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Citation: Chin. Phys. B . 2018, 27(1): 016702. **doi:** 10.1088/1674-1056/27/1/016702

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Two types of ground-state bright solitons in a coupled harmonically trapped pseudo-spin polarization Bose–Einstein condensate*

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(Received 7 August 2017; published online 28 November 2017)

We study two types of bright solitons in an attractive Bose–Einstein condensate with a spin–orbit interaction. By solving the coupled nonlinear Schrödinger equations with the variational method and the imaginary time evolution method, fundamental properties of solitons are carefully investigated in different parameter regimes. It is shown that the detuning between the Raman beam and energy states of the atoms dominates the ground state type and spin polarization strength. The soliton dynamics is also studied for various moving velocities for zero and nonzero detuning cases. We find that the shape of individual component solitons can be maintained when the moving speed of solitons is low and the detuning is small in the coupled harmonically trapped pseudo-spin polarization Bose–Einstein condensate.

Keywords: bright solitons, spin–orbit interaction, spin polarization

PACS: 67.85.−d, 03.75.Lm **DOI:** 10.1088/1674-1056/27/1/016702

1. Introduction

The effect of spin–orbit coupling (SOC) contacts the spin with velocity-dependent movement of particles. It is particularly important for new research fields in solid state systems, such as the spin Hall effect, topological insulator, spintronic devices, etc.[1–3] However, the adjustable region of SOC strength in practical materials is very small and can be inevitably effected by disorder and impurities.[4] On the other hand, the momentum of ultracold neutral atoms can also be coupled with their pseudo-spin state by the laser-coupling techniques.[5–7] The ultracold atomic researches have opened up a new way to explore novel states of matter. The multiple controllability in the Bose systems provides exclusive opportunities to investigate novel effects in SOC Bose–Einstein condensates (BECs).

Since the experimental realization of BECs in ultracold atom gases, matter wave solitons have been the object of intensive theoretical and experimental studies. Associated with the dispersion and nonlinearity, these matter wave solitons can be divided into two kinds of fundamental soltions: dark soliton with an envelope which comprises a rapid dip in the intensity with a phase jump across its intensity minimum in repulsive interaction systems, bright soliton with a fundamental envelope excitation identified by a localized intensity peak in an attractive interaction system.[8–10] For conventional BEC, the ground state wavefunction obeys the "no-node" theorem and is positive definite.[11] However, this is not true for an SOC-BEC system because of the violation of Galilean invariance.[12,13] The SOC strength and the detuning between the Raman beam and energy states of the atoms have important effects on the properties of solitons in a BEC with SOC.[14–16]

Compared with single-component BEC, multicomponent solitons have richer novel nonlinear effects in the SOC-BEC system.[17] For example, the intra- and inter-component attractive (repulsive) interaction types, the detuning, the strength of SOC, and velocities of matter waves are almost all tunable. These ample nonlinear structures, such as dark–dark solitons,[18,19] dark–bright solitons,[20–22] bright–bright solitons,[23,24] and gap bright–bright solitons[25] in a spatially periodic Zeeman field had become the focus of related SOC-BEC researches.

In this paper, we attempt to systematically study two different types of bright solitons in a quasi-one-dimensional SOC BEC trapped in an oscillator potential with attractive interaction. In particular, we are interested in how the Raman detuning of the system affects the type of spinor wavefunctions and spin polarization strength. It will be shown that the Raman detuning dominates the type of ground states and spin polarization intensity. Separation distance of the two solitons is simultaneously decided by the SOC, attractive interaction constant, and oscillating frequency for the case of zero detuning.

The paper is organized as follows. In Section 2, we introduce the model equation for the two types of bright solitons in the SOC Bose system. Section 3 is devoted to the computation of the ground-state wave functions via a numerical and variational method for zero detuning. We aim to gain a direct insight into the different ground-state densities in an interval of parameters. In Section 4, the bright solitons are calculated by numerical and variational methods for nonzero detuning.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11304270 and 11475144).

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In Section 5 we calculate the dynamic evolution for various parameters. Section 6 is a brief summary.

