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Spin-dependent balance equations in spintronics*

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It is commonly known that the hydrodynamic equations can be derived from the Boltzmann equation. In this paper, we derive similar spin-dependent balance equations based on the spinor Boltzmann equation. Besides the usual charge current, heat current, and pressure tensor, we also explore the characteristic spin accumulation and spin current as well as the spin-dependent pressure tensor and heat current in spintronics. The numerical results of these physical quantities are demonstrated using an example of spin-polarized transport through a mesoscopic ferromagnet.

Keywords: spin-dependent balance equations, spinor Boltzmann equation, spintronics, spin current

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1. Introduction

The hydrodynamic equations satisfied by mass density, pressure, flow velocity, heat, and entropy can be derived from the classical Boltzmann equation.[1] The quantum fluid equations derived by Madelung[2] from the Schrodinger equation have been applied to semiconductor device modeling.[3] The usual hydrodynamic equations describe the time and spatial variations of a fluid without spin. However, in some special fluids named Fermi liquids,[4–6] e.g., He^3, the spin plays an important role. Fermi liquids are governed by the Landau-Boltzmann equation,[7] and the conservation laws and spin dynamics from this equation indicate that the spin degrees of freedom have an important influence. In fact, one can describe the motion of a quasi-particle in a Fermi liquid by the spin hydrodynamic equations.[8,9] In the later, Landau's theory was extended to a degenerate electron liquid,[10,11] in particular ferromagnetic metal.[12–17]

Ferromagnetic metals, such as Fe, Co, and Ni, are the key ingredients in spintronics devices. The tunneling magnetoresistance[18,19] (TMR) and spin transfer effect[20–23] in spintronics are all dependent on the electronic spin-polarized transport in the ferromagnets,[24,25] and the electrons in a ferromagnet can be treated as a Fermi liquid. The spin current in a ferromagnetic metal satisfies the corresponding equations of continuity,[26–29] except for the charge current, which can be regarded as one of the spin-dependent hydrodynamic equations for charge density. The spin-dependent heat current has been studied in multilayers[30,31] and a new branch of spintronics, named spin caloritronics,[32,33] has emerged. However, the balance equation for the spin-dependent heat current has not been investigated. Moreover, it is worthwhile to further explore the spin-dependent balance equations corresponding to the momentum and entropy of spintronics.

The spinor Boltzmann equation is a powerful tool to study the spin-polarized transport of electrons in spintronics,[34,35] and includes spin freedom in the quantum Boltzmann equation. In 1957, the spinor Boltzmann equation was utilized to investigate a Fermi liquid by Silin.[10,11] As the usual hydrodynamic equations can be obtained from the classical Boltzmann equation, we expect that similar spin-dependent balance equations may also be obtained from the spinor Boltzmann equation,[34] and will be different from the usual hydrodynamic equations. In this paper, we derive the spin-dependent balance equations for the charge density, pressure, flow velocity, heat, and entropy. While some of the corresponding balance equations are not closed, as in the classical hydrodynamic equations, we obtained statistical expressions for the above spin-dependent physical quantities that are relevant to spintronics. In the final chapter, we provide a simple example to demonstrate the physical quantities in the spin-dependent balance equations.

2. Spin-dependent balance equations

The spinor Boltzmann equation is an equation of motion for the spinor density matrix. The spinor density matrix satisfies the quantum Liouville equation, and after the quantum Wigner transformation, the spinor density matrix can be related to a spinor distribution function $\hat{\boldsymbol{f}}$. The quantum Liouville equation reduces to the spinor Boltzmann equation by adopting a gradient approximation.[7,10,11,34] When an external electric field $\boldsymbol{E}$ is applied to the system, it can be written as[7,10,11,34]

$$\frac{\partial \hat{\boldsymbol{f}}}{\partial t}+\hat{\boldsymbol{V}}\cdot\frac{\partial \hat{\boldsymbol{f}}}{\partial \boldsymbol{r}}-\frac{e\boldsymbol{E}}{\hbar}\cdot\frac{\partial \hat{\boldsymbol{f}}}{\partial \boldsymbol{p}}+\frac{\mathrm{i}}{\hbar}[\widehat{\varepsilon},\hat{\boldsymbol{f}}]=-\left(\frac{\partial \hat{\boldsymbol{f}}}{\partial t}\right)_{\text{collision}}, \quad (1)$$

