_i^8 N_i},$$
$$N_{\rm lc}(+a,-b) = \frac{(N_3+N_4)}{\sum_i^8 N_i},$$
$$N_{\rm lc}(-a,+b) = \frac{(N_5+N_6)}{\sum_i^8 N_i}.$$

Repeat the same calculation procedure as in the antiparallel case we again have the extended BI $P_{\rm B}^{\rm lc} \leq 1$.

References

[1] Su H Y, Wu Y C and Chen J L 2013 *Phys. Rev. A* **88** 022124
[2] Kwiat P G, Barraza-Lopez S, Stefanov A, *et al.* 2001 *Nature* **409** 1014
[3] Popescu S 2010 *Nat. Phys.* **6** 151
[4] Nielsen M A and Chuang I L 2011 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press) pp. 1–59
[5] Bennett C H and Divincenzo D P 2000 *Nature* **404** 247
[6] Loss D and DiVincenzo D P 1998 *Phys. Rev. A* **57** 120
[7] Bell J S 1964 *Physics* **1** 195
[8] Waldherr G, Neumann P, Huelga S F, Jelezko F and Wrachtrup J 2011 *Phys. Rev. Lett.* **107** 090401
[9] Sakai H, Saito T, Ikeda T, *et al.* 2006 *Phys. Rev. Lett.* **97** 150405
[10] Pal K F and Vertesi T 2017 *Phys. Rev. A* **96** 022123
[11] Rowe M A, Kielpinski D, Meyer V, *et al.* 2001 *Nature* **409** 791
[12] Dada A C, Leach J, Buller G S, *et al.* 2012 *Nat. Phys.* **7** 677
[13] Gisin N and Peres A 1992 *Phys. Lett. A* **162** 15
[14] Brito S G A, Amaral B and Chaves R 2018 *Phys. Rev. A* **97** 022111
[15] Pozsgay V, Hirsch F, Branciard C, *et al.* 2017 *Phys. Rev. A* **96** 062128
[16] Groblacher S, Paterek T, Kaltenbaek R, *et al.* 2007 *Nature* **446** 871
[17] Buhrman H, Cleve R, Massar S, *et al.* 2010 *Rev. Mod. Phys.* **82** 665
[18] Cabello A and Sciarrino F 2012 *Phys. Rev. X* **2** 021010
[19] Wei L F, Liu Y X, and Nori F 2005 *Phys. Rev. B* **72** 104516
[20] Garcia-Patron R, Fiurasek J, Cerf N J, *et al.* 2004 *Phys. Rev. Lett.* **93** 130409
[21] Hess K, Raedt H D and Michielsen K 2017 *J. Mod. Phys.* **8** 57
[22] Clauser J F, Horne M A, Shimony A and Holt R A 1969 *Phys. Rev. Lett.* **23** 880
[23] Wigner E P 1970 *Am. J. Phys.* **38** 1005
[24] Sakurai J J, Tuan S F and Commins E D 1995 *Am. J. Phys.* **63** 93
[25] Home D, Saha D and Das S 2015 *Phys. Rev. A* **91** 012102
[26] Das D, Datta S, Goswami S, Majumdar A S and Home D 2017 *Phys. Lett. A* **381** 3396
[27] Pawlowski M, Paterek T, Kaszlikowski D, Scarani V, Winter A and Zukowski M 2009 *Nature* **461** 1101
[28] Song Z, Liang J Q and Wei L F 2014 *Mod. Phys. Lett. B* **28** 1450004
[29] Zhang H, Wang J, Song Z, *et al.* 2017 *Mod. Phys. Lett. B* **31** 1750032
[30] Grimme S 2003 *J. Chem. Phys.* **118** 9095
[31] Gu Y, Zhang H, Song Z, Liang J Q and Wei L F 2018 *Int. J. Quantum Inform.* **16** 1850041
[32] Hensen B, Bernien H, Dréau A E, *et al.* 2015 *Nature* **526** 682
[33] Hensen B, Kalb N, Blok M S, *et al.* 2016 *Sci. Rep.* **6** 30289
[34] Yin J, Cao Y, Li Y H, Liao S K, Zhang L, Ren J G, Cai W Q and Liu W Y 2017 *Science* **356** 1140
[35] Liang J Q and Wei L F 2011 *New Advances in Quantum Physics* (Beijing: Science Press) p. 56
[36] Bai X M, Gao C P, Li J Q and Liang J Q 2017 *Opt. Express* **25** 17051
[37] Zhao X Q, Liu N and Liang J Q 2014 *Phys. Rev. A* **90** 023622