2. Model equation

We study an elongated SOC BEC where transversal freedom is frozen due to larger trapping frequencies than weaker longitudinal confined frequency. The spin–orbit coupling is created through the momentum transfer between the Raman beams and the atoms. So one can consider a quasi-one-dimensional SOC BEC described by the following time-dependent reduced coupled nonlinear Schrödinger equation:[14]

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial T}\Phi_1 &= \left[-\frac{1}{2}\frac{\partial^2}{\partial X^2}+\delta+V_{\mathrm{trap}}-g\left(|\Phi_1|^2+|\Phi_2|^2\right)\right]\Phi_1 \\ &\quad +\gamma\frac{\partial}{\partial X}\Phi_2, \\ \mathrm{i}\frac{\partial}{\partial T}\Phi_2 &= \left[-\frac{1}{2}\frac{\partial^2}{\partial X^2}-\delta+V_{\mathrm{trap}}-g\left(|\Phi_1|^2+|\Phi_2|^2\right)\right]\Phi_2 \\ &\quad -\gamma\frac{\partial}{\partial X}\Phi_1, \end{aligned} \tag{1}$$

where Φ_1 and Φ_2 are the up or down pseudospin component respectively, δ represents the detuning between the Raman beam and energy states of the atoms, g denotes the interaction strength between atoms, γ is the SOC strength, and $V_{\mathrm{trap}} = \frac{1}{2}\Omega^2X^2$ is the harmonic-oscillator potential trap. For the sake of simplicity, the intra and inter interaction is set to be equal to g, which is in proportion to the s-wave scattering lengths respectively and the difference is very small.[19,26] The normalization wavefunction is determined via $\int_{-\infty}^{\infty}(|\phi_1|^2+|\phi_2|^2)\,\mathrm{d}x = 1$. Equation (1) is the starting point of the imaginary or real time evolution throughout this paper.

The total energy corresponding to Eq. (1) can be written as

$$\begin{aligned} E &= \int \mathrm{d}x\left[\frac{1}{2}|\partial_x\Phi_1|^2+\frac{1}{2}|\partial_x\Phi_2|^2+\delta\left(|\Phi_1|^2-|\Phi_2|^2\right)\right. \\ &\quad +\gamma\phi_1^*\partial_x\Phi_2-\gamma\phi_2^*\partial_x\Phi_1+\frac{1}{2}\Omega^2X^2\left(|\Phi_1|^2+|\Phi_2|^2\right) \\ &\quad \left.-\frac{g}{2}\left(|\Phi_1|^4+|\Phi_2|^4+2|\Phi_1|^2|\Phi_2|^2\right)\right]. \end{aligned} \tag{2}$$

Due to the lack of Galilean invariance in the SOC BEC system, its ground state is not single. It is demonstrated that there are two types of solitary-vortex condensates: semivortices and mixed modes in a two dimensional SOC-BEC with attractive interaction of atoms.[27] For the SOC BEC described by Eq. (1) there are also two kinds of analytical approximation bright soliton solutions via the variational method according to the different detuning setting. An appropriate variational ansatz for a zero detuning case is the following variational wavefunction:

$$\begin{aligned} \Phi_1 &= A\sin\left(N(X-X_1)\right)/\cosh\left(W(X-X_2)\right), \\ \Phi_2 &= A\sin\left(N(X+X_1)\right)/\cosh\left(W(X+X_2)\right). \end{aligned} \tag{3}$$

The other better variational function for the nonzero detuning case can be written as

$$\begin{aligned} \Phi_1 &= A_1\sin\left(NX\right)/\cosh\left(WX\right), \\ \Phi_2 &= A_2\cos\left(NX\right)/\cosh\left(WX\right). \end{aligned} \tag{4}$$

Compared with the variational ansatz (3) and ansatz (4), it is not difficult to find that the density distribution of the two bright soliton components depart a distance from the origin of the coordinate and have the same amplitude for the case of zero detuning. However, the density modes in the variational function (4) should be symmetrical about the origin of the coordinate and have different heights when the nonzero detuning is present. These observations will be proved by the following imaginary evolution methods. Under the normalization condition these unknown parameters in the function (3) and (4) can be obtained by minimizing the energy function in Eq. (2).

