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Truncated series solutions to the (2+1)-dimensional perturbed Boussinesq equation by using the approximate symmetry method*

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In this paper, the (2+1)-dimensional perturbed Boussinesq equation is transformed into a series of two-dimensional (2D) similarity reduction equations by using the approximate symmetry method. A step-by-step procedure is used to acquire Jacobi elliptic function solutions to these similarity equations, which generate the truncated series solutions to the original perturbed Boussinesq equation. Aside from some singular area, the series solutions are convergent when the perturbation parameter is diminished.

Keywords: approximate symmetry method, (2+1)-dimensional perturbed Boussinesq equation, series solutions, convergence of series solutions

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1. Introduction

Lie group theory[1,2] is one of the most effective tools for investigating nonlinear equations. By reducing the number of independent variables, the classical or nonclassical Lie group method[3] and the Lie symmetry method[4–6] can be effectively used to simplify partial differential equations (PDEs). The partial differential equations can also be dealt with by the direct reduction method,[7] although it is not directly concerned with group theory. Another extended direct method[8,9] produces various kinds of transform groups including the Lie symmetry group and discrete group.

For the nonclassical Lie group method, invariant curve conditions are appended to the original equation, then the classical Lie group method applies for the enlarged system. An enlarged system can be generated through other ways. By introducing auxiliary variables, nonlocal symmetries can be localized and a prolonged nonlinear system can be constructed and simplified by the Lie symmetry method. This scheme benefits the research into interactions among different types of nonlinear waves such as the solitons (or solitary waves), the cnoidal periodic waves, and Painlevé waves.[10–14]

Combined with perturbation theory,[15,16] the Lie group method facilitates the construction of approximate solutions to perturbed nonlinear equations through the following two ways. The first was to generalize the symmetry group generators to perturbation forms in Ref. [17] and the second was to apply the Lie symmetry method to the approximate equations from the perturbation expansion in Ref. [18]. The second one was further developed to achieve reduced equations with general forms in Refs. [19]–[22]. A further extension of the second method is to introduce a homotopy model to produce the approximate homotopy symmetry method,[23,24] which is superior to the original method for several aspects such as adjusting the convergence of the series solutions. These applications of the approximate symmetry method were only confined to (1+1)-dimensional perturbed equations and some problems such as convergence of series solutions remain to be studied. In this paper, a (2+1)-dimensional perturbed Boussinesq equation will be investigated towards these subjects.

The (2+1)-dimensional Boussinesq equation

$$u_{tt}+3(u^2)_{xx}-u_{xx}-u_{xxxx}-u_{yy}=0 \tag{1}$$

was derived by combining the classical Boussinesq equation with the weak dependence on the second spatial dimension.[25] It can be used to describe the propagation of gravity waves on water surfaces, in particular the head-on collision of oblique waves.

With higher order nonlinear perturbation terms such as u_{xxxxxx} considered, we come up with the perturbed equation

$$u_{tt}+3(u^2)_{xx}-u_{xx}-u_{xxxx}-u_{yy}=\varepsilon u_{xxxxxx}, \tag{2}$$

where ε is a perturbation parameter.

According to perturbation theory, a perturbation expansion

$$u=\sum_{k=0}^{\infty}u_k\varepsilon^k \tag{3}$$

is assumed for investigating perturbed nonlinear equations, so that the perturbed Boussinesq equation is readily decomposed

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into a series of equations

$$u_{k,tt}+6\sum_{i=0}^{k}(u_{k-i}u_{i,x})_x-u_{k,xx}-u_{k,xxxx}-u_{k,yy}=u_{k-1,xxxxxx},\quad (k=0,1,\ldots) \tag{4}$$

with $u_{-1}=0$. We name these equations approximate equations.

2. Similarity solutions to approximate equations

To study symmetry reductions of Eq. (4), we first construct the Lie point symmetry transformation of the vector form

$$V=X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+T\frac{\partial}{\partial t}+\sum_{k=0}^{\infty}U_k\frac{\partial}{\partial u_k}, \tag{5}$$

where X, Y, T, and U_k are functions with respect to $\{x,y,t,u_k,(k=0,1,\ldots)\}$. This Lie point symmetry transformation means that the approximate equations in Eq. (4) are invariant under the point transformation

$$\{x,y,t,u_k\}\to\{x+\varepsilon X,y+\varepsilon Y,t+\varepsilon T,u_k+\varepsilon U_k\},\quad (k=0,1,\ldots) \tag{6}$$

with the infinitesimal parameter ε. If the model equations are not explicitly space-time $\{x,y,t\}$ dependent, the symmetry in the vector form (5) can be written as a function form

$$\sigma_k=Xu_{k,x}+Yu_{k,y}+Tu_{k,t}-U_k,\ (k=0,1,\ldots) \tag{7}$$

which satisfies the linearized equations of Eq. (4)

$$\sigma_{k,tt}+6\sum_{i=0}^{k}(\sigma_{k-i}u_{i,x}+u_{k-i}\sigma_{i,x})_x-\sigma_{k,xx}-\sigma_{k,xxxx}-\sigma_{k,yy}=\sigma_{k-1,xxxxxx},\ (k=0,1,\ldots) \tag{8}$$

with $\sigma_{-1}=0$.