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where the spinor distribution function $\hat{\boldsymbol{f}}$ can be expressed as the equilibrium and nonequilibrium scalar distribution $f(p,X)$ and the vector distribution function $\boldsymbol{g}(p,X)$, and is given by[33]

$$\hat{\boldsymbol{f}}(p,X)=f^0(p)I+\left(-\frac{\partial f^0}{\partial\varepsilon}\right)[f(p,X)I+\boldsymbol{g}(p,X)\cdot\boldsymbol{\sigma}], \quad (2)$$

where $f^0(p)$ is the equilibrium Fermi distribution and $\boldsymbol{\sigma}$ is the Pauli matrix. The spinor energy $\hat{\varepsilon}$ in Eq. (1) is expressed as[33]

$$\hat{\varepsilon}(p)=\varepsilon(p)\hat{\boldsymbol{I}}+\frac{1}{2}\boldsymbol{J}(p)\cdot\boldsymbol{\sigma}, \quad (3)$$

where $\boldsymbol{J}(p)=J(p)\boldsymbol{M}(x)$ and $\boldsymbol{M}(x)$ is the unit vector of magnetization of the ferromagnet. The spinor velocity $\hat{\boldsymbol{V}}$ in Eq. (1) is defined as[33]

$$\hat{\boldsymbol{V}}=\frac{1}{\hbar}\frac{\partial\hat{\varepsilon}}{\partial\boldsymbol{k}}=\frac{1}{\hbar}\left(\frac{\partial\varepsilon}{\partial\boldsymbol{k}}I+\frac{1}{2}\frac{\partial J}{\partial\boldsymbol{k}}\boldsymbol{M}\cdot\boldsymbol{\sigma}\right)=\boldsymbol{V}\hat{\boldsymbol{I}}+\boldsymbol{u}\boldsymbol{M}\cdot\boldsymbol{\sigma}. \quad (4)$$

Similar to the derivation of the quantum Boltzmann equation by Kadanoff,[36,37] the spinor Boltzmann equation can also be derived by the nonequilibrium Green function method.[38,39] The collision term $(\frac{\partial\hat{\boldsymbol{f}}}{\partial t})_{\text{collision}}$ in Eq. (1) can be naturally expressed by the lesser (or greater) self-energies and Green functions. Their calculation is complicated, so the relaxation time approximation can be used to simplify the scattering term. In the relaxation time approximation, the collision term $(\frac{\partial\hat{\boldsymbol{f}}}{\partial t})_{\text{collision}}$ in Eq. (1) can be simplified as

$$\left(\frac{\partial\hat{\boldsymbol{f}}}{\partial t}\right)_{\text{collision}}=\left(-\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{fI+\boldsymbol{g}\cdot\boldsymbol{\sigma}-\langle fI+\boldsymbol{g}\cdot\boldsymbol{\sigma}\rangle}{\tau}+\frac{2\langle\boldsymbol{g}\cdot\boldsymbol{\sigma}\rangle}{\tau_{\text{sf}}}\right], \quad (5)$$

where $\langle\cdot\rangle$ denotes the average over the momentum, τ denotes the momentum relaxation time, and τ_{sf} is the spin flip relaxation time.

2.1. Spin-dependent equations of continuity for charge and spin currents

In this section, we will derive the balance equations from the spinor Boltzmann equation (1). Integrating both sides of Eq. (1) over momentum, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\int\hat{\boldsymbol{f}}\,\mathrm{d}p+\boldsymbol{\nabla}\cdot\int\hat{\boldsymbol{V}}\hat{\boldsymbol{f}}\,\mathrm{d}p\\&-\frac{e\boldsymbol{E}}{\hbar}\cdot\int\frac{\partial\hat{\boldsymbol{f}}}{\partial\boldsymbol{p}}\mathrm{d}p+\frac{\mathrm{i}}{\hbar}\int[\hat{\varepsilon},\hat{\boldsymbol{f}}]\mathrm{d}p\\&=-\int\left(\frac{\partial\hat{\boldsymbol{f}}}{\partial t}\right)_{\text{collision}}\mathrm{d}p. \quad (6)\end{aligned}$$

If we define the spinor density $\hat{\boldsymbol{\rho}}$ as

$$\hat{\boldsymbol{\rho}}=\int\hat{\boldsymbol{f}}\,\mathrm{d}p=\int f\,\mathrm{d}p+\int\boldsymbol{g}\cdot\boldsymbol{\sigma}\,\mathrm{d}p, \quad (7)$$

and spinor current $\hat{j}$ as

$$\hat{\boldsymbol{j}}=\int\hat{\boldsymbol{V}}\hat{\boldsymbol{f}}\,\mathrm{d}p=[(\boldsymbol{V}f+\boldsymbol{u}\boldsymbol{M}\cdot\boldsymbol{g})\hat{\boldsymbol{I}}+(\boldsymbol{V}\boldsymbol{g}+\boldsymbol{u}f\boldsymbol{M}+\mathrm{i}\boldsymbol{u}(\boldsymbol{M}\times\boldsymbol{g}))\cdot\boldsymbol{\sigma}], \quad (8)$$

