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Factorization method for inverse obstacle scattering problem in three-dimensional planar acoustic waveguides*

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In this paper, we consider the inverse scattering problem of reconstructing a bounded obstacle in a three-dimensional planar waveguide from the scattered near-field data measured on a finite cylindrical surface containing the obstacle and corresponding to infinitely many incident point sources also placed on the measurement surface. The obstacle is allowed to be an impenetrable scatterer or a penetrable scatterer. We establish the validity of the factorization method with the near-field data to characterize the obstacle in the planar waveguide by constructing an outgoing-to-incoming operator which is an integral operator defined on the measurement surface with the kernel given in terms of an infinite series.

Keywords: factorization method, planar waveguide, inverse scattering problem

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1. Introduction

Inverse problems play a central role in science and engineering, such as ocean acoustic, medical imaging, nondestructive testing, remote sensing and radar.[1,2] Especially, the inverse scattering problems of ocean acoustics have received much attention in recent years.[3–13] A finite deep ocean is popularly modeled by a waveguide bounded by two parallel planes. Time-harmonic acoustic waves in the waveguide are usually modeled by the Helmholtz equation inside it with suitable boundary conditions.[14] In this paper, we assume that the ocean has a pressure released surface and a rigid bottom, corresponding to the Dirichlet condition and Neumann condition, respectively. We concern with the inverse scattering problem of time-harmonic acoustic waves from a bounded obstacle in a planar waveguide at a fixed frequency. Let $\mathbb{R}^3_h = \{x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 : 0 < x_3 < h\}$ be the waveguide of thickness $h > 0$. The lower and upper boundaries of the waveguide $\mathbb{R}^3_h$ are denoted by

$$\Gamma_0 = \{x \in \mathbb{R}^3 : x_3 = 0\} \quad \text{and} \quad \Gamma_h = \{x \in \mathbb{R}^3 : x_3 = h\} \tag{1}$$

respectively. The bounded domain D in $\mathbb{R}^3_h$ with the boundary $\partial D \in C^2$ is occupied by the obstacle. The part of the waveguide not occupied by $\overline{D}$ is denoted by $\mathbb{R}^3_h \setminus \overline{D}$, which is assumed to be connected.

Then the acoustic wave u traveling inside the waveguide satisfies the following equations:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3_h \setminus \overline{D}, \tag{2}$$

$$u = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial u}{\partial x_3} = 0 \quad \text{on } \Gamma_h, \tag{3}$$

$$\mathscr{B}u = 0 \quad \text{on } \partial D, \tag{4}$$

where $k > 0$ is the wave number, and $\mathscr{B}$ denotes the boundary condition imposed on ∂D. Here, u is usually called the total field. In the simplest case, u satisfies the Dirichlet boundary condition

$$\mathscr{B}u := u = 0 \quad \text{on } \partial D, \tag{5}$$

which corresponds to a sound-soft obstacle, while for a sound-hard obstacle, u satisfies the Neumann boundary condition

$$\mathscr{B}u := \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial D. \tag{6}$$

Here, ν is the unit outward normal vector to ∂D. We also consider the impedance boundary condition

$$\mathscr{B}u := \frac{\partial u}{\partial \nu} + \rho(x)u = 0 \quad \text{on } \partial D, \tag{7}$$

which corresponds to an imperfect or partially coated obstacle. Here, $\rho(x) \in L^\infty(\partial D)$ is the impedance function.

Given incident fields, what we want to do is to get the information of the scattered field in the waveguide and to solve the inverse problem of reconstructing the shape and location of the domain D, which is occupied by the obstacle. This inverse problem is much more challenging than that in free-space. Due to the planar geometry, only a finite number of modes can propagate at a long distance, while the other modes are evanescent fields and undetectable away from the obstacles. This increases the ill posedness of the inverse problem. For this reason, the inverse scattering problem in a waveguide has received increasing attention.[5,6,9,10] We refer to Ref. [15] for the generalized dual space method, Ref. [11] for reverse time migration method, and Refs. [4], [6], and [13] for the linear sampling method. The authors of Ref. [16] used the factorization method to recover the scattering surface of a periodic structure.

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The purpose of this paper is to extend the factorization method[17–19] to reconstruct the obstacle D in the waveguide scattering. In this paper,we will utilize the factorization method to reconstruct the shape of an unknown obstacle embedded in a planar acoustic waveguide using the measurement of the scattered field near the obstacle. Following the idea of Ref. [17], we construct an auxiliary outgoing-to-incoming operator to derive the factorization of data operator. The main principle of factorization method is that a point z in the waveguide $\mathbb{R}^3_h$ belongs to the domain D if and only if the field generated by a special point source at z is in the range of a certain linear operator defined by the data, called the near-field operator. Finally, the range of this operator can be characterized by spectral properties.

The remaining part of this paper is organized as follows. In Section 2, we present the direct scattering problem in the 3D waveguide and some useful results. In Section 3, we derive the Fourier coefficients of the near-field operator. In Section 4, we give a justification of the factorization method for the case of a Dirichlet boundary condition in a waveguide. We define the outgoing-to-incoming operator in the first part and then give a description of the domain D. In Sections 5 and 6, we extend the factorization method with near-field data to the case of the impedance and Neumann boundary conditions and the inverse medium scattering case.

2. Direct scattering problem

In this section, we will present some notation and basic results to be used throughout the paper. Recall that the waveguide is denoted by $\mathbb{R}^3_h$. Noting that the x_3-axis is orthogonal to the waveguide, we combine the first two coordinates to write

$$x = (x_1, x_2, x_3)^\top = (\widetilde{x}, x_3)^\top, \quad x \in \mathbb{R}^3.$$

In this paper, we shall often work later in the bounded domain $\Omega_R := \{x \in \mathbb{R}^3_h : |\widetilde{x}|^2 < R^2\}$, where the radius R is assumed to be large enough such that $1 + |\widetilde{x}|^2 < R^2$ for all $x \in D$. This implies that $\overline{D}$ is contained in the interior of Ω_R. The part of the boundary $\partial\Omega_R$, contained in the waveguide $\mathbb{R}^3_h$, is denoted by the cylinder

$$C_R := \{x \in \mathbb{R}^3_h : |\widetilde{x}|^2 = R^2\}.$$

The incident field u^{i} we used is the solution to the Helmholtz equation $\Delta u^{\mathrm{i}} + k^2 u^{\mathrm{i}} = 0$ in $\mathbb{R}^3_h$, and satisfies the boundary conditions (3). The direct scattering problem is then to find the scattered field $u^{\mathrm{s}} = u - u^{\mathrm{i}}$ subject to Eqs. (2) and (3), when the total field u satisfies one of boundary conditions (5)–(7). To ensure the uniqueness of solution for direct scattering problem, the scattered field u^{s} have to satisfy additionally a radiation condition.[20]. Since $D \subset \Omega_R$, we have the following mode expansion by carrying out a separation of variables in $\widetilde{x}$ and x_3:

$$u^{\mathrm{s}}(x) = \sum_{m=1}^{\infty} \sin(\alpha_m x_3) u_m(\widetilde{x}), \quad \forall x \in \mathbb{R}^3_h \setminus \Omega_R, \tag{8}$$

where $\alpha_m := (2m-1)\pi/2h$, $m = 1, 2, \ldots$. Then the scattered field u^{s} automatically satisfies the waveguide boundary condition (3). The modes u_m satisfy the two-dimensional Helmholtz equation

$$\Delta u_m + k_m^2 u_m = 0, \quad |\widetilde{x}| > R, \quad m = 1, 2, \ldots, \tag{9}$$

where $k_m = \sqrt{k^2 - \alpha_m^2}$ if $k > \alpha_m$ and $k_m = \mathrm{i}\sqrt{\alpha_m - k^2}$ if $k < \alpha_m$. The radiation condition for the planar waveguide problem is then to impose the modes u_m to satisfy the outgoing Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u_m}{\partial r} - \mathrm{i}k_m u_m\right) = 0, \quad \text{with } r = |\widetilde{x}|, \tag{10}$$

uniformly for all direction $\widetilde{x}/r$. From the definition of k_m, we note that there is some positive integer $M(k)$ such that the modal wave numbers k_m are real, which correspond to modal frequencies of the waveguide, when $m \le M(k)$. While k_m are purely imaginary when $m > M(k)$, corresponding to exponentially decaying modes. However, the case $k_m = 0$ for some $m \in \mathbb{N}$ corresponds to an exceptional frequency. We always assume that

$$k \neq \alpha_m \quad \text{for all } m \in \mathbb{N}. \tag{11}$$

The Green's function[14] for the waveguide $\mathbb{R}^3_h$ with respect to the boundary condition (3) is given by

$$G(x,y) = \frac{\mathrm{i}}{2h}\sum_{m=1}^{\infty} \sin(\alpha_m x_3)\sin(\alpha_m y_3)\,\mathrm{H}_0^{(1)}(k_m|\widetilde{x}-\widetilde{y}|), \quad x, y \in \mathbb{R}^3_h, \widetilde{x} \neq \widetilde{y}. \tag{12}$$