3. Bright solitons for the case of zero detuning

In this section, we investigate the properties stationary bright solitons in the absence of detuning. By solving Eq. (1) numerically, one can obtain the two bright soliton solutions for each component. It is remarked that to solve Eq. (1), we can firstly differentiate it using a split-time-step Crank–Nicolson method with the zero boundary condition, and then evaluate several hundreds of thousands of steps in imaginary time until the lowest energy is reached.[28] The Crank–Nicolson method is a second-order method in time and is unconditionally stable. It preserves the unitarity of the long time evolution and yields good convergence of the solutions.[29] In the calculation, the problem is numerically solved in the interval of $[-10,10]$ with a spatial size of 0.02 and time step of 0.0002. Figures 1(a1) and 1(b1) show the first bright soliton component and figures 1(a2) and 1(b2) are related to the second bright soliton component in different SOC strengths.

As shown in Fig. 1, the bright soliton densities evolve from a no node density distribution to one node when the SOC constant γ is increased. The up or down pseudospin components share the same amplitude and hold center symmetry. Hence, the spin polarization for these bright solitons should be zero, as we will discuss this below. The peaks of densities have zero-point motions in coordinate space, which is obviously different from the density distribution in common BEC. When we increase the SOC strength continually, and we find the nodes of density are synchronously increased. "No-node" theorem is gradually ruled out in the ground state of the Bose system with the increase in the SOC γ. In fact, once the SOC is presented in Eq. (1), the spatial symmetry of the ground state density is broken at the same time when the detuning is zero. In view of Figs. 1(b1) or 1(b2), the density distribution of

SOC BEC does not hold bilateral symmetry, while the ground state of conventional BEC must be bilateral symmetry about the central peak of density.

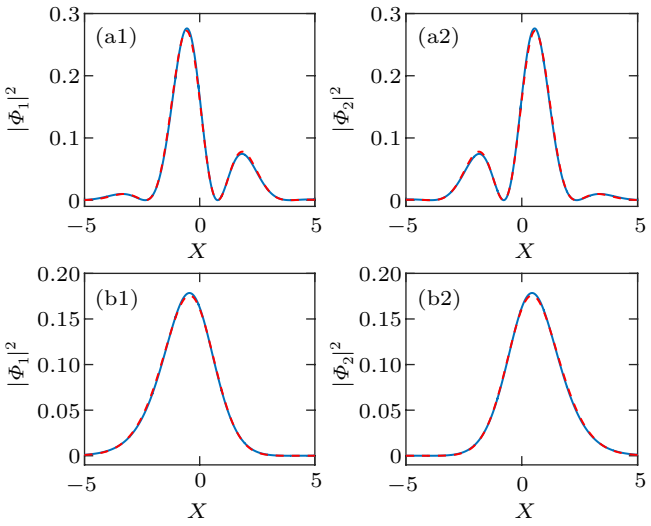


Fig. 1. (color online) (a1), (b1) The first bright soliton component density of the SOC BEC. (a2), (b2) The second bright soliton component. The blue solid lines are the variational method results, and the red dotted lines are the numerical densities by the imaginary time evolution method. The SOC is $\gamma = 1.0$ and 0.2 for panels (a) and (b), respectively. The other parameters for panels (a) and (b) are $g = 1$, $\Omega^2 = 0.02$, and $\delta = 0$.

In order to compare the results obtained by the imaginary-time method, we also calculate the density profiles for the SOC BEC using the variational method by minimizing the energy in Eq. (2) according to the variational ansatz (3). These parameters A, N, and W can be regarded as the original properties of solitons, such as amplitude, numbers of nodes, and width. It can be seen that the variational curve of $|\Phi_{1,2}|^2$ agrees very well with the soliton densities obtained by the imaginary-time method for a different γ case (see Fig. 1).