and then equation (6) can be rewritten as

$$\frac{\partial}{\partial t}\hat{\boldsymbol{\rho}}+\boldsymbol{\nabla}\cdot\hat{\boldsymbol{j}}=\frac{\mathrm{i}}{\hbar}\int[\hat{\varepsilon},\hat{\boldsymbol{f}}]\mathrm{d}p-\int\left(\frac{\partial\hat{\boldsymbol{f}}}{\partial t}\right)_{\text{collision}}\mathrm{d}p, \quad (9)$$

which is the equation of continuity for the spinor density and spinor current and the terms on the right-hand side are the source terms.

The expansion of the spinor and spinor current densities by the unit and Pauli matrices are

$$\hat{\boldsymbol{\rho}}=\rho\hat{\boldsymbol{I}}+\boldsymbol{m}\cdot\boldsymbol{\sigma}, \quad (10)$$

$$\hat{\boldsymbol{j}}=\boldsymbol{j}\hat{\boldsymbol{I}}+\vec{\boldsymbol{j}}_m\cdot\boldsymbol{\sigma}, \quad (11)$$

where ρ is the charge density, $\boldsymbol{m}$ is the spin accumulation, $\boldsymbol{j}$ is the charge current density, and $\vec{\boldsymbol{j}}_m$ is the spin current density and is a second-order tensor. Comparing Eqs. (7) and (10), we obtain

$$\rho=\int f\,\mathrm{d}p, \quad (12)$$

$$\boldsymbol{m}=\int\boldsymbol{g}\,\mathrm{d}p. \quad (13)$$

Similarly, as for Eq. (8) for the spinor current, the charge current and spin current densities in Eq. (11) can be expressed as

$$\boldsymbol{j}=\int(\boldsymbol{V}f+\boldsymbol{u}(\boldsymbol{M}\cdot\boldsymbol{g}))\mathrm{d}p, \quad (14)$$

$$\vec{\boldsymbol{j}}_m=\int[\boldsymbol{V}\boldsymbol{g}+\boldsymbol{u}f\boldsymbol{M}+\mathrm{i}\boldsymbol{u}(\boldsymbol{M}\times\boldsymbol{g})]\mathrm{d}p. \quad (15)$$

Finally, the equation of continuity, Eq. (9), for the spinor density and spinor current can be expressed as the equation of continuity for the charge density and charge current in the relaxation time approximation

$$\frac{\partial}{\partial t}\rho+\boldsymbol{\nabla}\cdot\boldsymbol{j}=\int\left(\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{f-\langle f\rangle}{\tau}\right]\mathrm{d}p. \quad (16)$$

The continuity equation for the spin accumulation and spin current is then given by

$$\frac{\partial}{\partial t}\boldsymbol{m}+\boldsymbol{\nabla}\cdot\vec{\boldsymbol{j}}_m=-\frac{\mathrm{i}}{\hbar}\int(\boldsymbol{J}(k)\times\boldsymbol{g})\mathrm{d}p+\int\left(\frac{\partial f^0}{\partial\varepsilon}\right)\times\left[\frac{\boldsymbol{g}-\langle\boldsymbol{g}\rangle}{\tau}+\frac{2\langle\boldsymbol{g}\rangle}{\tau_{\text{sf}}}\right]\mathrm{d}p. \quad (17)$$

Equations (16) and (17) were given in Refs. [26]–[30] except the terms $-\frac{\mathrm{i}}{\hbar}\int(\boldsymbol{J}(k)\times\boldsymbol{g})\mathrm{d}p$ which originate from the ferromagnet, and is an additional term in our expression. Equations (16) and (17) constitute the spin-dependent balance equations for the charge and spin current.

2.2. Spin-dependent momentum balance equation

In the formalism of the spinor Boltzmann equation, the spinor momentum density can be defined as

$$\hat{\rho}\langle \boldsymbol{V}\rangle = m\int \hat{\boldsymbol{f}}\boldsymbol{V}\,\mathrm{d}p. \tag{18}$$

Differentiating the above expression with respect to time t and making use of the spinor Boltzmann equation (Eq. (1)), we have

$$\begin{aligned}\frac{\partial \hat{\rho}\langle \boldsymbol{V}\rangle}{\partial t} &= -m\int\left[\hat{\boldsymbol{V}}\cdot\frac{\partial \hat{\boldsymbol{f}}}{\partial \boldsymbol{r}} - \frac{e\boldsymbol{E}}{\hbar}\cdot\frac{\partial \hat{\boldsymbol{f}}}{\partial \boldsymbol{p}} + \frac{\mathrm{i}}{\hbar}[\widehat{\varepsilon},\hat{\boldsymbol{f}}]\right.\\ &\quad\left.+\left(\frac{\partial \hat{\boldsymbol{f}}}{\partial t}\right)_{\text{collision}}\right]\boldsymbol{V}\,\mathrm{d}p.\end{aligned} \tag{19}$$