Equations (4), (7), and (8) are to be combined to determine X, Y, T, and U_k, $(k=0,1,\ldots)$. Only the first few equations are considered in Eq. (4) for convenience. Restriction of k to $k\in\{0,1,2\}$ in Eq. (4) is adequate for studying the symmetry reduction of the general form. In this case, it is easily seen that the variables of X, Y, T, U_0, U_1, and U_2 are confined to $\{x,y,t,u_0,u_1,u_2\}$. We first insert Eq. (7) into Eq. (8), then eliminate $\{u_{0,tt},u_{1,tt},u_{2,tt}\}$ with Eq. (4), and vanish all derivatives of $\{u_0,u_1,u_2\}$. A large system of determining equations for $\{X,Y,T,U_0,U_1,U_2\}$ will be obtained. We follow a step-by-step procedure to solve these determining equations.

For $k=0$ in Eqs. (4), (7), and (8), 447 determining equations for the functions $\{X,Y,T,U_0\}$ can be achieved and solved as

$$X=\frac{b}{2}x+a_0,$$
$$Y=by+at+a_2,\ T=ay+bt+a_1,\ U_0=b\Big(\frac{1}{6}-u_0\Big), \tag{9}$$

where $\{a,b,a_0,a_1,a_2\}$ are arbitrary constants.

Similarly, for $k=1$ in Eqs. (4), (7), and (8) with Eq. (9), we can obtain 95 determining equations for U_1, which hold only if $U_1=-2bu_1$. Taking $k=2$, in the same way we can gain 95 determining equations for U_2, from which we get $U_2=-3bu_2$. It is thus inferred that

$$X=\frac{b}{2}x+a_0,\ \ Y=by+at+a_2,$$
$$T=ay+bt+a_1,\ \ U_k=\frac{1}{6}b\delta_{k,0}-(k+1)bu_k, \tag{10}$$

with $k=1,2,\ldots$, where the notation $\delta_{i,j}$ satisfies $\delta_{0,0}=1$ and $\delta_{k,0}=0$, and $(k\neq 0)$ also applies in the following text.

The constant b in U_k should be distinguished in the following two subsections.

2.1. The case for $b\neq 0$

The constants $\{a_0,a_1,a_2\}$ in Eq. (10) that signify translational invariance are negligible. Without loss of generality, we adjust the constants $a\mapsto ab$ and $b\mapsto 2b$. Similarity solutions to Eq. (4) can be solved as

$$u_k=\frac{P_k(\xi,\eta)}{x^{2(k+1)}}+\frac{1}{6}\delta_{k,0},$$
$$\xi=\frac{x^{a-2}(t-y)}{2},\ \eta=\frac{x^{-(a+2)}(t+y)}{2},\ (k=0,1,\ldots) \tag{11}$$

from $\sigma_k=0$ in Eq. (7) with Eq. (10).

These similarity solutions transform Eq. (4) into reduction equations with a general form, which is shown in Eq. (A1) in Appendix A. We consider particular solutions to these reduction equations such as Jacobi elliptic function solutions and make the assumption

$$P_k=\sum_{i=0}^{n_k}f_{k,i}(\xi,\eta)\,\mathrm{sn}^i(g(\xi),m), \tag{12}$$

where $\{f_{k,i},g\}$ with the indicated variables are functions to be determined.

Taking $k=0$ in Eq. (A1), considering Eq. (12) and balancing the highest degree of

$$\{\mathrm{sn}(g(\xi,\eta),m),\mathrm{cn}(g(\xi),m),\mathrm{dn}(g(\xi),m)\}$$

from $P_{0,\xi\xi\xi\xi}$ and $P_{0,\xi}^2$, we see that $n_0+4=2n_0+2$. The same rule applies for $P_{k,\xi\xi\xi\xi}$ and $P_{k-1,\xi\xi\xi\xi\xi\xi}$ in every equation of Eq. (A1) to give $n_k+4=n_{k-1}+6$, leading to $n_k=2(k+1)$.

The determination of P_k in Eq. (12) is implemented through a step-by-step procedure.

Step 1 Considering zero-order reduction equation ($k=0$ in Eq. (A1) and its undetermined solution ($k=0$ in Eq. (12)), we vanish the coefficients of

$$\{\mathrm{sn}(g(\xi),m),\mathrm{cn}(g(\xi),m),\mathrm{dn}(g(\xi),m)\}$$

after the substitution.