Using the method of images,[14] one can derive an equivalent representation of the Green's function in $\mathbb{R}^3_h$,

$$G(x,y) = \frac{1}{4\pi}\sum_{m=-\infty}^{+\infty}(-1)^m\left\{\frac{\mathrm{e}^{\mathrm{i}k|x-y_m|}}{|x-y_m|} - \frac{\mathrm{e}^{\mathrm{i}k|x-y'_m|}}{|x-y'_m|}\right\}, \quad x \neq y \in \mathbb{R}^3_h, \tag{13}$$

where the source image points are given by

$$y_m = (y_1, y_2, y_3)^\top + (0, 0, 2mh)^\top, \quad y'_m = (y_1, y_2, -y_3)^\top + (0, 0, 2mh)^\top, \ m \in \mathbb{Z}. \tag{14}$$

Usually, the formula (13) is called the normal mode representation of Green's function in the waveguide and we call Eq. (13) the ray representation. From Eq. (13), we observe that the Green's function can be written as

$$G(x,y) = \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} + \widetilde{G}(x,y), \quad x \neq y \in \mathbb{R}^3_h.$$

It may be see that the first part of the superposition

$$\Phi(x,y) = \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|}$$

is the fundamental solution of Helmholtz equation in free space, while the remaining part $\widetilde{G}(x,y)$ is analytic.[4] Consequently, the jump relations and regularity properties of single-layer potential $\int_{\partial D} G(x,y)\varphi(y)\,\mathrm{d}s(y)$ and double-layer potential

$$\int_{\partial D} \frac{\partial G(x,y)}{\partial \nu(y)}\varphi(y)\,\mathrm{d}s(y)$$

are the same as for the potentials with kernel $\Phi(x,y)$, respectively.

From Ref. [20], we know that the Green's formulas are still holds in the waveguide $\mathbb{R}^3_h$, that is, for any function u satisfying Eqs. (2), (3), (4), and (10), it holds that

$$\int_{\partial D}\left\{u(y)\frac{\partial G(x,y)}{\partial \nu(y)} - G(x,y)\frac{\partial u}{\partial \nu}(y)\right\}\mathrm{d}s(y) = \begin{cases} 0, & \text{if } x\in D,\\ u(x), & \text{if } x\in \mathbb{R}^3_h\setminus\overline{D}. \end{cases} \tag{15}$$

Moreover, by seeking a solution in the form of a combined double- and single-layer potential

$$w(x) = \int_{\partial D}\left\{\frac{\partial G(x,y)}{\partial \nu(y)} - \mathrm{i}\eta G(x,y)\right\}\varphi(y)\,\mathrm{d}s(y), \tag{16}$$

we know that the w is a solution of Eqs. (2), (3), (4), and (10), provided that the density φ is the solution of the boundary integral equation

$$\varphi(x) + 2\int_{\partial D}\frac{\partial G(x,y)}{\partial \nu(y)}\varphi(y)\,\mathrm{d}s(y) - 2\mathrm{i}\eta\int_{\partial D} G(x,y)\varphi(y)\,\mathrm{d}s(y) = -2u^{\mathrm{i}}(x), \quad \text{for } x\in\partial D. \tag{17}$$

Thus, the problem of Eqs. (2), (3), (4), and (10) is uniquely solvable, due to the boundary integral equation is uniquely solvable.[20]

We remark that under some certain geometric assumptions on a sound-soft obstacle, the uniqueness of the solution to the acoustic waveguide scattering problem was proved in Ref. [21]. For the sound-hard obstacle, it has been proved the existence of non-trivial waves for the homogeneous problem, called trapped modes.[22] The authors in Ref. [3] proved out that these scattering problems are uniquely solvable except possibly for an infinite series of exceptional values of the wavenumber with no finite accumulation point.

3. Fourier coefficients of near-field operator

In this paper, we consider the point source wave as incident field $u^{\mathrm{i}}(x,y) = G(x,y)$ which is generated at the source position $y\in C_R$, and the corresponding scattered field is denoted by $u^{\mathrm{s}}(x,y)$ which depends on the point source y. Then the measurement data we need in this paper are taken on the cylinder C_R. Therefore, the set $\{u^{\mathrm{s}}(x,y) : x,y\in C_R\}$ are the data for the inverse problem, or equivalently, the so-called ***near-field operator*** $N: L^2(C_R)\to L^2(C_R)$ is given by

$$(Ng)(x) = \int_{C_R} u^{\mathrm{s}}(x,y)g(y)\,\mathrm{d}s(y), \quad x\in C_R. \tag{18}$$

Under cylindrical coordinates on $C_1 := \{x\in\mathbb{R}^3_h : |\widetilde{x}|^2 = 1\}$, the functions

$$Y_{m,n}(\varphi_x, x_3) = \mathrm{e}^{\mathrm{i}n\varphi_x}\sin(\alpha_m x_3), \quad m\in\mathbb{Z}^+, n\in\mathbb{Z}.$$

form a complete orthogonal system in $L^2(C_R)$. Thus, for any $g\in L^2(C_R)$, we have the expansion

$$\begin{aligned} g(x) &= \sum_{m=1}^{\infty}\sum_{n=-\infty}^{+\infty} g_{m,n}\mathrm{e}^{\mathrm{i}n\varphi_x}\sin(\alpha_m x_3),\\ x &= (R\cos\varphi_x, R\sin\varphi_x, x_3)\in C_R \end{aligned} \tag{19}$$

with

$$g_{m,n} = \frac{1}{\pi hR}\int_{C_R} g(x)\,\mathrm{e}^{-\mathrm{i}n\varphi_x}\sin(\alpha_m x_3)\,\mathrm{d}s(x), \tag{20}$$

where the coefficients $g_{m,n}\in\mathbb{C}$ are regarded as the Fourier coefficients of g with respect to the functions $\{Y_{m,n} : m\in\mathbb{Z}^+, n\in\mathbb{Z}\}$. The Fourier coefficients of an L^2 function on C_R are understood in this sense throughout the paper. We observe that

$$\|g\|^2_{L^2(C_R)} = \pi hR\sum_{m=1}^{\infty}\sum_{n=-\infty}^{+\infty}|g_{m,n}|^2.$$

Therefore, We define the operator $\mathscr{F}_R : L^2(C_R)\to\ell^2$ by

$$\mathscr{F}_R(g) = \boldsymbol{g}, \quad \boldsymbol{g} := \{g_{m,n} : m\in\mathbb{Z}^+, n\in\mathbb{Z}\}\in\ell^2. \tag{21}$$

Conversely, for $\boldsymbol{g} = \{g_{m,n} : m\in\mathbb{Z}^+, n\in\mathbb{Z}\}\in\ell^2$ we can define the operator $\mathscr{F}_R^{-1} : \ell^2\to L^2(C_R)$ by

$$\mathscr{F}_R^{-1}(\boldsymbol{g}) = \sum_{m=1}^{\infty}\sum_{n=-\infty}^{+\infty} g_{m,n}\mathrm{e}^{\mathrm{i}n\varphi_x}\sin(\alpha_m x_3). \tag{22}$$

Then, we can readily deduce from Eqs. (21) and (22) that

$$\begin{aligned} &\mathscr{F}_R^{-1}\mathscr{F}_R = I_{L^2(C_R)}, \quad \mathscr{F}_R\mathscr{F}_R^{-1} = I_{\ell^2},\\ &\mathscr{F}_R^{*} = \frac{1}{\pi hR}\mathscr{F}_R^{-1}, \quad \left(\mathscr{F}_R^{-1}\right)^{*} = \pi hR\mathscr{F}_R, \end{aligned} \tag{23}$$

where $I_{L^2(C_R)}$ and I_{ℓ^2} denote the identity operator on $L^2(C_R)$ and ℓ^2, respectively.

We have already known that the boundary condition (3) and the radiation condition (10) lead to a series representation of any radiating solution $u^{\mathrm{s}} \in H^1_{loc}(\mathbb{R}^3_h \setminus \Omega_R)$,[4]

$$u^{\mathrm{s}}(x) = \sum_{m=1}^{\infty} \sum_{n=-\infty}^{+\infty} U_{m,n} \mathrm{H}_n^{(1)}(k_m r) \mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_m x_3), \tag{24}$$

for $x = (r\cos\varphi_x, r\sin\varphi_x, x_3)^\top \in \mathbb{R}^3_h \setminus \Omega_R$, where $\mathrm{H}_n^{(1)}$ denotes the Hankel function of the first kind of order n.

Let J_n be the Bessel functions of order n. Set

$$u^{\mathrm{i}}_{m,n}(x) = \mathrm{J}_n(k_m|\widetilde{x}|) \mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_m x_3), \quad x \in \mathbb{R}^3_h, m \in \mathbb{Z}^+, n \in \mathbb{Z}.$$

We can verify that $u^{\mathrm{i}}_{m,n}$ satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^3_h$ and the waveguide boundary condition (3). If we consider $u^{\mathrm{i}}_{m,n}$ as the incident field, the corresponding scattered field is denoted by $u^{\mathrm{s}}_{m,n}$. Then from the series representation (24), we know that $u^{\mathrm{s}}_{m,n}$ has the expansion in the waveguide $\mathbb{R}^3_h$ outside Ω_R,

$$u^{\mathrm{s}}_{m,n}(x) = \sum_{p=1}^{\infty} \sum_{q=-\infty}^{+\infty} U^{m,n}_{p,q} \mathrm{H}_q^{(1)}(k_p|\widetilde{x}|) \mathrm{e}^{\mathrm{i}l\varphi_x} \sin(\alpha_p x_3), \quad x \in \mathbb{R}^3_h \setminus \Omega_R, \tag{25}$$

where $U^{m,n}_{p,q} \in \mathbb{C}$ for all $m \in \mathbb{Z}^+, n \in \mathbb{Z}$. Therefore,

$$\mathscr{F}_R\big(u_{m,n}|_{C_R}\big) = \Big\{ U^{m,n}_{p,q} \mathrm{H}_q^{(1)}(k_p R) : p \in \mathbb{Z}^+, q \in \mathbb{Z} \Big\} \in \ell^2.$$

for all $m \in \mathbb{Z}^+, n \in \mathbb{Z}$. By using the Fourier coefficients of Ng, we will consider the operator

$$\mathscr{N} := \mathscr{F}_R N \mathscr{F}_R^{-1} : \ \ell^2 \to \ell^2 \tag{26}$$

instead of the near-field operator N. An explicit expression of $\mathscr{N}$ is given as follows.