Now, let us check the fundamental characters of bright solitons in absence of the detuning. With the increase in the nonlinear attractive interaction g, the separation distance and width of the two pseudospin components decrease, while the height increases and the number of nodes almost keeps a constant (see Fig. 2(a)). It is clear that the attractive interaction constant has a squeezing effect on the overall shape of bright solitons when the interaction constant increases, which is also a common result in a conventional BEC system. Especially, it can be found that the bigger separation distance of the two condenses prefers a smaller attractive interaction system. We also notice that the influence of the harmonic-oscillator trap frequency Ω^2 on the characters of bright solitons is precisely the same as nonlinear attractive interaction g, except that a rapid changes occurs when the frequency is very low (see Fig. 2(c)). Properties of solitons in Fig. 2(b) show that the SOC γ have almost no effect on the width and height of bright solitons, which maintain constant in the parameter regime. However, the number of nodes increases for the stronger SOC, and the separation distance between the two bright solitons is also reduced.

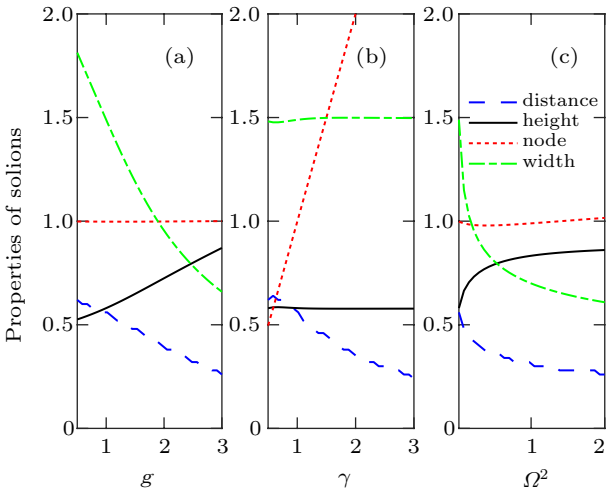


Fig. 2. (color online) The fundamental properties of bright–bright solitons versus the main parameters of the SOC-BEC system. Panels (a)–(c) show these properties in relation with the interaction strength g, the SOC γ, and the frequency Ω^2 of the harmonic-oscillator trap, respectively. Distance represents the half of the two bright solitons, height is the amplitude of any one bright soliton, node denotes the number nodes of bright solitons, and width is the width of bright solitons. The other parameters of panel (a) are $\gamma = 1.0$, $\Omega^2 = 0.02$, panel (b) are $g = 1.0$, $\Omega^2 = 0.02$, and panel (c) are $\gamma = 1.0$, $g = 1.0$. The detuning constant is $\delta = 0$ for all the cases.

4. Bright solitons for the case of nonzero detuning

In this section, we check the spin polarization and densities mode for the case of nonzero detuning. The atom number of the two bright solitons is generally unbalanced when the detuning is presented. The SOC-BEC density distribution can be effected by the detuning between the Raman beam and energy levels of the atoms, which has a non-negligible heating effect at zero detuning and finite detuning can be used to suppress the effect.[5] Different from that in the zero detuning system, the number of atoms in the two components is strictly equal. The major characteristic feature of bright solitons in SOC BEC for the nonzero case admits a polarization mode, which can be checked by defining the pseudo-spin polarization for these bright solitons

$$P = \int \left(|\Phi_1|^2 - |\Phi_2|^2 \right) \mathrm{d}X. \tag{5}$$

The results of variational ansatz (4) and imaginary time evolution method with the finite detuning is shown in Fig. 3, where it can be found that they are in excellent agreement. In the calculation, we firstly substitute variational ansatz (4) into Eq. (2), minimize the energy and get the variational solution of Eq. (1). Then we regard the variational solution as the initial value of Eq. (1), evolve the equation in imaginary time and find the imaginary time solution. This kind of soliton in the high dimensional region is called the half-quantum vortex

or semivortex.[13,27] This characteristic makes SOC BEC very different from conventional BEC, and there are two kinds of ground states for different initial values. If the initial states are prepared with variational solution just like ansatz (3), we will obtain the result shown in Fig. 1. If we use the initial states just like ansatz (4), then we get the results of Fig. 3. When the SOC becomes gradually stronger, the nodes increase little by little and the bright solitons shown in Fig. 3 will become a strip soliton.[25,30–32] At the same time, the type of bright soliton holds a unique spin-parity symmetry, the spin-parity operator is defined by $P\sigma_z$, where P is the parity operator, σ_z is the Pauli matrix, and its eigenvalue is -1.[14] However, the spin-parity symmetry is not present when the detuning is zero.