From the coefficient of $\mathrm{sn}^6(g(\xi),m)$, we obtain

$$f_{0,2}=2(a-2)^2m^2\xi^2g_\xi^2. \tag{13}$$

From the coefficient of $\mathrm{sn}^5(g(\xi),m)$ and Eq. (13), we see that

$$f_{0,1}=0. \tag{14}$$

From the coefficient of $\mathrm{sn}^4(g(\xi),m)$, Eqs. (13) and (14), we obtain

$$f_{0,0}=-\frac{1}{6}(a-2)^2\xi^2\left[4(m^2+1)g_\xi^2-\frac{12g_{\xi\xi\xi}}{g_\xi}-27\left(\frac{g_{\xi\xi}}{g_\xi}\right)^2\right]+15\xi(a-3)(a-2)\frac{g_{\xi\xi}}{g_\xi}+\frac{(a-3)(13a-43)}{2}. \tag{15}$$

After the substitution of Eqs. (13), (14), and (15) into the coefficient of $\mathrm{sn}^3(g(\xi),m)$, the unknown function $g(\xi)$ can be determined as

$$g=c_0+c_1\xi^{1/(a-2)}, \tag{16}$$

where we can specify the arbitrary constants as $c_0=0$, $c_1=1$ directly.

Other coefficients of $\mathrm{sn}(g(\xi),m)$ vanish automatically due to Eqs. (13)–(16). Meanwhile, the zero-order equation in Eq. (A1) subjects to the solution

$$P_0=\frac{2}{3}\xi^{2/(a-2)}[-1-m^2+3m^2\mathrm{sn}^2(\xi^{1/(a-2)},m)]. \tag{17}$$

Step 2 Given Eqs. (16) and (17), considering one-order reduction equation ($k=1$ in Eq. (A1)) and its undetermined solution ($k=1$ in Eq. (12)), we still vanish the coefficients of $\{\mathrm{sn}(\xi^{1/(a-2)},m),\mathrm{cn}(\xi^{1/(a-2)},m),\mathrm{dn}(\xi^{1/(a-2)},m)\}$ after the substitution.

The coefficient of $\mathrm{sn}^8(\xi^{1/(a-2)},m)$ leads to

$$f_{1,4}=-30m^4\xi^{4/(a-2)}. \tag{18}$$

From the coefficient of $\mathrm{sn}^6(\xi^{1/(a-2)},m)$ and Eq. (18), we gain

$$f_{1,2}=20m^2(1+m^2)\xi^{4/(a-2)}. \tag{19}$$

From the coefficient of $\mathrm{sn}^5(\xi^{1/(a-2)},m)$, Eqs. (18) and (19), we see that

$$\begin{aligned}f_{1,1}=\frac{1}{2m^2}\Big[&(a+2)^2\xi^{2/(2-a)}\eta^2f_{1,3,\eta\eta}-2(a+2)(a-2)\xi^{(a-4)/(a-2)}\eta f_{1,3,\xi\eta}+(a-2)^2\xi^{2(a-3)/(a-2)}f_{1,3,\xi\xi}\\&+(a+2)(a+11)\xi^{2/(2-a)}\eta f_{1,3,\eta}+(a-2)(a-11)\xi^{(a-4)/(a-2)}f_{1,3,\xi}+(-1-m^2+20\xi^{2/(2-a)})f_{1,3}\Big].\end{aligned} \tag{20}$$

From the coefficient of $\mathrm{sn}^4(\xi^{1/(a-2)},m)$, Eqs. (18), (19), and (20), we obtain

$$f_{1,0}=\frac{2}{3}(4-19m^2+4m^4)\xi^{4/(a-2)}. \tag{21}$$

Other coefficients of $\{\mathrm{sn}(\xi^{1/(a-2)},m),\mathrm{cn}(\xi^{1/(a-2)},m),\mathrm{dn}(\xi^{1/(a-2)},m)\}$ give rise to partial differential equations with only one function $f_{1,3}$ under Eqs. (18)–(21), from which we see that $f_{1,3}=0$. Accordingly, the one-order reduction equation in Eq. (A1) admits the solution

$$P_1=\frac{2}{3}\xi^{4/(a-2)}\left[4-19m^2+4m^4+30m^2(1+m^2)\mathrm{sn}^2(\xi^{1/(a-2)},m)-45m^4\,\mathrm{sn}^4(\xi^{1/(a-2)},m)\right]. \tag{22}$$

Step 3 Similar to Steps 1 and 2, given Eq. (16), Eq. (17), and Eq. (22), considering the two-order reduction equation ($k=2$ in Eq. (A1)) and its undetermined solution ($k=2$ in Eq. (12)), we can solve $f_{2,j}$, ($j=6,5,4,2,1,0$) from the coefficients of

$$\mathrm{sn}^l(\xi^{1/(a-2)},m),\quad l=10,9,8,6,5,4$$

one by one:

$$f_{2,6}=930m^6\xi^{6/(a-2)}, \tag{23}$$

$$f_{2,5}=0, \tag{24}$$

$$f_{2,4}=-930m^4(1+m^2)\xi^{6/(a-2)}, \tag{25}$$

$$f_{2,2}=30m^2(2+29m^2+2m^4)\xi^{6/(a-2)}, \tag{26}$$

$$f_{2,1}=\frac{1}{2m^2}\Big[(a+2)^2\xi^{2/(2-a)}\eta^2 f_{2,3,\eta\eta}-2(a+2)(a-2)\cdot\xi^{(a-4)/(a-2)}\eta f_{2,3,\xi\eta}+(a-2)^2\xi^{2(a-3)/(a-2)}f_{2,3,\xi\xi}+(a+2)(a+15)\xi^{2/(2-a)}\eta f_{2,3,\eta}+(a-2)(a-15)\xi^{(a-4)/(a-2)}f_{2,3,\xi}+(-1-m^2+42\xi^{2/(2-a)})f_{2,3}\Big], \tag{27}$$