Lemma 1 Let $g \in L^2(C_R)$. $g_{m,n}, \boldsymbol{g}$, and $U^{m,n}_{p,q}$ are given as in Eqs. (20), (21), and (25), respectively. Then we have

$$\mathscr{N}\boldsymbol{g} = \left\{ \frac{\mathrm{i}\pi R}{2} \mathrm{H}_q^{(1)}(kR) \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} U^{m,n}_{p,q} \mathrm{H}_n^{(1)}(k_m R) g_{m,n} : p \in \mathbb{Z}^+, q \in \mathbb{Z} \right\}. \tag{27}$$

Proof By definition, we know that $\mathscr{N}\boldsymbol{g} = \mathscr{F}_R N g$. Thus we only need to derive the Fourier coefficients of the near-field data Ng on C_R with respect to $\{Y_{m,n} : m \in \mathbb{Z}^+, n \in \mathbb{Z}\}$.

Set

$$U(x) := \int_{C_R} u^{\mathrm{s}}(x,y) g(y) ds(y), \quad x \in \mathbb{R}^3_h \setminus \Omega_R. \tag{28}$$

Then we observe that Ng is the restriction on C_R of U, and U is the scattered field corresponding to the incident field $U^{\mathrm{i}}(x) := \int_{C_R} G(x,y) g(y) \mathrm{d}s(y)$, for $|\widetilde{x}| < R$. From the representation of Green's function (12), we note that $(\mathrm{i}/4)\mathrm{H}_0^{(1)}(k_m|\widetilde{x}-\widetilde{y}|)$ is just the fundamental solution of two-dimensional Helmholtz equation (9). Recall from Ref. [1], the Hankel function $\mathrm{H}_0^{(1)}(k_m|\widetilde{x}-\widetilde{y}|)$ has the expansion

$$\mathrm{H}_0^{(1)}(k_m|\widetilde{x}-\widetilde{y}|) = \sum_{n=-\infty}^{+\infty} \mathrm{J}_n(k_m|\widetilde{x}|) \mathrm{H}_n^{(1)}(k_m|\widetilde{y}|) \mathrm{e}^{-\mathrm{i}n(\varphi_x - \varphi_y)}, \tag{29}$$

and the series on the right side is absolutely and uniformly convergent on compact subsets of $|\widetilde{x}| < |\widetilde{y}|$. Thus we have the expansion of Green's function as follows:

$$G(x,y) = \frac{\mathrm{i}}{2h} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} \sin(\alpha_m x_3) \sin(\alpha_m y_3) \cdot \mathrm{J}_n(k_m|\widetilde{x}|) \mathrm{H}_n^{(1)}(k_m|\widetilde{y}|) \mathrm{e}^{\mathrm{i}n(\varphi_y - \varphi_x)}, \tag{30}$$

for $|\widetilde{x}| < |\widetilde{y}|$. Since $\mathrm{J}_{-n}(t) = (-1)^n \mathrm{J}_n(t)$ and $\mathrm{H}_{-n}^{(1)}(t) = (-1)^n \mathrm{H}_n^{(1)}(t)$ for $n \in \mathbb{Z}$, we can rewrite the previous identity as

$$G(x,y) = \frac{\mathrm{i}}{2h} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} \sin(\alpha_m x_3) \sin(\alpha_m y_3) \cdot \mathrm{J}_n(k_m|\widetilde{x}|) \mathrm{H}_n^{(1)}(k_m|\widetilde{y}|) \mathrm{e}^{\mathrm{i}n(\varphi_x - \varphi_y)}. \tag{31}$$

This implies that for $|\widetilde{x}| < R$,

$$\begin{aligned} U^{\mathrm{i}}(x) &= \frac{\mathrm{i}}{2h} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} \sin(\alpha_m x_3) \mathrm{e}^{\mathrm{i}n\varphi_x} \mathrm{J}_n(k_m|\widetilde{x}|) \int_{C_R} g(y) \sin(\alpha_m y_3) \mathrm{H}_n^{(1)}(k_m|\widetilde{y}|) \mathrm{e}^{-\mathrm{i}n\varphi_y} ds(y) \\ &= \frac{\mathrm{i}}{2h} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} \pi h R g_{m,n} \mathrm{H}_n^{(1)}(k_m R) \sin(\alpha_m x_3) \mathrm{e}^{\mathrm{i}n\varphi_x} \mathrm{J}_n(k_m|\widetilde{x}|) \\ &= \frac{\mathrm{i}\pi R}{2} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} g_{m,n} \mathrm{H}_n^{(1)}(k_m R) u^{\mathrm{i}}_{m,n}(x) \end{aligned} \tag{32}$$

with $g_{m,n}$ defined by Eq. (20). Then, by linear superposition we conclude from Eqs. (25) and (32) that

$$U(x) = \frac{\mathrm{i}\pi R}{2} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} g_{m,n} \mathrm{H}_n^{(1)}(k_m R) u^{\mathrm{s}}_{m,n}(x) \tag{33}$$

$$= \frac{\mathrm{i}\pi R}{2} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} g_{m,n} \mathrm{H}_n^{(1)}(k_m R) \sum_{p=1}^{+\infty} \sum_{q=-\infty}^{+\infty} U_{p,q}^{m,n} \mathrm{H}_q^{(1)}(k_p|\tilde{x}|) \mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_p x_3) \tag{34}$$

$$= \frac{\mathrm{i}\pi R}{2} \sum_{p=1}^{+\infty} \sum_{q=-\infty}^{+\infty} \mathrm{H}_q^{(1)}(k_p|\tilde{x}|) \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} U_{p,q}^{m,n} \mathrm{H}_n^{(1)}(k_m R) g_{m,n} \mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_p x_3) \tag{35}$$

for $|x| > R$. Here, interchanging the order of summation is allowed since the two series are absolutely and uniformly convergent on compact subsets of $|x| > R$. Therefore, the Fourier coefficients of $U(x)\big|_{C_R}$ yield the expression (27). □

4. Dirichlet boundary condition

In this section, we will utilize the factorization method to reconstruct a sound-soft obstacle embedded in a planar acoustic waveguide using the measurement of the scattered field on C_R. The key ingredients in this part is the construction of an auxiliary unitary mapping T_1 and an appropriate factorization of the operator T_1.

For the sound-soft obstacle, we consider the general Dirichlet problem

Problem 1 (Dirichlet boundary value problem)
Given $f \in H^{1/2}(\partial D)$, we find a solution v to the following problem:

$$\Delta v + k^2 v = 0, \ \text{ in } \mathbb{R}_h^3 \setminus \overline{D},$$
$$v = f \ \text{ on } \partial D,$$
$$v = 0 \ \text{ on } \Gamma_0, \quad \frac{v}{\partial x_3} = 0 \ \text{ on } \Gamma_h,$$
The radiation condition (10).

4.1. Factorization of near-field operator

First, we define the incidence operator

$$H_{\mathrm{Dir}} : \ell^2 \to H^{1/2}(\partial D)$$

for the Dirichlet boundary problem in the waveguide as follows:

$$H_{\mathrm{Dir}}(g) := U^{\mathrm{i}}\big|_{\partial D} = \frac{\mathrm{i}\pi R}{2} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathrm{H}_n^{(1)}(k_m R) g_{m,n} u_{m,n}^{\mathrm{i}}(x), \quad x \in \partial D. \tag{36}$$

Here H_{Dir} is a Herglotz-like operator, which maps a superposition of the incident waves $u_{m,n}^{\mathrm{i}}$ with the weight $g_{m,n}$ into its trace on ∂D. The adjoint operator $H_{\mathrm{Dir}}^* : \mathrm{H}^{-1/2}(\partial D) \to \ell^2$ is

$$H_{\mathrm{Dir}}^* \psi = \left\{ -\frac{\mathrm{i}\pi R}{2} \overline{\mathrm{H}_n^{(1)}(k_m R)} \int_{\partial D} \psi(y) \overline{\mathrm{J}_n(k_m|\tilde{y}|)} \mathrm{e}^{-\mathrm{i}n\varphi_y} \sin(\alpha_m y_3) \mathrm{d}s(y) : n \in \mathbb{Z}^+, m \in \mathbb{Z} \right\}. \tag{37}$$

Recall the expression of the Bessel functions of order n:[1]

$$\mathrm{J}_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!} \left(\frac{t}{2}\right)^{n+2p}, \quad n = 0, 1, 2, \ldots. \tag{38}$$

We observe that $\overline{\mathrm{J}_n(\mathrm{i}t)} = (-1)^n \mathrm{J}_n(\mathrm{i}t)$. By the definition of k_m, we already know that k_m is real, if $m \le M(k)$; otherwise, k_m is purely imaginary. Therefore,

$$\overline{\mathrm{J}_n(k_m|\tilde{x}|)} = \begin{cases} \mathrm{J}_n(k_m|\tilde{x}|), & m \le M(k), \\ (-1)^n \mathrm{J}_n(k_m|\tilde{x}|), & m > M(k). \end{cases} \tag{39}$$

Thus, we could avoid the conjugate of Bessel function J_n inside the integral of Eq. (37).