We define the spinor pressure tensor $\hat{\vec{\boldsymbol{P}}} = m\int \hat{\boldsymbol{V}}\hat{\boldsymbol{f}}\boldsymbol{V}\,\mathrm{d}p$, and equation (19) can be rearranged as

$$\begin{aligned}&\frac{\partial \hat{\rho}\langle \boldsymbol{V}\rangle}{\partial t}\\ &= -\boldsymbol{\nabla}\cdot\hat{\vec{\boldsymbol{P}}} - m\int\left[\frac{\mathrm{i}}{\hbar}[\widehat{\varepsilon},\hat{\boldsymbol{f}}] + \left(\frac{\partial \hat{\boldsymbol{f}}}{\partial t}\right)_{\text{collision}}\right]\boldsymbol{V}\,\mathrm{d}p,\end{aligned} \tag{20}$$

which is the balance equation for the spinor momentum.

If we further expand the spinor pressure tensor by the Pauli matrices,

$$\hat{\vec{\boldsymbol{P}}} = \vec{\boldsymbol{P}}\hat{\boldsymbol{I}} + \vec{\vec{\boldsymbol{P}}}_m\cdot\boldsymbol{\sigma}, \tag{21}$$

we have

$$\vec{\boldsymbol{P}} = m\int(\boldsymbol{V}f + \boldsymbol{u}\boldsymbol{M}\cdot\boldsymbol{g})\boldsymbol{V}\,\mathrm{d}p, \tag{22}$$

$$\vec{\vec{\boldsymbol{P}}}_m = m\int[\boldsymbol{V}\boldsymbol{g} + \boldsymbol{u}f\boldsymbol{M} + \mathrm{i}\boldsymbol{u}(\boldsymbol{M}\times\boldsymbol{g})]\boldsymbol{V}\,\mathrm{d}p, \tag{23}$$

where $\vec{\boldsymbol{P}}$ is a second-order tensor and $\vec{\vec{\boldsymbol{P}}}_m$ a third-order tensor. The balance equation (Eq. (20)) for the spinor momentum can then be decomposed as the usual momentum balance equation in the relaxation time approximation

$$\begin{aligned}\frac{\partial \rho\langle \boldsymbol{V}\rangle}{\partial t} &= -\boldsymbol{\nabla}\cdot\vec{\boldsymbol{P}} - m\frac{\mathrm{i}e\boldsymbol{E}}{\hbar}\rho\\ &\quad + m\int\left(\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{f-\langle f\rangle}{\tau}\right]\boldsymbol{V}\,\mathrm{d}p,\end{aligned} \tag{24}$$

and the spin-dependent part is

$$\begin{aligned}\frac{\partial \boldsymbol{m}\langle \boldsymbol{V}\rangle}{\partial t} &= -\boldsymbol{\nabla}\cdot\vec{\vec{\boldsymbol{P}}}_m - m\frac{e\boldsymbol{E}}{\hbar}\boldsymbol{m} - m\frac{\mathrm{i}}{\hbar}\int(\boldsymbol{J}(k)\times\boldsymbol{g})\boldsymbol{V}\,\mathrm{d}p\\ &\quad + m\int\left(\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{\boldsymbol{g}-\langle\boldsymbol{g}\rangle}{\tau} + \frac{2\langle\boldsymbol{g}\rangle}{\tau_{\text{sf}}}\right]\boldsymbol{V}\,\mathrm{d}p.\end{aligned} \tag{25}$$

Equation (24) corresponds to the well-known Navier–Stokes equation, while equation (25) is a new equation which originates from the spin accumulation in spintronic device. The spin-dependent pressure tensor $\vec{\vec{\boldsymbol{P}}}_m$ is induced by spin, from the complete spin-dependent momentum balance equation, and varies according to Eqs. (20), (24), and (25).