$$f_{2,0}=-30m^2(1+m^2)\xi^{6/(a-2)}. \tag{28}$$

Other coefficients of

$$\{\mathrm{sn}(\xi^{1/(a-2)},m),\mathrm{cn}(\xi^{1/(a-2)},m),\mathrm{dn}(\xi^{1/(a-2)},m)\}$$

are simplified into PDEs with only one function $f_{2,3}$, which hold only if $f_{2,3}=0$. Accordingly, the two-order reduction equation in Eq. (A1) admits the solution

$$P_2=30\xi^{6/(a-2)}\Big[-m^2(1+m^2)+m^2(2+29m^2+2m^4)\cdot\mathrm{sn}^2(\xi^{1/(a-2)},m)-31m^4(1+m^2)\,\mathrm{sn}^4(\xi^{1/(a-2)},m)+31m^6\,\mathrm{sn}^6(\xi^{1/(a-2)},m)\Big]. \tag{29}$$

Step 4 Similar to Steps 1, 2, and 3, for $k=3$ in Eqs. (A1) and (12) under Eqs. (16), (17), (22), and (29), the unknown functions $f_{3,j}$, $(j=8,7,6,5,4,2,1,0)$ can be determined from the coefficients of

$$\mathrm{sn}^l(\xi^{1/(a-2)},m),\quad l=12,11,10,9,8,6,5,4$$

one by one:

$$f_{3,8}=-49662m^8\xi^{8/(a-2)}, \tag{30}$$

$$f_{3,7}=0, \tag{31}$$

$$f_{3,6}=66216m^6(1+m^2)\xi^{8/(a-2)}, \tag{32}$$

$$f_{3,5}=0 \tag{33}$$

$$f_{3,4}=-12\xi^{8/(a-2)}m^4(1753+6524m^2+1753m^4), \tag{34}$$

$$f_{3,2}=24\xi^{8/(a-2)}m^2(1+m^2)(103+619m^2+103m^4), \tag{35}$$

$$f_{3,1}=\frac{1}{2m^2}\big[(a+2)^2\xi^{2/(2-a)}\eta^2 f_{3,3,\eta\eta}-2(a+2)(a-2)\cdot\xi^{(a-4)/(a-2)}\eta f_{3,3,\xi\eta}+(a-2)^2\xi^{2(a-3)/(a-2)}f_{3,3,\xi\xi}+(a+2)(a+19)\xi^{2/(2-a)}\eta f_{3,3,\eta}+(a-2)(a-19)\cdot\xi^{(a-4)/(a-2)}f_{3,3,\xi}+(-1-m^2+72\xi^{2/(2-a)})f_{3,3}\big], \tag{36}$$

$$f_{3,0}=-6m^2\xi^{8/(a-2)}(206-77m^2+206m^4). \tag{37}$$

Other coefficients of

$$\{\mathrm{sn}(\xi^{1/(a-2)},m),\mathrm{cn}(\xi^{1/(a-2)},m),\mathrm{dn}(\xi^{1/(a-2)},m)\}$$

naturally give rise to partial differential equations with only one function $f_{3,3}$ whose solution is $f_{3,3}=0$. Accordingly, three-order reduction equation ($k=3$ in Eq. (A1)) admits the following solution

$$P_3=6m^2\xi^{8/(a-2)}\Big[-206+77m^2-206m^4+4(1+m^2)(103+619m^2+103m^4)\,\mathrm{sn}^2(\xi^{1/(a-2)},m)-2m^2(1753+6524m^2+1753m^4)\,\mathrm{sn}^4(\xi^{1/(a-2)},m)+11036m^4(1+m^2)\,\mathrm{sn}^6(\xi^{1/(a-2)},m)-8277m^6\,\mathrm{sn}^8(\xi^{1/(a-2)},m)\Big]. \tag{38}$$

More solutions to reduction equations (A1) can be solved through this procedure. It is easily seen that these solutions grow more and more complicated.

Remark 1 It can be verified that the replacement of $g(\xi)$ with $g(\xi,\eta)$ in Eq. (12) makes no difference, so does the replacement of $f_{k,i}(\xi,\eta)$ with $f_{k,i}(\xi)$ in Eq. (12). All the solutions P_k obtained before are independent of the similarity variable η, but these solutions are still $\{x,y,t\}$ relevant, since it is seen from Eq. (11) that the original variables x,y,t are involved in the similarity variable ξ when $a\neq 2$.