Let v be the unique outgoing solution to the Problem 1 with the boundary value $f \in \mathrm{H}^{1/2}(\partial D)$. Suppose that on C_R, v has the following expansion

$$v(x)\big|_{C_R} = \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} b_{m,n} \mathrm{H}_n^{(1)}(k_m R) \mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_m x_3), \quad b_{m,n} \in \mathbb{C}. \tag{40}$$

Then the solution operator $G_{\mathrm{Dir}} : \mathrm{H}^{1/2}(\partial D) \to \ell^2$ is defined by the Fourier coefficients of $v\big|_{C_R}$, i.e.,

$$G_{\mathrm{Dir}}(f) := \left\{ b_{m,n} \mathrm{H}_n^{(1)}(k_m R) : n \in \mathbb{Z}^+, m \in \mathbb{Z} \right\}. \tag{41}$$

From the definition of $\mathcal{N}$, H_{Dir}, and G_{Dir}, the following relation holds:

$$\mathcal{N} = -G_{\mathrm{Dir}} H_{\mathrm{Dir}}. \tag{42}$$

In the rest of this section, the single-layer operator and single-layer potential are essential tools. They are give by

$$(S\psi)(x) = \int_{\partial D} G(x,y)\psi(y)\mathrm{d}s(y), \quad x \in \partial D, \tag{43}$$

$$(V\psi)(x) = \int_{\partial D} G(x,y)\psi(y)\mathrm{d}s(y), \quad x \in \mathbb{R}_h^3 \tag{44}$$

for $\psi \in \mathrm{H}^{-1/2}(\partial D)$, respectively. From the expansion of Green's function (30), we obtain the expansion of single-layer potential

$$(V\psi)(x) = \frac{\mathrm{i}}{2h} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} \sin(\alpha_m x_3) \mathrm{e}^{\mathrm{i}n\varphi_x} \mathrm{H}_n^{(1)}(k_m|\tilde{x}|) \times \int_{\partial D} \psi(y) \mathrm{J}_n(k_m|\tilde{y}|) \mathrm{e}^{-\mathrm{i}n\varphi_y} \sin(\alpha_m y_3) \mathrm{d}s(y). \tag{45}$$

Together with the definition of G_{Dir} and the jump relation for single-layer potentials in the waveguide,[20] we have the following relation

$$G_{\rm Dir}\big((V\psi)\big|_{\partial D}\big) = G_{\rm Dir}S\psi = \left\{\frac{\mathrm{i}}{2h}\,\mathrm{H}_n^{(1)}(k_mR)\int_{\partial D}\psi(y)\,\mathrm{J}_n(k_m|\tilde{y}|)\,\mathrm{e}^{-\mathrm{i}n\varphi_y}\sin(\alpha_m y_3)\,\mathrm{d}s(y) : m\in\mathbb{N}^+,\ n\in\mathbb{Z}\right\}. \tag{46}$$

In order to find an appropriate factorization of $\mathscr{N}$, we observe Eqs. (37) and (41) that

$$\pi RhT_0G_{\rm Dir}S = H^*_{\rm Dir},\quad \pi RhS^*G^*_{\rm Dir}T_0^* = H_{\rm Dir}. \tag{47}$$

with the operator $T_0:\ell^2\to\ell^2$ defined by

$$\begin{aligned}(T_0\boldsymbol{g})_{m,n} &= -\frac{\overline{\mathrm{H}_n^{(1)}(k_mR)}}{\mathrm{H}_n^{(1)}(k_mR)}g_{m,n}, 1\le m\le M,\\ (T_0\boldsymbol{g})_{m,n} &= (-1)^{n+1}\frac{\overline{\mathrm{H}_n^{(1)}(k_mR)}}{\mathrm{H}_n^{(1)}(k_mR)}g_{m,n}, m>M\end{aligned} \tag{48}$$

for $\boldsymbol{g}=\{g_{m,n}: n\in\mathbb{Z}^+, m\in\mathbb{Z}\}$. Here $(T_0\boldsymbol{g})_{m,n}$ represent the component of sequence $T_0\boldsymbol{g}\in\ell^2$, $m\in\mathbb{Z}^+, n\in\mathbb{Z}$. Moreover, it is seen from Eq. (48) that $T_0T_0^*=T_0^*T_0=I_{\ell^2}$, that is, T_0 is unitary operator on ℓ^2.

From Eq. (42) and the second relation in Eq. (47), we observe that

$$T_0\mathscr{N} = -T_0G_{\rm Dir}H_{\rm Dir} = -\pi Rh(T_0G_{\rm Dir})S^*(T_0G_{\rm Dir})^*. \tag{49}$$

Consequently, we obtain a factorization of the near-field operator N as follows.

Theorem 1 For the near-field operator N, we have the factorization

$$T_1N = -\mathbb{G}_{\rm Dir}S^*\mathbb{G}^*_{\rm Dir},\quad \mathbb{G}_{\rm Dir} := \mathscr{F}_R^{-1}T_0G, \tag{50}$$

where $T_1 := \mathscr{F}_R^{-1}T_0\mathscr{F}_R : L^2(C_R)\to L^2(C_R)$ takes the form:

$$\big(T_1g\big)(x) = \int_{C_R}K(x,y)g(y)\,\mathrm{d}s(y), \tag{51}$$

for $g\in L^2(\Gamma_{\pm R})$, with the kernel

$$\begin{aligned}K(x,y) = &-\frac{1}{\pi hR}\sum_{m=1}^{M}\sum_{n=-\infty}^{+\infty}\frac{\overline{\mathrm{H}_n^{(1)}(k_mR)}}{\mathrm{H}_n^{(1)}(k_mR)}\,\mathrm{e}^{\mathrm{i}n(\varphi_x-\varphi_y)}\sin(\alpha_mx_3)\sin(\alpha_my_3)\\ &-\frac{1}{\pi hR}\sum_{m=M+1}^{+\infty}\sum_{n=-\infty}^{+\infty}(-1)^{n+1}\frac{\overline{\mathrm{H}_n^{(1)}(k_mR)}}{\mathrm{H}_n^{(1)}(k_mR)}\,\mathrm{e}^{\mathrm{i}n(\varphi_x-\varphi_y)}\sin(\alpha_mx_3)\sin(\alpha_my_3).\end{aligned} \tag{52}$$

Here, M denotes $M(k)$ as above.

Proof By the definition of $\mathscr{F}_R$, N, and $\mathscr{N}$, we know that $N=\mathscr{F}_R^{-1}\mathscr{N}\mathscr{F}_R$. With the factorization (49) and the definition of T_1, we derive that

$$\begin{aligned}T_1N &= \mathscr{F}_R^{-1}T_0\mathscr{N}\mathscr{F}_R = -\pi hR\mathscr{F}_R^{-1}T_0G_{\rm Dir}S^*(T_0G_{\rm Dir})^*\mathscr{F}_R = -\big(\mathscr{F}_R^{-1}T_0G_{\rm Dir}\big)S^*G^*_{\rm Dir}*T_0^*\big(\mathscr{F}_R^{-1}\big)^*\\ &= -\mathbb{G}_{\rm Dir}S^*\mathbb{G}^*_{\rm Dir},\end{aligned} \tag{53}$$

with the third equality follows the last relation in Eq. (23), and the operator $\mathbb{G}_{\rm Dir}$ in the last equality is given as above. Then the first part of the theorem is proved.