2.3. Spin-dependent heat balance equation

If we define the spinor internal energy density as

$$\hat{\rho}u = \frac{1}{2}m\int\hat{\boldsymbol{f}}\boldsymbol{V}^2\,\mathrm{d}p, \tag{26}$$

then making use of the spinor Boltzmann equation (Eq. (1)), we can obtain the equation of motion satisfying Eq. (26)

$$\begin{aligned}\frac{\partial \hat{\rho}u}{\partial t} &= -\boldsymbol{\nabla}\cdot\hat{\boldsymbol{j}}_q + \frac{m}{2}\frac{\mathrm{i}}{\hbar}\int(\boldsymbol{J}(k)\times\boldsymbol{g})\cdot\boldsymbol{\sigma}\boldsymbol{V}^2\,\mathrm{d}p\\ &\quad - \frac{m}{2}\int\left(\frac{\partial \hat{\boldsymbol{f}}}{\partial t}\right)_{\text{collision}}\boldsymbol{V}^2\,\mathrm{d}p.\end{aligned} \tag{27}$$

Here $\hat{\boldsymbol{j}}_q = \frac{1}{2}m\int\hat{\boldsymbol{V}}\hat{\boldsymbol{f}}\boldsymbol{V}^2\,\mathrm{d}p$ is the spinor thermal current density and can be decomposed as

$$\hat{\boldsymbol{j}}_q = \boldsymbol{j}_q\hat{\boldsymbol{I}} + \vec{\boldsymbol{j}}_{qm}\cdot\boldsymbol{\sigma}, \tag{28}$$

where

$$\boldsymbol{j}_q = -\frac{1}{2}m\int(\boldsymbol{V}f + \boldsymbol{u}\boldsymbol{M}\cdot\boldsymbol{g})\boldsymbol{V}^2\,\mathrm{d}p, \tag{29}$$

$$\vec{\boldsymbol{j}}_{qm} = -\frac{1}{2}m\int[\boldsymbol{V}\boldsymbol{g} + \boldsymbol{u}f\boldsymbol{M} + \mathrm{i}\boldsymbol{u}(\boldsymbol{M}\times\boldsymbol{g})]\boldsymbol{V}^2\,\mathrm{d}p. \tag{30}$$

$\vec{\boldsymbol{j}}_{qm}$ is a second-order tensor. Equation (27) is the heat balance equation for the spinor internal density. It can also be expressed as the heat balance equation about $\boldsymbol{j}_q$ with a spin-dependent part about $\vec{\boldsymbol{j}}_{qm}$ in the relaxation time approximation, as we did in the previous section,

$$\begin{aligned}\frac{\partial \rho u}{\partial t} &= -\boldsymbol{\nabla}\cdot\boldsymbol{j}_q + m\frac{e\boldsymbol{E}}{\hbar}\cdot\int(\boldsymbol{V}f + \boldsymbol{u}\boldsymbol{M}\cdot\boldsymbol{g})\,\mathrm{d}p\\ &\quad + \frac{1}{2}m\int\left(\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{f-\langle f\rangle}{\tau}\right]\boldsymbol{V}^2\,\mathrm{d}p,\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{\partial \boldsymbol{m}u}{\partial t} &= -\boldsymbol{\nabla}\cdot\vec{\boldsymbol{j}}_{qm} + m\frac{e\boldsymbol{E}}{\hbar}\\ &\quad\cdot\int[\boldsymbol{V}\boldsymbol{g} + \boldsymbol{u}f\boldsymbol{M} + \mathrm{i}\boldsymbol{u}(\boldsymbol{M}\times\boldsymbol{g})]\,\mathrm{d}p\\ &\quad + \frac{m}{2}\frac{\mathrm{i}}{\hbar}\int(\boldsymbol{J}(k)\times\boldsymbol{g})\boldsymbol{V}^2\,\mathrm{d}p\\ &\quad + \frac{m}{2}\int\left(\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{\boldsymbol{g}-\langle\boldsymbol{g}\rangle}{\tau} + \frac{2\langle\boldsymbol{g}\rangle}{\tau_{\text{sf}}}\right]\boldsymbol{V}^2\,\mathrm{d}p.\end{aligned} \tag{32}$$

Equation (31) is the usual heat balance equation, while equation (32) is the balance equation about the spin-dependent thermal current $\vec{\boldsymbol{j}}_{qm}$ that has been widely studied both theoretically and experimentally.[19–22] The spin-dependent balance equation (Eqs. (27), (31), and (32)) will be useful for the study of spin caloritronics.[21]