2.2. The case for $b=0$

In this case, it is inferred from Eq. (10) that the condition $a_0\neq 0$ is necessary to make sure $X\neq 0$. The constants $\{a_1,a_2\}$ in Eq. (10) that signify translational invariance are negligible. Without loss of generality, we adjust the constant $a\mapsto a_0 a$. Similarity solutions to Eq. (4) can be solved as

$$u_k=P_k(\xi,\eta),\quad \xi=x-\frac{1}{a}\ln(y+t),\quad \eta=t^2-y^2 \tag{39}$$

with $k=0,1,\ldots,$ which transform the approximate equations (4) into a general form of reduction equations as given below

$$4\eta P_{k,\eta\eta}+6\sum_{i=0}^{k}(P_{k-i,\xi}P_{i,\xi}+P_{k-i}P_{i,\xi\xi})+4P_{k,\eta}-\frac{4}{a}P_{k,\xi\eta}-P_{k,\xi\xi\xi}-P_{k,\xi\xi\xi\xi\xi}-P_{k-1,\xi\xi\xi\xi\xi\xi}=0. \tag{40}$$

Similar to the last subsection, the assumption (12) with $n_k = 2(k+1)$ also serves for the above reduction equations. The coefficients $f_{k,i}$ in Eq. (12) can be determined after we substitute Eq. (12) into Eq. (40) and vanish the coefficients of $\{\mathrm{sn}(g(\xi),m),\mathrm{cn}(g(\xi),m),\mathrm{dn}(g(\xi),m)\}$.

Step 1 We consider $k=0$ for Eqs. (12) and (40). The coefficients of $\mathrm{sn}^l(g(\xi),m)$, $l=6,5,4,3$ lead to

$$f_{0,2} = 2m^2 g_\xi^2, \tag{41}$$

$$f_{0,1} = 0, \tag{42}$$

$$f_{0,0} = \frac{1}{6} - \frac{2}{3}(1+m^2)g_\xi^2 + \frac{2g_{\xi\xi\xi}}{g_\xi} + \frac{9}{2}\left(\frac{g_{\xi\xi}}{g_\xi}\right)^2, \tag{43}$$

$$g = c_0 + c_1\xi, \tag{44}$$

where we can specify the arbitrary constants as $c_0=0$, $c_1=1$ directly.

Other coefficients of $\mathrm{sn}(g(\xi),m)$ vanish due to Eqs. (41)–(44). The zero-order solution to Eq. (40) is thus obtained:

$$P_0 = -\frac{1}{2} - \frac{2}{3}m^2(1-3\mathrm{sn}^2(\xi,m)). \tag{45}$$

Step 2 We consider $k=1$ for Eqs. (12) and (40). The coefficients of $\mathrm{sn}^l(\xi,m)$, $l=8,6,5,4$ lead to

$$f_{1,4} = -30m^4, \tag{46}$$

$$f_{1,2} = 20m^2(1+m^2), \tag{47}$$

$$f_{1,1} = \frac{1}{2m^2}[f_{1,3\xi\xi} - (1+m^2)f_{1,3}], \tag{48}$$

$$f_{1,0} = \frac{2}{3}(4-19m^2+4m^4). \tag{49}$$

Other coefficients of $\mathrm{sn}(\xi,m)$, $\mathrm{cn}(\xi,m)$, and $\mathrm{dn}(\xi,m)$ give rise to $f_{1,3}=0$. The one-order solution to Eq. (40) is thus obtained:

$$P_1 = \frac{2}{3}[4-19m^2+4m^4+30m^2(1+m^2) \cdot \mathrm{sn}^2(\xi,m) - 45m^4\,\mathrm{sn}^4(\xi,m)]. \tag{50}$$

Step 3 We consider $k=2$ for Eqs. (12) and (40). The coefficients of $\mathrm{sn}^l(\xi,m)$, $l=10,9,8,6,5,4$ lead to

$$f_{2,6} = 930m^6, \tag{51}$$

$$f_{2,5} = 0 \tag{52}$$

$$f_{2,4} = -930m^4(1+m^2), \tag{53}$$

$$f_{2,2} = 30(2+29m^2+2m^4)m^2, \tag{54}$$

$$f_{2,1} = \frac{1}{2m^2}[f_{2,3,\xi\xi} - (1+m^2)f_{2,3}], \tag{55}$$

$$f_{2,0} = -30m^2(1+m^2). \tag{56}$$

Other coefficients of $\mathrm{sn}(\xi,m)$, $\mathrm{cn}(\xi,m)$, and $\mathrm{dn}(\xi,m)$ give rise to $f_{2,3}=0$. The two-order solution to Eq. (40) is thus obtained:

$$P_2 = 30m^2[-1-m^2+(2+29m^2+2m^4)\,\mathrm{sn}^2(\xi,m) - 31m^2(1+m^2)\,\mathrm{sn}^4(\xi,m)+31m^4\,\mathrm{sn}^6(\xi,m)]. \tag{57}$$