From the definition of T_1 and T_0, it follows that for $g\in L^2(C_R)$,

$$\begin{aligned}(T_1g)(x) = \big(\mathscr{F}_RT_0\boldsymbol{g}\big)(x) = &-\sum_{m=1}^{M}\sum_{n=-\infty}^{+\infty}\left[\frac{\overline{\mathrm{H}_n^{(1)}(k_mR)}}{\mathrm{H}_n^{(1)}(k_mR)}g_{m,n}\right]\mathrm{e}^{\mathrm{i}n\varphi_x}\sin(\alpha_mx_3)\\ &-\sum_{m=M+1}^{\infty}\sum_{n=-\infty}^{+\infty}(-1)^n\left[\frac{\overline{\mathrm{H}_n^{(1)}(k_mR)}}{\mathrm{H}_n^{(1)}(k_mR)}g_{m,n}\right]\mathrm{e}^{\mathrm{i}n\varphi_x}\sin(\alpha_mx_3),\quad x\in C_R\end{aligned}$$

with

$$g_{m,n} = \frac{1}{\pi hR}\int_{C_R}g(x)\,\mathrm{e}^{-\mathrm{i}n\varphi_x}\sin(\alpha_mx_3)\,\mathrm{d}s(x).$$

Substitute the expression of $g_{m,n}$ into the last equality,

$$\begin{aligned}(T_1g)(x) = -\frac{1}{\pi hR}\Bigg[&\sum_{m=1}^{M}\sum_{n=-\infty}^{+\infty}\frac{\overline{\mathrm{H}_n^{(1)}(k_mR)}}{\mathrm{H}_n^{(1)}(k_mR)}\,\mathrm{e}^{\mathrm{i}n\varphi_x}\sin(\alpha_mx_3)\int_{C_R}g(y)\,\mathrm{e}^{-\mathrm{i}n\varphi_y}\sin(\alpha_my_3)\,\mathrm{d}s(y)\\ &+\sum_{m=M+1}^{\infty}\sum_{n=-\infty}^{+\infty}(-1)^n\frac{\overline{\mathrm{H}_n^{(1)}(k_mR)}}{\mathrm{H}_n^{(1)}(k_mR\,\mathrm{e}^{\mathrm{i}n\varphi_x})}\sin(\alpha_mx_3)\int_{C_R}g(y)\,\mathrm{e}^{-\mathrm{i}n\varphi_y}\sin(\alpha_my_3)\,\mathrm{d}s(y)\Bigg].\end{aligned}$$

Interchanging the order of summation and integral, we can reformulate the last identity as

$$(T_1 g)(x) = \int_{C_R} K(x,y) g(y) \mathrm{d}s(y),$$

where the kernel is given by Eq. (52). Thus, we finish the proof. □

Remark 1 Let v be an outgoing solution to the Helmholtz equation in the waveguide $\mathbb{R}^3_h$, whose modes satisfy the 2D outgoing Sommerfeld radiation condition (10). Suppose v has the expansion

$$v(x) = \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} v_{m,n} \mathrm{H}_n^{(1)}(k_m|\widetilde{x}|)\mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_m x_3), \quad \text{for all } x \in \mathbb{R}^3_h \setminus \Omega_{R_0},$$

where $R_0 > 0$. Then we define the incoming field

$$\widetilde{v}(x) = -\sum_{m=1}^{M} \sum_{n=-\infty}^{+\infty} v_{m,n} \overline{\mathrm{H}_n^{(1)}(k_m R)}\mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_m x_3) - \sum_{m=M+1}^{\infty} \sum_{n=-\infty}^{+\infty} (-1)^n v_{m,n} \overline{\mathrm{H}_n^{(1)}(k_m R)}\mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_m x_3).$$

Then the two parts of the function $\widetilde{v}$ both satisfy the Helmholtz equation in $\mathbb{R}^3_h$, the waveguide boundary condition (3) and the incoming condition. Namely, the modes $\widetilde{v}_m$ satisfy the 2D Helmholtz equation (9) and the incoming condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial \widetilde{v}_m}{\partial \nu} + \mathrm{i}k_m \widetilde{v}_m\right) = 0, \quad r = |\widetilde{x}|$$

uniformly for all direction $\widetilde{x}/r$. Then by Eq. (52) we can readily verify that

$$T_1\left(v|_{C_R}\right) = \widetilde{v}|_{C_R}, \quad \text{for all } R > R_0.$$

In particular, we have

$$T_1\left(G(x,z)|_{x\in C_R}\right) = \overline{G(x,z)}|_{x\in C_R},$$

for all $R > |z|$.

4.2. Characteristic function of D

We first give the properties of the solution operator G_{Dir} defined by Eq. (41) and the modified solution operator $\mathbb{G}_{\mathrm{Dir}}$.

Lemma 2 The solution operator $G_{\mathrm{Dir}} : \mathrm{H}^{1/2}(\partial D) \to \ell^2$ is compact, one-to-one with a dense range in ℓ^2.

Accordingly, the operator $\mathbb{G}_{\mathrm{Dir}} : \mathrm{H}^{1/2}(\partial D) \to L^2(C_R)$ is compact, one-to-one with a dense range in $L^2(C_R)$

Proof First, injectivity of G_{Dir} follows from the uniqueness of the Dirichlet boundary value problem.[20]

From Ref. [3], we have the well-posedness of the three-dimensional waveguide scattering Problem 1 in $H^1_{\mathrm{loc}}(\mathbb{R}^3_h \setminus \overline{D})$. Together with the compact embedding property of $\mathrm{H}^{1/2}(C_R)$ into $L^2(C_R)$, we conclude that G_{Dir} is a compact operator.

Then we will prove that the operator G_{Dir} has a dense range in ℓ^2. For some $P \in \mathbb{Z}^+$, define the sequence $\boldsymbol{g}^{(P)}$ as

$$\boldsymbol{g}^{(P)} = \left\{g^{(P)}_{m,n} : m \in \mathbb{Z}^+, n \in \mathbb{Z}\right\},$$
$$g^{(P)}_{m,n} = \begin{cases} g_{m,n}, & \text{if } m \le P, |n| \le P, \\ 0, & \text{otherwise.} \end{cases}$$

for $\boldsymbol{g} = \left\{g_{m,n} : m \in \mathbb{Z}^*, n \in \mathbb{Z}\right\} \in \ell^2$. Thus, for any $\varepsilon > 0$ there exists a integer $P_\varepsilon > 0$ such that $||\boldsymbol{g}^{(P_\varepsilon)} - \boldsymbol{g}||_{\ell^2} \le \varepsilon$. Now we define the function v by

$$v(x) = \sum_{m=1}^{P_\varepsilon} \sum_{n=-P_\varepsilon}^{+P_\varepsilon} \left[\frac{1}{\mathrm{H}_n^{(1)}(k_m R)} g_{m,n}\right] \mathrm{H}_n^{(1)}(k_m|\widetilde{x}|)\mathrm{e}^{\mathrm{i}n\varphi_x} \sin(\alpha_m x_3), \quad \text{for } \widetilde{x} \neq 0.$$

Obviously, v is a solution to the Helmholtz equation $\Delta v + k^2 v = 0$ in $\mathbb{R}^3_h \setminus \{x \in \mathbb{R}^3_h : \widetilde{x} = 0\}$, and satisfies the outgoing radiation condition (10). From the definition of the solution operator G_{Dir}, we know that $G_{\mathrm{Dir}}\left(v|_{\partial D}\right) = \boldsymbol{g}^{(P_\varepsilon)}$, hence

$$\left\|G_{\mathrm{Dir}}\left(v|_{\partial D}\right) - \boldsymbol{g}\right\|_{\ell^2} < \varepsilon.$$

Thus, we obtain the denseness of the range of G_{Dir} in ℓ^2.

Finally, we notice the fact that T_0 is an unitary operator in ℓ^2 and $\mathscr{F}_R$ is an isomorphism. Then, the properties of $\mathbb{G}_{\mathrm{Dir}}$ follows. □

Lemma 3 Let $\mathbb{G}_{\mathrm{Dir}}$ be given as in Eq. (50). For $z \in \Omega_R$, we define the function

$$\phi_z(x) = \overline{G(x,z)}|_{C_R} \in L^2(C_R).$$

Then $z \in D$ if and only if ϕ_z belongs to the range $\mathscr{R}(\mathbb{G}_{\mathrm{Dir}})$ of $\mathbb{G}_{\mathrm{Dir}}$.

Proof First, we assume that $z \in D$. Clearly, $G(\cdot,z)|_{\mathbb{R}^3_h \setminus \overline{D}}$ is analytic, hence it is the unique radiating solution to the Problem 1 with the boundary data $f := G(\cdot,z)|_{\partial D}$. From the definition of G_{Dir}, $\mathbb{G}_{\mathrm{Dir}}$, and T_1, it follows that $G_{\mathrm{Dir}}(f) = \mathscr{F}_R\left(G(\cdot,z)|_{C_R}\right)$

$$\mathbb{G}_{\mathrm{Dir}}(f) = \mathscr{F}_R^{-1} T_0 G_{\mathrm{Dir}}(f) = \mathscr{F}_R^{-1} T_0 \mathscr{F}_R \left(G(\cdot,z)|_{C_R}\right) = T_1\left(G(\cdot,z)|_{C_R}\right). \tag{54}$$

We notice that $z \in D \subset \Omega_R$, then $|\widetilde{z}| < R = |\widetilde{x}|$ holds for $x \in C_R$. Thus, from the expansion (31) for the Green's function $G(x,z)$ with $|x| > |z|$, the complex conjugate of Green's function can rewrite as

$$\overline{G(x,z)}|_{x\in C_R} = -\frac{\mathrm{i}}{2h} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{+\infty} \overline{\mathrm{H}_n^{(1)}(k_m R)} \sin(\alpha_m x_3) \sin(\alpha_m z_3) \overline{\mathrm{J}_n(k_m|\widetilde{z}|)}\mathrm{e}^{-\mathrm{i}n(\varphi_z - \varphi_x)}$$

$$= -\frac{\mathrm{i}}{2h}\sum_{m=1}^{M}\sum_{n=-\infty}^{+\infty}\overline{\mathrm{H}_n^{(1)}(k_mR)}\sin(\alpha_m x_3)\sin(\alpha_m z_3)\,\mathrm{J}_n(k_m|\tilde{z}|)\,\mathrm{e}^{\mathrm{i}n(\varphi_x-\varphi_z)}$$
$$-\frac{\mathrm{i}}{2h}\sum_{m=+1}^{\infty}\sum_{n=-\infty}^{+\infty}(-1)^n\overline{\mathrm{H}_n^{(1)}(k_mR)}\sin(\alpha_m x_3)\sin(\alpha_m z_3)\,\overline{\mathrm{J}_n(k_m|\tilde{z}|)}\,\mathrm{e}^{\mathrm{i}n(\varphi_x-\varphi_z)}, \tag{55}$$

where $M = M(k)$, such that k_m is real if $m \le M$ while k_m is purely imaginary if $m > M$. The last equality holds with the fact $\overline{\mathrm{J}_n(\mathrm{i}t)} = (-1)^n\mathrm{J}_n(\mathrm{i}t)$. Observing Eq. (55), we find that

$$\overline{G(x,z)}\big|_{x\in C_R} = T_1\big(G(x,z)\big|_{x\in C_R}\big). \tag{56}$$