2.4. Spin-dependent entropy balance equation

The last balance equation is the entropy balance equation, and we now derive the spin-dependent form. If we define the spinor entropy density as

$$\hat{\rho}\bar{s} = -k\int \hat{\boldsymbol{f}}\ln\hat{\boldsymbol{f}}\,\mathrm{d}p, \tag{33}$$

and differentiating it with respect to t, using the spinor Boltzmann equation (Eq. (1)), we have

$$\begin{aligned}\frac{\partial\hat{\rho}\bar{s}}{\partial t} &= -\boldsymbol{\nabla}\cdot\hat{\boldsymbol{j}}_s - \frac{\mathrm{i}}{\hbar}\int(\boldsymbol{J}(k)\times\boldsymbol{g})\cdot\boldsymbol{\sigma}\,\mathrm{d}p\\ &\quad -k\int\left(\frac{\partial\hat{\boldsymbol{f}}}{\partial t}\right)_{\text{collision}}\mathrm{d}p,\end{aligned} \tag{34}$$

where $\hat{\boldsymbol{j}}_s = -k\int\hat{\boldsymbol{V}}\hat{f}\ln\hat{\boldsymbol{f}}\,\mathrm{d}p$ is the spinor entropy current density. Equation (34) is the required spinor entropy balance equation. We can further derive the corresponding spin-independent and spin-dependent parts. As in the derivation of the previous section, the spinor entropy current can be derived as

$$\hat{\boldsymbol{j}}_s = \boldsymbol{j}_s\hat{\boldsymbol{I}} + \vec{\boldsymbol{j}}_{sm}\cdot\boldsymbol{\sigma}, \tag{35}$$

and by comparing with the definition, we have

$$\begin{aligned}\boldsymbol{j}_s &= -k\int[a_1(\boldsymbol{V}f+\boldsymbol{u}\boldsymbol{M}\cdot\boldsymbol{g})\\ &\quad +(\boldsymbol{V}\boldsymbol{g}+\boldsymbol{u}f\boldsymbol{M}+\mathrm{i}\boldsymbol{u}(\boldsymbol{M}\times\boldsymbol{g}))\cdot\boldsymbol{a}_2]\mathrm{d}p,\end{aligned} \tag{36}$$

$$\begin{aligned}\vec{\boldsymbol{j}}_{sm} &= -k\int[\boldsymbol{a}_2(\boldsymbol{V}f+\boldsymbol{u}\boldsymbol{M}\cdot\boldsymbol{g})\\ &\quad +a_1(\boldsymbol{V}\boldsymbol{g}+\boldsymbol{u}f\boldsymbol{M}+\mathrm{i}\boldsymbol{u}(\boldsymbol{M}\times\boldsymbol{g}))\\ &\quad +\mathrm{i}(\boldsymbol{u}\boldsymbol{g}+\boldsymbol{u}f\boldsymbol{M}+\mathrm{i}\boldsymbol{u}(\boldsymbol{M}\times\boldsymbol{g}))\times\boldsymbol{a}_2]\mathrm{d}p,\end{aligned} \tag{37}$$

where $\vec{\boldsymbol{j}}_{sm}$ is a second-order tensor, and

$$a_1 = \frac{1}{2}\mathrm{Tr}\ln(f\hat{\boldsymbol{I}}+\boldsymbol{g}\cdot\boldsymbol{\sigma}), \tag{38}$$

$$\boldsymbol{a}_2 = (a_{2x}, a_{2y}, a_{2z}), \tag{39}$$

with

$$a_{2i} = \frac{1}{2}\mathrm{Tr}[\sigma_i\ln(f\hat{\boldsymbol{I}}+\boldsymbol{g}\cdot\boldsymbol{\sigma})],\quad (i=x,y,z). \tag{40}$$

In the relaxation time approximation, equation (34) can be then be decomposed as

$$\frac{\partial\rho\bar{s}}{\partial t} = -\boldsymbol{\nabla}\cdot\boldsymbol{j}_s + k\int\left(\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{f-\langle f\rangle}{\tau}\right]\mathrm{d}p, \tag{41}$$

$$\begin{aligned}\frac{\partial\boldsymbol{m}\bar{s}}{\partial t} &= -\boldsymbol{\nabla}\cdot\vec{\boldsymbol{j}}_{sm} - \frac{\mathrm{i}}{\hbar}k\int(\boldsymbol{J}(k)\times\boldsymbol{g})\mathrm{d}p\\ &\quad +k\int\left(\frac{\partial f^0}{\partial\varepsilon}\right)\left[\frac{\boldsymbol{g}-\langle\boldsymbol{g}\rangle}{\tau}+\frac{2\langle\boldsymbol{g}\rangle}{\tau_{sf}}\right]\mathrm{d}p.\end{aligned} \tag{42}$$

Equation (41) is the usual entropy balance equation, while equation (42) corresponds to the spin-dependent entropy balance equation. As the spin-dependent charge and thermal currents have been observed experimentally, and the entropy current are usually carried by the charge and thermal current, the study of the spin-dependent entropy current is interesting. Equations (34), (41), and (42) form the complete spin-dependent entropy balance equations.