Step 4 We consider $k=3$ for Eqs. (12) and (40). The coefficients of $\mathrm{sn}^l(\xi,m)$, $l=12,11,10,9,8,6,5,4$ lead to

$$f_{3,8} = -49662m^8, \tag{58}$$

$$f_{3,7} = 0, \tag{59}$$

$$f_{3,6} = 66216m^6(1+m^2), \tag{60}$$

$$f_{3,5} = 0, \tag{61}$$

$$f_{3,4} = -12m^4(1753+6524m^2+1753m^4), \tag{62}$$

$$f_{3,2} = 24m^2(1+m^2)(103+619m^2+103m^4), \tag{63}$$

$$f_{3,1} = \frac{1}{2m^2}[f_{3,3,\xi\xi} - (1+m^2)f_{3,3}], \tag{64}$$

$$f_{3,0} = -6m^2(206-77m^2+206m^4). \tag{65}$$

Other coefficients of $\mathrm{sn}(\xi,m)$, $\mathrm{cn}(\xi,m)$, and $\mathrm{dn}(\xi,m)$ give rise to $f_{3,3}=0$. The three-order solution to Eq. (40) is thus obtained:

$$P_3 = 6m^2\left[-206+77m^2-206m^4+4(1+m^2)(103+619m^2+103m^4)\,\mathrm{sn}^2(\xi,m) - 2m^2(1753+6524m^2+1753m^4)\,\mathrm{sn}^4(\xi,m)+11036m^4(1+m^2)\,\mathrm{sn}^6(\xi,m)-8277m^6\,\mathrm{sn}^8(\xi,m)\right]. \tag{66}$$

Finally, truncated series solutions to the perturbed Boussinesq equation (2) can be composed from the similarity solutions to the approximate equations (4) for Subsections 2.1 and 2.2.

3. Convergence of series solutions

For the case of Subsection 2.1 in the last section, a general form of series solutions to the perturbed Boussinesq equation (2) can be formulated from Eqs. (3), (11), (12), (17), (22), (29), and (38) as follows:

$$u = \frac{1}{6} + \sum_{k=0}^{\infty} A_k B_k, \quad A_k = \sum_{i=0}^{2k+2} c_{k,i}\mathrm{sn}^i\left(\left(\frac{t-y}{2}\right)^{1/(a-2)}, m\right), \quad B_k = \left(\frac{t-y}{2}\right)^{2(k+1)/(a-2)}\varepsilon^k, \tag{67}$$

where $c_{k,i}$ are constants that contain m. A necessary condition for this series solution to be convergent is that $A_k B_k$ must tend to zero.

It is known that the Jacobi elliptic function in Eq. (67) is bounded in the interval $[-1,1]$, but as the subscript k increases, the increased number of terms of the Jacobi elliptic function and the incremental magnitude for coefficients $c_{k,i}$ of these terms in A_k mean that the growth of A_k is uncontrollable. In order that the series solutions are convergent, two conditions for B_k should be considered. First, the perturbation parameter ε should be sufficiently small. Second, $a-2$ should be negative to make sure that $|[(t-y)/2]^{2(k+1)/(a-2)}|$ decreases when $|t-y|$ or the index k grows. In this case, divergence exists when t and y get close.

For the case of Subsection 2.2 in the last section, the general form of series solutions to the perturbed Boussinesq equation (2) can also be constructed. Similarly, a series solution in a simpler form can be given from Eqs. (45), (50), (57), and (66)

$$u=\sum_{k=0}^{\infty} P_k \varepsilon^k,\ P_k=\sum_{i=0}^{2k+2} d_{k,i}\,\mathrm{sn}^i\left(x-\frac{\ln(y+t)}{a},m\right), \tag{68}$$

where $d_{k,i}$ are also constants that contain m.

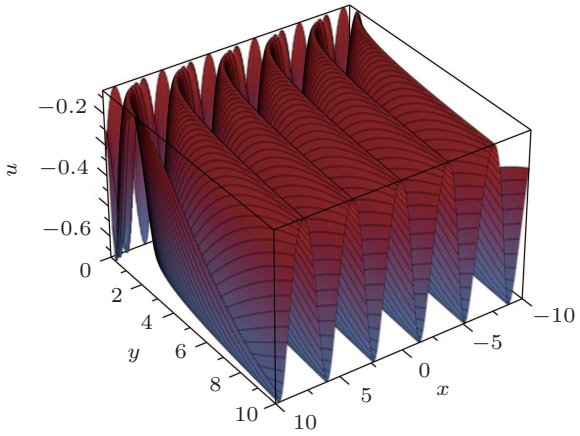


Fig. 1. (color online) Profile of truncated series solutions (68) at $t=0$ with $\varepsilon=0.01$, $m=0.5$, and $a=1$.

The rapid growth of P_k should also be compensated by the perturbation parameter ε. Figure 1 shows one shot of the truncated series solution (68) at $t=0$, where we diminish the perturbation parameter to $\varepsilon=0.01$. Singularity appears when $y\to 0$, since $\ln(y+t)$ is contained in the solution. It can be observed in Fig. 2 that the curve shape of the profile tends to straighten when it evolves, showing that the singular area around $y=0$ moves faster when the whole profile propagates. The reason for the straightening is that the singularity for $\ln(y+t)$ around $y=0$ fades when the time t goes away from 0. The straightening also indicates the improvement for the convergence of the series solution as it evolves.