Combing Eqs. (54) and (55) yields

$$\phi_z(\cdot) = \overline{G(\cdot,z)}\big|_{C_R} \in \mathscr{R}(\mathbb{G}_{\mathrm{Dir}}).$$

On the other hand, let $z \in \Omega_R \subset \mathbb{R}^3_h$ and assume that $\mathbb{G}_{\mathrm{Dir}}(f) = \phi_z$. Since T_0 is unitary on ℓ^2, we have

$$G_{\mathrm{Dir}} = T_0^{-1}\mathscr{F}_R\mathbb{G}_{\mathrm{Dir}}, \text{ and } T_1^* = \mathscr{F}_R^{-1}T_0^{-1}\mathscr{F}_R.$$

Then,

$$\mathscr{F}_R^{-1}G_{\mathrm{Dir}}(f) = \mathscr{F}_R^{-1}T_0^{-1}\mathscr{F}_R\mathbb{G}_{\mathrm{Dir}}(f) = \mathscr{F}_R^{-1}T_0^{-1}\mathscr{F}_R(\phi_z)$$
$$= T_1^*(\phi_z) = G(\cdot,z)\big|_{C_R}.$$

The last identity follows from Eq. (56). Therefore, $G_{\mathrm{Dir}}(f) = \mathscr{F}_R\big(G(\cdot,z)\big|_{C_R}\big)$ holds. Let v be the solution of Problem 1 with the boundary f such that

$$G_{\mathrm{Dir}}(f) = \mathscr{F}_R(v\big|_{C_R}) = \mathscr{F}_R(G(\cdot,z)\big|_{C_R}).$$

Due to the uniqueness of solutions to the Problem 1, we get $v = G(\cdot,z)$ in $\mathbb{R}^3_h \setminus \Omega_R$. Therefore, $v = G(\cdot,z) = \mathbb{R}^3_h \setminus (\overline{D}\cup\{z\})$ holds by analytic continuation. If $z \in \mathbb{R}^3_h \setminus \overline{D}$, this contradicts the fact that v is analytic in $\mathbb{R}^3_h \setminus \overline{D}$ but $G(\cdot,z)$ is singular at z. It is impossible that $z \in \partial D$ since $v\big|_{\partial D} \in H^{1/2}(\partial D)$ but $G(\cdot,z)\big|_{\partial D} \notin H^{1/2}(\partial D)$. Hence, we conclude that $z \in D$, which complete the proof. □

We now give the properties of the middle operator S.

Lemma 4 Some properties of single-layer operator S in the waveguide $\mathbb{R}^3_h$ are given as follows:

(i) The operator S is an isomorphism from space $\mathrm{H}^{-1/2}(\partial D)$ into $\mathrm{H}^{1/2}(\partial D)$.

(ii) Let S_i be defined by Eq. (43) with $k = i$. Then S_i is self-adjoint and coercive as an operator from $\mathrm{H}^{-1/2}(\partial D) \to \mathrm{H}^{1/2}(\partial D)$.

(iii) $\mathrm{Im}\langle\varphi, S\varphi\rangle < 0$ for all $\varphi \in \mathrm{H}^{-1/2}(\partial D)$ with $\varphi \neq 0$. Here, $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathrm{H}^{1/2}(\partial D)$ and $\mathrm{H}^{-1/2}(\partial D)$ extending the L^2-product in $L^2(\partial D)$.

(iv) The difference $S - S_i$ is compact from $\mathrm{H}^{-1/2}(\partial D)$ into $\mathrm{H}^{1/2}(\partial D)$.

Proof From Ref. [20], we realize that the single-layer potential in the waveguide $\mathbb{R}^3_h$ has the same properties as the single-layer potential in the free space, such as the jump relation and the Green's theorems. In the case of free space $\mathbb{R}^3$, the same conclusion holds, see Lemma 1.14 in Ref. [18]. Thus, from the same process, Lemma 4 can be proved. □

Now, we have shown all prerequisites which are necessary to apply Theorem 2.15 in Ref. [18].

Theorem 2 Let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space U and reflexive Banach space X such that the embedding is dense. Furthermore, let Y be a second Hilbert space and let $F : Y \to Y$, $G : X \to Y$ and $T : X^* \to X$ be linear bounded operators such that

$$F = GTG^*. \tag{57}$$

We make the following assumptions:

i) G is compact with dense range.

ii) $\mathrm{Re}T$ has the form $\mathrm{Re}T = T_0 + T_1$ with some coercive operator T_0 and some compact operator $T_1 : X^* \to X$.

iii) Im T is nonnegative on X, i.e., $\langle\varphi, (\mathrm{Im}T)\varphi\rangle > 0$.

iv) T is injective.

Then the operator $F^\sharp = \big|\mathrm{Re}[\mathrm{e}^{\mathrm{i}t}T]\big| + |\mathrm{Im}\,F|$ is positive, self-adjoint, and compact, and the range of $G : X \to Y$ and $F_\sharp^{1/2} : Y \to Y$ coincide. Here $|\mathrm{Re}F|$ is defined in the standard way via the spectral representation for compact self-adjoint operators, and the parts $\mathrm{Re}F$ and $\mathrm{Im}F$ are defined by

$$\mathrm{Re}\,F = \frac{1}{2}(F + F^*), \quad \mathrm{Im}F = \frac{1}{2\mathrm{i}}(F - F^*).$$

From this result on range identities in factorizations applied to the factorization $F = GTG^*$, we obtain the final result of this section, which give an explicit reconstruction of D.

Theorem 3 Assume that $k_m \neq \alpha_m$ for all $m \in \mathbb{N}$. For $z \in \Omega_R$, ϕ_z is given as in Lemma 3. Denote by $(\lambda_j, \psi_j)_{j\in\mathbb{N}}$ an eigensystem of the normal operator $(T_1N)_\sharp := |\mathrm{Re}(T_1N)| + |\mathrm{Im}(T_1N)|$. Then

$$z \in D \Longleftrightarrow \phi_z \in \mathscr{R}\Big[\big(T_1N\big)_\sharp^{1/2}\Big]$$
$$\Longleftrightarrow W(z) := \left[\sum_j \frac{\big|\big(\phi_z, \psi_j\big)_{L^2(C_R)}\big|^2}{|\lambda_j|}\right]^{-1} > 0. \tag{58}$$

Proof According to the Lemmas 2, 3, and 4, the theorem is a direct corollary of Theorem 5. □

This theorem presents a sufficient and necessary computational criterion for precisely characterizing the region occupied by the sound-soft obstacles. In other words, the characteristic function χ_D of D is given by $\chi_D(z) = \mathrm{sign}(W(z))$. In

conclusion, we establish the factorization method for reconstructing a sound-soft obstacle in the waveguide $\mathbb{R}^3_h$ with the auxiliary outgoing-to-incoming operator T_1. When the boundary condition is sound-hard or impedance, the similar result can be deduced with the same means.

5. Impedance boundary condition

In this section we talk about the obstacle with respect to impedance boundary conditions. Then the scattered field in this case satisfies the following equations:

Problem 2 [Impedance boundary value problem] Given $f \in H^{-1/2}(\partial D)$, find a solution to the problem:

$$\Delta v + k^2 v = 0 \quad \text{in } \mathbb{R}^3_h \setminus \overline{D},$$
$$\mathscr{B} v := \frac{\partial v}{\partial \nu} + \rho(x) v = f \quad \text{on } \partial D,$$
$$v = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \Gamma_h,$$
the radiation conditioin,

where the impedance function $\rho(x) \in L^\infty(\partial D)$ is complex-valued satisfies that $\mathrm{Im}(\rho) \geq 0$ almost everywhere on ∂D.

Noting that for fixed $y \in \mathbb{R}^3_h \setminus \overline{D}$, when we choose

$$f = -\frac{\partial G(x,y)}{\partial \nu(x)} - \rho(x) G(x,y), \quad x \in \partial D,$$

the solution of Problem 2 is the scattered field with the incident field $G(x,y)$ whose source point at $y \in \mathbb{R}^3_h \setminus \overline{D}$.