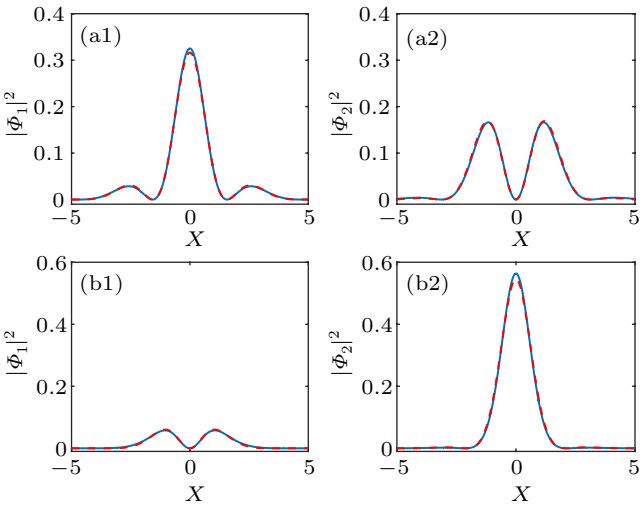


Fig. 3. (color online) (a1), (b1) The first bright soliton component density of the SOC BEC for the nonzero detuning case. (a2), (b2) The second bright soliton component. The blue solid lines are variational method results, and the red dotted lines are the numerical densities by the imaginary time evolution method. In panels (a) and (b) the SOC is $\delta = 0.01, 0.5$, respectively. The other parameters for panels (a) and (b) are $g = 1$, $\Omega^2 = 0.02$, and $\gamma = 1$.

We investigate the influence on the properties of bright solitons by the detuning, and the results are shown in Fig. 4(a). It is easily seen that the width and distance between the two bright solitons are wider for bigger detuning δ, while the tendency of node numbers declines with the increase in the detuning. It is worthwhile mentioning that the amplitude of the two bright solitons are identical and the spin polarization should be zero. However, the tendency of the two bright solitons amplitude begins to differentiate and the spin polarization becomes strong when nonzero detuning is turned on (see Fig. 4(a)). It shows that the detuning dominates the spin polarization in Fig. 4(b). It is also indicated that the SOC strength has an effect on the spin polarization at the same time, and a smaller polarization is found with a smaller SOC strength when the same detuning is present. Especially, for different detuning $\delta = 0.01, 0.50$, the spin polarization is $P = 0.06, 0.48$ for the red circle in Fig. 4(b), respectively. Under this circumstance, we find a zero point of polarization when the detuning is $\delta = 0.06$. It is demonstrated that the polarization does not monotonously increase with the detuning and there is a transition point when the SOC is $\gamma = 1.0$.

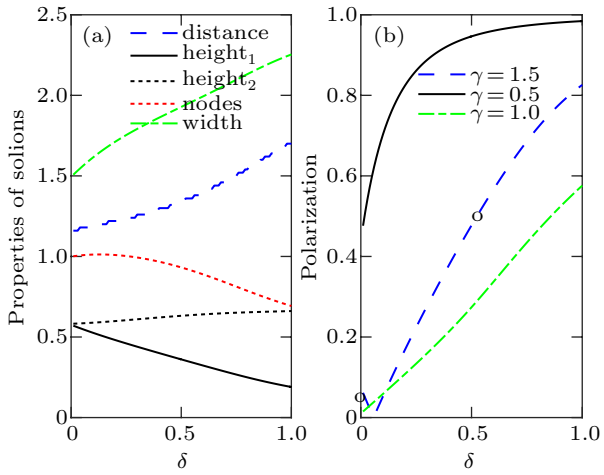


Fig. 4. (color online) The fundamental properties of bright–bright solitons versus main parameters of the SOC-BEC system for the nonzero detuning case. Panels (a) and (b) show the relation of these properties with the detuning δ. The meanings of these defined parameters: distance, height, node, and width, are the same as those used in Fig. 2. The other parameters of panels (a) and (b) are $\gamma = 1.0$, $\Omega^2 = 0.02$, and $g = 1.0$. The red circle marked in panel (b) is related to the density modes in Fig. 3.