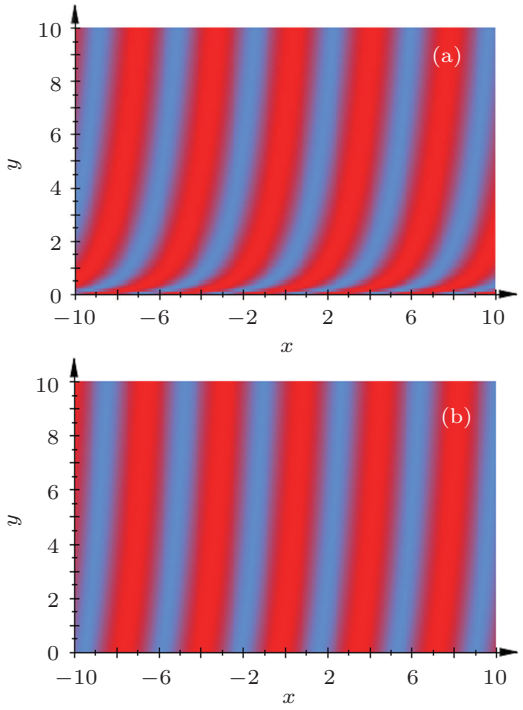


Fig. 2. (color online) Density plots of truncated series solutions (68) with $\varepsilon=0.01$, $m=0.5$, and $a=1$ for (a) $t=0$ and (b) $t=5$.

4. Summary

The combination of perturbation theory and the symmetry method achieve the construction of truncated series solutions to the (2+1)-dimensional perturbed Boussinesq equation. The reduction equations and the related reduction solutions are regular and accordant, from which two types of tidy series solutions are classified.

A trial and summary methodology is applied to solve both the approximate equations by the Lie symmetry method and the reduction equations by the Jacobi elliptic function expansion method. Although the computation is concerned with rapidly growing complexity, the truncated series solutions are finally constructed and displayed visually.

Aside from some particular area, convergence for the series solutions can generally be guaranteed when the perturbation parameter is diminished. Perturbation terms improve the accuracy of the model equations to reality, so the approximate solutions truncated from the convergent series solutions are more reasonable to get the exact solutions to unperturbed equations.

For the first time, the perturbation term εu_{xxxxxx} is added to the (2+1)-dimensional Boussinesq equation in order to investigate the truncated series solutions. In fact, various forms of perturbation terms can similarly be considered to produce more suitable models on different occasions, where the approximate symmetry method could be more significative. Furthermore, the method is also applicable to three-dimensional perturbed nonlinear equations.

Appendix A: The similarity equations in Subsection 2.1

In the following, we give the similarity equations for the case $b \neq 0$ in Subsection 2.1