We use the similar means in the case of Dirichlet boundary conditions, we first introduce the incidence operator $\ell^2 \to H^{-1/2}(\partial D)$ by

$$\big(H_{\mathrm{imp}} \boldsymbol{g}\big)(x) := \frac{\mathrm{i}\pi R}{2} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathrm{H}_n^{(1)}(k_m R) g_{m,n} \left(\frac{\partial}{\partial \nu(x)} + \rho(x) \right) \left[\mathrm{J}_n(k_m |\widetilde{x}|) \mathrm{e}^{\mathrm{i} n \varphi_x} \sin(\alpha_m x_3) \right], \quad x \in \partial D, \tag{59}$$

with $\boldsymbol{g} \in \ell^2$ given by Eq. (16). The adjoint operator $H^*_{\mathrm{imp}} : H^{1/2}(\partial D) \to \ell^2$ takes the form

$$H^*_{\mathrm{imp}} \psi = \left\{ -\frac{\mathrm{i}\pi R}{2} \overline{\mathrm{H}_n^{(1)}(k_m R)} \int_{\partial D} \left(\frac{\partial}{\partial \nu(y)} + \overline{\rho(y)} \right) \left[\overline{\mathrm{J}_n(k_m |\widetilde{y}|)} \mathrm{e}^{-\mathrm{i} n \varphi_y} \sin(\alpha_m y_3) \right] \psi(y) \mathrm{d}s(y) : n \in \mathbb{Z}^+, m \in \mathbb{Z} \right\} \tag{60}$$

for all $\psi \in H^{1/2}(\partial D)$. Further, we define the solution operator $G_{\mathrm{imp}} : \mathrm{H}^{-1/2}(\partial D) \to \ell^2$ by

$$G_{\mathrm{imp}}(f) = \left\{ b_{m,n} \mathrm{H}_n^{(1)}(k_m R) : n \in \mathbb{Z}^+, m \in \mathbb{Z} \right\}, \tag{61}$$

where $b_{m,n} \mathrm{H}_n^{(1)}(k_m R)$ are the Fourier coefficients of $u\big|_{C_R}$ with u the radiating solution to the Problem 2.

Define the layer-potential operators K, K', and J, respectively, by

$$K\psi(x) = \int_{\partial D} \frac{\partial G(x,y)}{\partial \nu(y)} \psi(y) \mathrm{d}s(y) \quad \text{for } x \in \partial D, \tag{62}$$

$$K'\psi(x) = \int_{\partial D} \frac{\partial G(x,y)}{\partial \nu(x)} \psi(y) \mathrm{d}s(y) \quad \text{for } x \in \partial D, \tag{63}$$

$$J\psi(x) = \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial G(x,y)}{\partial \nu(y)} \psi(y) \mathrm{d}s(y) \quad \text{for } x \in \partial D. \tag{64}$$

Since the Green's function can be written as the superposition of the fundamental solution $\Phi(x,y)$ in free space and an analytic function, the layer-potential operators has the same properties as the ones with kernel Φ. It follows from Refs. [20] and [23] that the operators $K : \mathrm{H}^{1/2}(\partial D) \to \mathrm{H}^{1/2}(\partial D)$, $K' : \mathrm{H}^{-1/2}(\partial D) \to \mathrm{H}^{-1/2}(\partial D)$ and $\mathrm{J} : \mathrm{H}^{1/2}(\partial D) \to \mathrm{H}^{-1/2}(\partial D)$ are all bounded.

Theorem 4 Let T_0 and $\mathscr{N}$ be given by Eqs. (48) and (26), respectively. $T_1 = \mathscr{F}_R^{-1} T_0 \mathscr{F}_R$ has been defined in Theorem 2. Then

$$T_0 \mathscr{N} = -\pi R h \big(T_0 G_{\mathrm{imp}}\big) T^*_{\mathrm{imp}} \big(T_0 G_{\mathrm{imp}}\big)^*, \tag{65}$$

$$T_1 N = -\mathbb{G}_{\mathrm{imp}} T^*_{\mathrm{imp}} \mathbb{G}^*_{\mathrm{imp}}, \quad \mathbb{G}_{\mathrm{imp}} = \mathscr{F}_R^{-1} T_0 G_{\mathrm{imp}}, \tag{66}$$

where $T_{\mathrm{imp}} : \mathrm{H}^{1/2}(\partial D) \to \mathrm{H}^{-1/2}(\partial D)$ is defined as $T_{\mathrm{imp}} = J + \mathrm{i}(\mathrm{Im}\ \rho) I + K' \overline{\rho} + \rho K + \rho S \overline{\rho}$.

Proof Define

$$V(x) := \int_{\partial D} \left[\frac{\partial G(x,y)}{\partial \nu(y)} + \overline{\rho(y)} G(x,y) \right] \psi(y) \mathrm{d}s(y), \quad x \in \mathbb{R}^3_h \setminus \partial D. \tag{67}$$

From Eq. (30), we can see that

$$V(x)\big|_{C_R} = \frac{\mathrm{i}}{2h} \sum_{m=1}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathrm{H}_n^{(1)}(k_m R) \sin(\alpha_m x_3) \mathrm{e}^{\mathrm{i} n \varphi_x} \int_{\partial D} \left(\frac{\partial}{\partial \nu(y)} + \overline{\rho(y)} \right) \left[\mathrm{J}_n(k_m |\widetilde{y}|) \sin(\alpha_m y_3) \mathrm{e}^{-\mathrm{i} n \varphi_y} \right] \psi(y) \mathrm{d}s(y).$$

We recall that $\mathrm{J}_n(k_m|\widetilde{x}|)$ has the conjugate property (39), together with the expression (60) of H^*_{imp}, we can deduce that

$$\pi RhT_0 G_{\mathrm{imp}}\big(\mathscr{B}V\big|_{\partial D}\big) = H^*_{\mathrm{imp}}\psi. \tag{68}$$

Using the jump relations of layer-potentials, we have

$$\begin{aligned}\mathscr{B}V\big|_{\partial D} &= J\psi + i(\mathrm{Im}\,\rho)\psi + K'(\overline{\rho}\psi) + \rho K\psi + \rho S(\overline{\rho}\psi)\\ &= T_{\mathrm{imp}}\psi\end{aligned} \tag{69}$$

Then, by Eq. (68) we have

$$\pi RhT_0 G_{\mathrm{imp}} T_{\mathrm{imp}} = H^*_{\mathrm{imp}} \ \text{ and } \ \pi RhT^*_{\mathrm{imp}} G^*_{\mathrm{imp}} T^*_0 = H_{\mathrm{imp}}. \tag{70}$$

Recalling the definition of the operator N in Eq. (26), we observe that

$$\mathscr{N} = -G_{\mathrm{imp}} H_{\mathrm{imp}}$$

under the impedance boundary condition, from which the relation

$$T_0\mathscr{N} = -\pi Rh\big(T_0 G_{\mathrm{imp}}\big)T^*_{\mathrm{imp}}\big(T_0 G_{\mathrm{imp}}\big)^* \tag{71}$$

follows. This completes the proof of Eq. (65). Since $N = \mathscr{F}_R^{-1}\mathscr{N}\mathscr{F}_R$, the factorization (66) can be justified in the same manner as in the case of Dirichlet boundary conditions. □

Theorem 5 Assume that $k_m \neq \alpha_m$ for all $m \in \mathbb{N}$ and k^2 is not an eigenvalue of $-\Delta$ in $D \subset \mathbb{R}^3_h$ with respect to the impedance boundary condition.

(I) The operator $\mathbb{G}_{\mathrm{imp}}$ is compact, one-to-one with a dense range in $L^2(S_R)$.

(II) The operator T_{imp} is an isomorphism from the space $\mathrm{H}^{1/2}(\partial D)$ into $\mathrm{H}^{-1/2}(\partial D)$.

(III) Let J_i be define by Eq. (64) with $k = i$. Then J_i is self-adjoint and coercive as an operator from $\mathrm{H}^{1/2}(\partial D)$ into $\mathrm{H}^{-1/2}(\partial D)$.

(IV) $\mathrm{Im}\big\langle T_{\mathrm{imp}}\varphi, \varphi\big\rangle > 0$ for all $\varphi \in \mathrm{H}^{-1/2}(\partial D)$ with $\varphi \neq 0$.

(V) The difference $T_{\mathrm{imp}} - J_i$ is compact from $\mathrm{H}^{1/2}(\partial D)$ into $\mathrm{H}^{-1/2}(\partial D)$.

Since the difference between the Green's function in waveguide and the fundamental solution in free space is an analytic function, the theorem above is a trivial extension of Theorem 4.2 in Ref. [17].

Similarly to Theorem 3 for the sound-soft obstacle case, we can prove the following result.

Theorem 6 Assume that $k_m \neq \alpha_m$ for all $m \in \mathbb{N}$ and k^2 is not an eigenvalue of $-\Delta$ in D with respect to the impedance boundary condition. For $z \in \Omega_R$, define the function $\phi_z(\cdot) = \overline{G(\cdot,z)}\big|_{C_R}$. Then

$$z \in D \Longleftrightarrow \phi_z \in \mathscr{R}\left[(T_1 N)^{1/2}_\sharp\right]$$

$$\Longleftrightarrow W(z) := \left[\sum_j \frac{\big|(\phi_z, \psi_j)_{L^2(C_R)}\big|^2}{\lambda_j}\right]^{-1} > 0, \tag{72}$$

where $\lambda_j \in \mathbb{C}$ are the eigenvalues of the normal operator

$$(T_1 N)_\sharp := |\mathrm{Re}(T_1 N)| + |\mathrm{Im}(T_1 N)|$$

with the corresponding normalized eigenfunctions $\psi_j \in L^2(C_R)$

We remark that when we take $\rho(x) = 0$, the factorization method with near-field data in the waveguide extends straightly to the Neumann boundary condition, provided for any $m \in \mathbb{Z}^+$, $k \neq (2m-1)\pi/2h$.