5. The dynamics of bright solitons for different velocity

We investigate the dynamics of a soliton in SOC BEC by adding a trivial phase on the stationary bright solitons. The dynamics property is decided by the following equations:

$$\begin{aligned} \Psi_1(X,T) &= \Phi_1(X - vT)\exp(\mathrm{i}vX), \\ \Psi_2(X,T) &= \Phi_2(X - vT)\exp(\mathrm{i}vX), \end{aligned} \tag{6}$$

where the wave functions $\Phi_{1,2}$ are the stationary solutions of Eq. (1) by an imaginary time evolution method and the moving speed of bright soliton is v. By setting a different speed v, we can get the effect of different moving speeds on the dynamic property of solitons. As we can see that the Hamiltonian of SOC BEC in Eq. (1) has violated the Galilean invariant, which means that the dynamic of bright solitons may occur with anisotropic diffusion behavior (see below). When the SOC is turned on, the dynamic properties are significantly different from those in the absence of a spin–orbit coupling system,[33] where fundamental properties of solitons can be maintained for different moving velocities.

The coupled nonlinear Eq. (1) can be numerically solved by using a split-time-step Crank–Nicolson method in real time for the case of zero detuning. The results are shown in Fig. 5, in which the first (second) component of the bright solitons density evolution is denoted in the top (bottom) panel, respectively. The moving speed for Fig. 5(a) is a smaller velocity $v = 0.2$, for Fig. 5(b) is $v = 0.5$, and for Fig. 5(c) is a bigger

speed $v = 1$. We see clearly that fundamental properties (amplitude, width, number peaks, etc) of the moving solitons is maintained during the long time evolution for different speeds, and these results show ample spin mixing dynamics in the parameters region. During the real time evolution, the atom number of the two components is continually exchanged but the total atom number is almost unchanged. As shown by the top panel of Fig. 5, the density maximum of the first bright soliton has a tendency to bend to the left direction, while the second component has an opposite tendency, see the bottom panel. It is proved that Galilean invariance is broken in the SOC-BEC system and the dynamics of bright solitons shows strong anisotropy. Due to the confinement of the external oscillator trap, the soliton with lower moving speed cannot reach the inner edge of the trap (see Fig. 5(a)). When the velocity of solitons is bigger, the soliton may spill the trap and reach the boundary of the system (see Fig. 5(c)). Another interesting finding is that the oscillating frequency of the three cases in Fig. 5 is almost the same, which is mainly related to the oscillator trap frequency Ω^2. The oscillating frequency is greater when the oscillator frequency is bigger, which is a common result for soliton dynamics in the oscillator potential trap.