$$
\begin{aligned}
&-(a+2)^6\eta^6 P_{k-1,\eta\eta\eta\eta\eta\eta} - (a-2)^4\xi^4 P_{k,\xi\xi\xi\xi} - (a+2)^4\eta^4 P_{k,\eta\eta\eta\eta} - 3(a+2)^5(5a+4k+15)\eta^5 P_{k-1,\eta\eta\eta\eta\eta} \\
&+4(a+2)^3(a-2)\xi\eta^3 P_{k,\xi\eta\eta\eta} + 6\sum_{i=0}^{k}\{(a+2)\eta\left[(a+2)\eta P_{i,\eta} + 2(i+1)P_i - (a-2)\xi P_{i,\xi}\right]P_{k-i,\eta} \\
&+(a-2)\xi\left[(a-2)\xi P_{i,\xi} - 2(i+1)P_i - (a+2)\eta P_{i,\eta}\right]P_{k-i,\xi} + P_{k-i}\left[(a-2)^2\xi^2 P_{i,\xi\xi} + (a+2)^2\eta^2 P_{i,\eta\eta}\right. \\
&\left.-2(a^2-4)\xi\eta P_{i,\xi\eta} + (a-2)(a-2k-2i-9)\xi P_{i,\xi} + (a+2)(a+2k+2i+9)\eta P_{i,\eta} + 2(i+1)(2k+5)P_i\right]\} \\
&-(a+2)^2\eta^2[31a^4 + (180k+473)a^3 + (420k^2+2130k+2689)a^2 + (480k^3+3480k^2+8400k+6747)a \\
&+30(k+2)(2k+5)(4k^2+18k+21)]P_{k-1,\eta\eta} - 5(a+2)^3(3a+4k+11)(6a^2+12ka+39a+8k^2+44k+63)\eta^3 P_{k-1,\eta\eta\eta} \\
&-(a+2)(a+4k+9)(a^2+11a+4ka+40+36k+8k^2)\eta P_{k,\eta} - 30(a+2)^2(a-2)^3(a-4k-15)\xi^3\eta^2 P_{k-1,\xi\xi\xi\eta\eta} \\
&+15(a+2)(a-2)^2\xi^2\eta(a-4k-11)(2a^2-4ka-13a+8k^2+44k+63)P_{k-1,\xi\xi\eta} - 3(a-2)^5(5a-4k-15)\xi^5 P_{k-1,\xi\xi\xi\xi\xi} \\
&-6(a^2-4)^2\xi^2\eta^2 P_{k,\xi\xi\eta\eta} - (a-2)(a-4k-7)\xi[a^4 - (8k+18)a^3 + (28k^2+118k+119)a^2 - 2(4k+9)(6k^2+23k+19)a \\
&+12(k+1)(k+2)(2k+3)(2k+5)]P_{k-1,\xi} - 5(a+2)^4\eta^4(13a^2+82a+24ka+129+12k^2+78k)P_{k-1,\eta\eta\eta\eta} \\
&-5(a-2)^3\xi^3(3a-4k-11)(6a^2-12ka-39a+8k^2+63+44k)P_{k-1,\xi\xi\xi} + 20(a^2-4)^3\xi^3\eta^3 P_{k-1,\xi\xi\xi\eta\eta\eta} \\
&-4(k+1)(k+2)(2k+3)(2k+5)(4k^2P_{k-1} + 2kP_{k-1} + P_k) + 6(a+2)^5(a-2)\xi\eta^5 P_{k-1,\xi\eta\eta\eta\eta} \\
&+6(a+2)(a-2)^5\xi^5\eta P_{k-1,\xi\xi\xi\xi\xi\eta} + 20\xi\eta^3(a+2)^3(a-2)(4a^2+41a+12ka+129+12k^2+78k)P_{k-1,\xi\eta\eta} \\
&-15\xi^4\eta^2(a-2)^4(a+2)^2 P_{k-1,\xi\xi\xi\xi\eta\eta} + 15\xi\eta^2(a+2)^2(a-2)(a+4k+11)(2a^2+4ka+13a+8k^2+44k+63)P_{k-1,\xi\eta\eta} \\
&+15\xi\eta^4(a-2)(a+2)^4(3a+4k+15)P_{k-1,\xi\eta\eta\eta} + 15\xi^4\eta(a+2)(a-2)^4(3a-4k-15)P_{k-1,\xi\xi\xi\xi\eta} - (a-2)^6\xi^6 P_{k-1,\xi\xi\xi\xi\xi\xi} \\
&-30(a^2-4)^2\xi^2\eta^2(a^2+12k^2+78k+129)P_{k-1,\xi\xi\eta\eta} - (a-2)(a-4k-9)(a^2-11a-4ka+40+36k+8k^2)\xi P_{k,\xi} \\
&-2(a+2)^3(3a+4k+13)\eta^3 P_{k,\eta\eta\eta} + 2(a^2-4)\xi\eta[a^4 + (60k^2+270k+299)a^2 + 30(k+2)(2k+5)(4k^2+18k+21)]P_{k-1,\xi\eta} \\
&-2(a-2)^3(3a-4k-13)\xi^3 P_{k,\xi\xi\xi} + 6\xi^2\eta(a-2)^2(a+2)(a-4k-13)P_{k,\xi\xi\eta} + 6\xi\eta^2(a-2)(a+2)^2(a+4k+13)P_{k,\xi\eta\eta} \\
&+[1+2(a+2)(a-2)(a^2+24k^2+132k+183)\xi\eta]P_{k,\xi\eta} - 5(a-2)^4(13a^2-82a-24ka+12k^2+78k+129)\xi^4 P_{k-1,\xi\xi\xi\xi} \\
&-(a-2)^2(7a^2-24ka-70a+24k^2+132k+183)\xi^2 P_{k,\xi\xi} - (a+2)^2(7a^2+24ka+70a+24k^2+132k+183)\eta^2 P_{k,\eta\eta} \\
&-(a-2)^2\xi^2[31a^4 - (180k+473)a^3 + (420k^2+2130k+2689)a^2 - (480k^3+3480k^2+8400k+6747)a \\
&+30(k+2)(2k+5)(4k^2+18k+21)]P_{k-1,\xi\xi} + 4\xi^3\eta(a+2)(a-2)^3 P_{k,\xi\xi\xi\eta} - 15(a+2)^4(a-2)^2\xi^2\eta^4 P_{k-1,\xi\xi\eta\eta\eta\eta} \\
&-30(a+2)^3(a-2)^2(a+4k+15)\xi^2\eta^3 P_{k-1,\xi\xi\eta\eta\eta} - (a+2)(a+4k+7)\eta[a^4 + (8k+18)a^3 + (28k^2+118k+119)a^2 \\
&+2(4k+9)(6k^2+23k+19)a + 12(k+1)(k+2)(2k+3)(2k+5)]P_{k-1,\eta} \\
&+20(a+2)(a-2)^3(4a^2-41a-12ka+12k^2+78k+129)\xi^3\eta P_{k-1,\xi\xi\xi\eta} = 0. \qquad \text{(A1)}
\end{aligned}
$$

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