3. Numerical results

As an example, we consider the spin-dependent transport through a one-dimensional mesoscopic ferromagnet. At a certain bias, the system will arrive at a steady state after the relaxation time. In this paper, we only investigate the physical quantities in the steady state, where $\frac{\partial f(p,X)}{\partial t} = 0$ and $\frac{\partial\boldsymbol{g}(p,X)}{\partial t} = 0$. Since $J(p)$ usually varies slowly with momentum, then $\boldsymbol{u}$ is very small compared to the electron velocity. By using Eq. (1) and neglecting the u_x term, we can express Eq. (1) as the equation of the scalar and vector distribution in the steady state[22]

$$V_x\frac{\partial f}{\partial x} + eEV_x = -\frac{f-\langle f\rangle}{\tau}, \tag{43}$$

$$V_x\frac{\partial\boldsymbol{g}}{\partial x} + \frac{J}{\hbar}\boldsymbol{M}\times\boldsymbol{g} = -\frac{\boldsymbol{g}-\langle\boldsymbol{g}\rangle}{\tau} - \frac{2\langle\boldsymbol{g}\rangle}{\tau_{sf}}. \tag{44}$$

Equations (43) and (44) can be solved by the Fourier transformation method. Taking the Fourier transform of both sides of Eqs. (43) and (44), we have

$$\begin{aligned}&\mathrm{i}k\frac{p}{m}F(k,p) + eE\frac{p}{m}\delta(k)\\ &= -\frac{F(k,p)-\langle F(k,p)\rangle}{\tau},\end{aligned} \tag{45}$$

$$\begin{aligned}&\mathrm{i}k\frac{p}{m}\boldsymbol{G}(k,p) + \frac{J}{\hbar}\boldsymbol{M}\times\boldsymbol{G}(k,p)\\ &= -\frac{\boldsymbol{G}(k,p)-\langle\boldsymbol{G}(k,p)\rangle}{\tau},\end{aligned} \tag{46}$$

where

$$F(k,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(p,x)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x, \tag{47}$$

$$\boldsymbol{G}(k,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\boldsymbol{g}(p,x)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x. \tag{48}$$

Equations (47) and (48) are the Fourier transformations of the scalar and vector distribution functions, and are easily obtained from Eqs. (45) and (46) via an algebra calculation. After taking the inverse Fourier transformation, we finally obtain the scalar and vector distribution function

$$f(p,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\frac{m}{\mathrm{i}p}\left[\frac{\langle F(k,p)\rangle}{\tau} - eE\frac{p}{m}\delta(k)\right]}{\left(k+\frac{m}{\mathrm{i}p\tau}\right)}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k, \tag{49}$$

$$g_{x(y,z)}(p,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{A_{x(y,z)}}{A}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k, \tag{50}$$

where

$$A=\begin{vmatrix} \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} & \frac{J}{\hbar}M_y & -\frac{J}{\hbar}M_z \\ \frac{J}{\hbar}M_z & \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} & -\frac{J}{\hbar}M_x \\ -\frac{J}{\hbar}M_y & \frac{J}{\hbar}M_x & \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} \end{vmatrix}, \tag{51}$$

$$A_x=\begin{vmatrix} \frac{\langle G_x(k,p)\rangle}{\tau} & \frac{J}{\hbar}M_y & -\frac{J}{\hbar}M_z \\ \frac{\langle G_y(k,p)\rangle}{\tau} & \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} & -\frac{J}{\hbar}M_x \\ \frac{\langle G_z(k,p)\rangle}{\tau} & \frac{J}{\hbar}M_x & \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} \end{vmatrix}, \tag{52}$$

$$A_y=\begin{vmatrix} \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} & \frac{\langle G_x(k,p)\rangle}{\tau} & -\frac{J}{\hbar}M_z \\ \frac{J}{\hbar}M_z & \frac{\langle G_y(k,p)\rangle}{\tau} & -\frac{J}{\hbar}M_x \\ -\frac{J}{\hbar}M_y & \frac{\langle G_z(k,p)\rangle}{\tau} & \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} \end{vmatrix}, \tag{53}$$

$$A_z=\begin{vmatrix} \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} & \frac{J}{\hbar}M_y & \frac{\langle G_x(k,p)\rangle}{\tau} \\ \frac{J}{\hbar}M_z & \mathrm{i}k\frac{p}{m}+\frac{1}{\tau} & \frac{\langle G_y(k,p)\rangle}{\tau} \\ -\frac{J}{\hbar}M_y & \frac{J}{\hbar}M_x & \frac{\langle G_z(k,p)\rangle}{\tau} \end{vmatrix}. \tag{54}$$

The integrals of Eqs. (49) and (50) can be further calculated by the residue theorem. After a self-consistent iteration procedure, we obtain the solutions for the scalar and vector distributions.