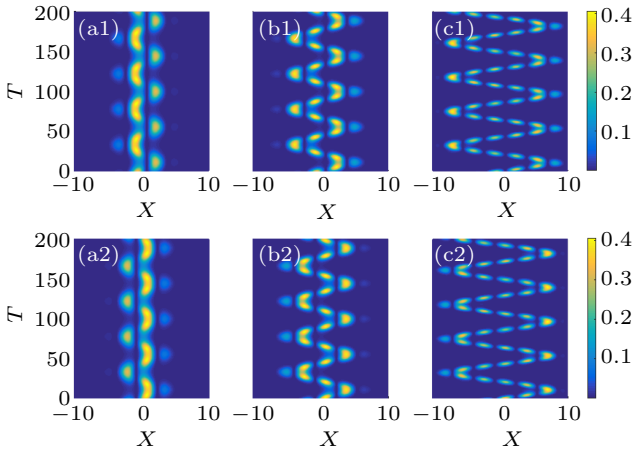


Fig. 5. (color online) Dynamical evolution of bright solitons for different velocity in the case of zero detuning. The detuning is $\delta = 0$, the SOC is $\gamma = 1$, the oscillator frequency is $\Omega^2 = 0.02$, and interaction strength is $g = 1$. Top (bottom) panel: the first (second) bright solitons density evolution result. The moving velocity of bright solitons for panels (a1) and (a2) is $v = 0.2$, (b1) and (b2) is $v = 0.5$, (c1) and (c2) is $v = 1$.

Especially, the bright soliton for the case of nonzero detuning evolves Fig. 6(a) with the maintained shape as the initial wavefunction in a low velocity interval. If we compare the velocity of the exchange atom number at the same moving speed, one can find the case of nonzero detuning is slower than the zero detuning case. This can be understood in the following manner. When the detuning is turned on at the initial time, then the stable exchange mode is established. The case of zero detuning needs more time to build a stable mode, so the detuning between the Raman beam and energy levels of the atoms is critically important for building stable soliton dynamics in SOC BEC. With the increasing of the moving speed of bright solitons in the case of nonzero detuning, the density distribution of the bright soliton tends to be a fragile BEC mode, as shown in Fig. 6(c). The evolution result shows no regular pattern and anisotropy in SOC BEC is not obvious when the bigger velocity is present. So we conclude that small detuning and low moving speed are beneficial for maintaining the shape of the bright soliton in pseudo-spin polarization SOC-BEC dynamics.

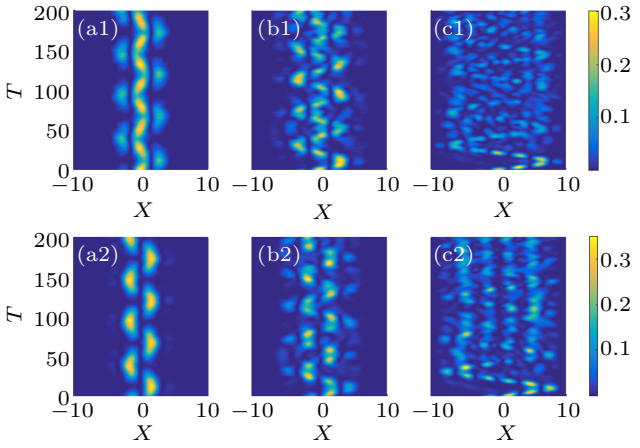


Fig. 6. (color online) Dynamical evolution of bright solitons for different velocities in the case of nonzero detuning. The detuning is $\delta = 0.12$ and other parameters are the same as Fig. 5.

6. Summary

In summary, we have investigated the two types of ground state bright solitons in an attractive interacting boson in one-dimensional SOC-BEC system by the variational method and the imaginary time evolution method. The main focus is to consider the influence of the detuning, the spin–orbit coupling strength, attractive interaction strength, and outer oscillator potential trap on the fundamental properties of the two component bright solitons. Our results verify that the type of ground states and spin polarization strength are mainly decided by the detuning. Two types of solitons are found in zero and nonzero detuning parameter regions. The SOC γ has almost no effect on the width and height of bright solitons, but the number of nodes increases with the stronger SOC for the two types of bright solitons. The attractive interaction constant and harmonic-oscillator trap frequency have a squeezing effect on the overall shape of bright solitons when the corresponding parameter value increases. Especially, the separation distance of the two solitons decreases with the increasing of the SOC, attractive interaction constant and oscillating frequency for the case of zero detuning. The dynamics of solitons show ample spin mixing dynamics that the atom number of the two components is continually exchanged, while the velocity of the exchange atom number at the same moving speed for zero and nonzero detuning cases has a little difference. It is found that

small detuning and low moving speed can better keep the fundamental properties of a bright soliton in SOC-BEC dynamics. Our research might provide a way to better understand soliton dynamics for the SOC-BEC system.

Acknowledgment

We thank Biao Wu for our useful discussion.

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