6. Inverse medium scattering problem

In this section, we consider that D is a penetrable with the refraction index $n(x)$. Assume that $\mathrm{Re}\,(n) \geq 0$, $\mathrm{Im}\ (n) \geq 0$, $n = 1$ in background $\mathbb{R}^3_h \setminus \overline{D}$ and $n \neq 1$ in D. Then the direct scattering in a planar waveguide with a penetrable medium is modeled simply by the equation

$$\Delta u + k^2 n(x) u = 0 \tag{73}$$

with the waveguide boundary condition (3) and modal radiation condition (10). The incident field u^{i} in this section is still the point source at $y \in C_R$. Then the scattered field u^{s} is the solution of the following problem with $f = u^{\mathrm{i}}\big|_D \in L^2(D)$ corresponding to the incident field $u^{\mathrm{i}}(\cdot) = G(\cdot, y)$.

Problem 3 [Inhomogeneous medium problem]

Given $f \in L^2(D)$, find the solution v to the following problems:

$$\begin{aligned}&\Delta v + k^2 n(x) v = k^2(1-n(x))f, \quad \text{in } \mathbb{R}^3_h,\\ &v = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \Gamma_h,\\ &\text{the radiation condition (10).}\end{aligned}$$

As in the case of sound-soft obstacle, we also define the solution operator $G_{\mathrm{pen}} : L^2(D) \to \ell^2$ and the incidence operator $H_{\mathrm{pen}} : \ell^2 \to L^2(D)$ for the Problem 3 as follows:

$$G_{\mathrm{pen}}(f) = \big\{b_{m,n}\mathrm{H}^{(1)}_n(k_m R) : m \in \mathbb{Z}^+, n \in \mathbb{Z}\big\}, \tag{74}$$

$$H_{\mathrm{pen}}(\boldsymbol{g}) = \frac{\mathrm{i}\pi R}{2}\sum_{m=1}^{+\infty}\sum_{n=-\infty}^{+\infty}\mathrm{H}^{(1)}_n(k_m R) g_{m,n} u^{\mathrm{i}}_{m,n}(x), \ \ x \in D, \tag{75}$$

where $b_{m,n}\mathrm{H}^{(1)}_n(k_m R)$ are the Fourier coefficients of $v\big|_{C_R}$ in the expansion (40). And $u^{\mathrm{i}}_{m,n}$, $\boldsymbol{g}$ are given as in Section 3. We still have the relation $\mathscr{N} = -G_{\mathrm{pen}} H_{\mathrm{pen}}$. The adjoint operator $H^*_{\mathrm{pen}} : L^2(D) \to \ell^2$ can be readily given by

$$H^*_{\mathrm{Dir}}\psi = \left\{-\frac{\mathrm{i}\pi R}{2}\overline{\mathrm{H}^{(1)}_n(k_m R)}\int_{\partial D}\psi(y)\overline{\mathrm{J}_n(k_m|\widetilde{x}|)}\,\mathrm{e}^{-\mathrm{i}n\varphi_x}\sin(\alpha_m x_3)\,\mathrm{d}s(x) : n \in \mathbb{Z}^+, \ \ m \in \mathbb{Z}\right\}. \tag{76}$$

Then, by the unitary operator T_0 and expansion (30) of Green's function, we derive that

$$\left(T_0^* H_{\text{pen}}^* \varphi\right)(x) = \pi h R \mathscr{F}_R\left(\int_D G(x,y)\varphi(y)dy\big|_{C_R}\right). \quad (77)$$

For the penetrable obstacle, define the middle operator $T_{\text{pen}}: L^2(D) \to L^2(D)$ by

$$T_{\text{pen}}\varphi = \frac{1}{k(n-1)}\varphi - \int_D G(x,y)\varphi(y)\,\mathrm{d}s(y). \quad (78)$$

It is easy to verify that $\int_D G(x,y)\varphi(y)\,\mathrm{d}y$ is the solution of Problem 3 with $f = T_{\text{pen}}\varphi$. Then, we derive the relations as follows:

$$T_0^* H_{\text{pen}}^* = \pi h R G_{\text{pen}} T_{\text{pen}} \quad \text{and} \quad H_{\text{pen}} T_0 = \pi h R T_{\text{pen}}^* G_{\text{pen}}^*. \quad (79)$$

Therefore, we get the factorization

$$T_0 \mathscr{N} = -\pi h R (T_0 G_{\text{pen}}) T_{\text{pen}}^* (T_0 G_{\text{pen}})^* \quad (80)$$

Here, $\mathscr{N}$ is defined as in Eq. (26). By defining the operator $\mathbb{G}_{\text{pen}} := \mathscr{F}_R^{-1} T_0 G_{\text{pen}}$, together with the properties (23) of Fourier operator $\mathscr{F}$, we have the factorization of $T_1 N$,

$$T_1 N = -\mathbb{G}_{\text{pen}} T_{\text{pen}} \mathbb{G}_{\text{pen}}^*. \quad (81)$$

where T_1 is given in Theorem 1.

The middle operator T_{pen} of the factorization (81) satisfies all the properties of Theorem 5 from Refs. [2], [20], and [24]. Using the similar argument as in Section 4, we have the following result.

Theorem 7 Let $n(x) \neq 1$ for all $x \in D$. For $z \in \Omega_R$, define $\phi_z(\cdot) = \overline{G(\cdot,z)}\big|_{C_R}$. Then

$$\begin{aligned} z \in D &\Longleftrightarrow \phi_z \in \mathscr{R}\left[\left(T_1 N\right)_\sharp^{1/2}\right] \\ &\Longleftrightarrow W(z) := \left[\sum_j \frac{\left|\left(\phi_z, \psi_j\right)_{L^2(C_R)}\right|^2}{|\lambda_j|}\right]^{-1} > 0, \end{aligned} \quad (82)$$

where $\lambda_j \in \mathbb{C}$ are the eigenvalues of the normal operator $(T_1 N)_\sharp := |\mathrm{Re}\,(T_1 N)| + |\mathrm{Im}\,(T_1 N)|$ with the corresponding normalized eigenfunctions $\psi_j \in L^2(C_R)$.

7. Conclusion

In this paper, we establish the factorization method for the impenetrable and penetrable obstacles, with the idea of defining the outgoing-to-incoming operator T_1 to find an appropriate factorization of $T_1 N$. From the final results, Theorems 6, 9, and 10, we know that the characteristic functions of D only depend on the near-field operator, and T_1 is determined by the waveguide $\mathbb{R}_h^3$ and wavenumber k. Finally, we observe that the inversion scheme is independent of the physical properties of the underlying obstacles. This is an advantage to practical application.

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References

[1] Colton D and Kress R 2013 *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd edn. (New York: Springer) p. 1

[2] Kirsch A 2011 *An introduction to mathematical theory of inverse problem* (Berlin: Springer) p. 1

[3] Arens T, Gintides and Lechleiter A 2008 *Math. Method. Appl. Sci.* **31** 821

[4] Arens T, Gintides and Lechleiter A 2011 *SIAM J. Appl. Math.* **71** 753

[5] Bourgeois L, Chambeyron C and Kusiak S 2007 *J. Comput. Appl. Math.* **204** 387

[6] Bourgeois L and Lunéville E 2008 *Inverse Probl.* **24** 015018

[7] Ikehata M, Makrakis G N and Nakamura G 2004 *Math. Method Appl. Sci.* **27** 1367

[8] Lechleiter A and Nguyen D L 2012 *IMA J. Numer. Anal.* **32** 813

[9] Dediu S and Mclaughlin J R 2006, *Inverse Probl.* 22 1227

[10] Gilbert R P, Werby M and Xu Y Z 2001 *J. Comput. Acoust.* **9** 1025

[11] Chen Z M and Huang G H 2015 *Sci. Chin. Math.* **58** 1811

[12] Liu K J, Xu Y Z and Zou J 2014 *Appl. Math. Comput.* **235** 364

[13] Sun J G and Zheng C X 2013 *Contemp. Math.* **586** 341

[14] Ahluwalia D S and Keller J B 1977 "Exact and asymptotic representations of the sound field in a stratified ocean", in *Wave Propagation and Underwater Acoustics* (Berlin, Heidelberg: Springer) **70** p. 14

[15] Xu Y Z, Mawata C and Lin W 2000 *Inverse Probl.* **16** 1761

[16] Arens T and Kirsh A 2002 *Inverse Probl.* **19** 1195

[17] Hu G H, Yang J Q, Zhang B and Zhang H W 2014 *Inverse Probl.* **30** 095005

[18] Kirsch A and Grinberg N 2008 *The Factorization Method for Inverse Problems* (Oxford: Oxford University Press) p. 1

[19] Yin T, Hu G H, Xu L W and Zhang B 2016 *Inverse Probl.* **32** 015003

[20] Xu Y Z 1990 *Appl. Anal.* **35** 129

[21] Ramm A G and Makrakis G N 1998 in *Spectral and Scattering Theory* (New York: Plenum Publishers) p. 89

[22] Linton C M and Mclver P 2007 *Wave Motion* 45 16

[23] McLean W 2000 *Strongly elliptic systems and boundary integral equations* (Cambridge: Cambridge University Press) p. 1

[24] Lechleiter A 2009 *Inverse Probl. Imaging* **3** 123