The numerical results for the charge density and charge current as a function of position are shown in Figs. 1 and 2, and both exponentially decreased with position, due to the resistance of the ferromagnet to an injected charge current. We also plotted the spin accumulation and spin current versus position in Figs. 3 and 4, and the three components of spin accumulation and spin current all oscillated with position. There was a reasonable phase delay between the three components. In Fig. 5, we plotted the usual thermal current, which decreased at a certain bias. The spin-dependent part of the thermal current is shown in Fig. 6, and the three components oscillated with position with a phase delay. As there was no temperature gradient in our system, the thermal current mainly originated from the Joule heat of the charge current. The usual and spin-dependent parts of the pressure tensor are shown in Figs. 7 and 8, the usual pressure tensor decreases with position, while the three components of the spin-dependent part oscillate with position, which indicates the existence of pressure in the Fermi liquid produced by the electrons in the ferromagnet.

4. Summary and discussion

In this paper, we derived the spin-dependent balance equations from the spinor Boltzmann equation. The charge density and charge current, spin accumulation and spin current, and spin-dependent pressure tensor, as well as the thermal and entropy currents were investigated in spintronics. The numerical results for these physical quantities are shown in Figs. 1–8. In our paper, we did not consider the temperature gradient in the system and the thermal current, with a spin-dependent part, originated from the Joule heat. We will consider the temperature gradient in future research.

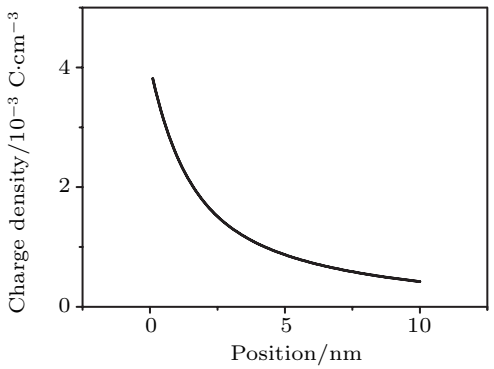


Fig. 1. (color online) Charge density as a function of position. The momentum relaxation time is $\tau = 0.01$ ps, and the electric field $E =$ 1 mV/nm.

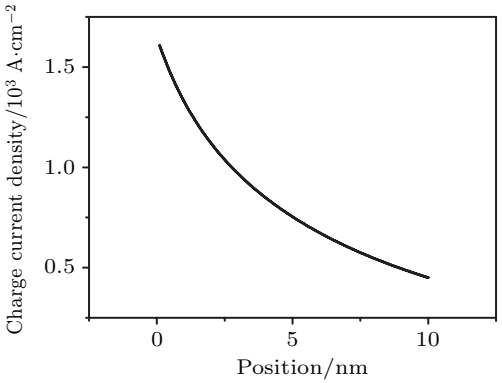


Fig. 2. (color online) Charge current versus position. The momentum relaxation time is $\tau = 0.01$ ps, and the electric field $E = 1$ mV/nm.

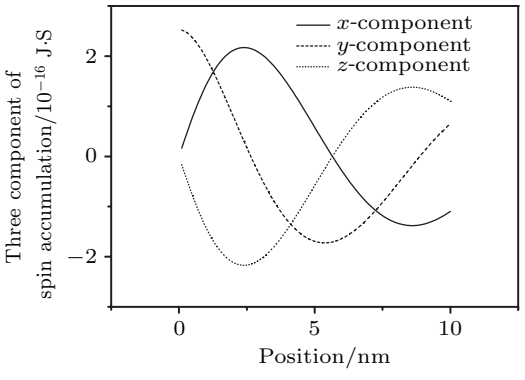


Fig. 3. (color online) Three components of spin accumulation versus position. The momentum relaxation time is $\tau = 0.01$ ps, and the electric field $E = 1$ mV/nm.

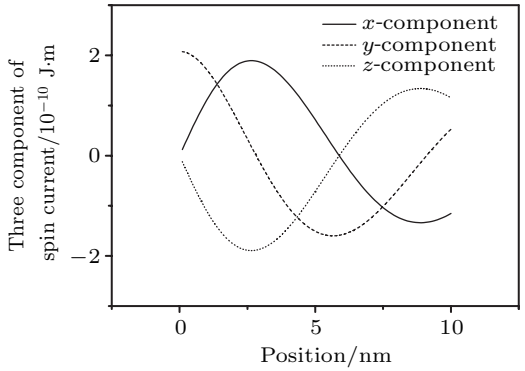


Fig. 4. (color online) Three components of spin current versus position. The momentum relaxation time is $\tau = 0.01$ ps, and the electric field $E = 1$ mV/nm.

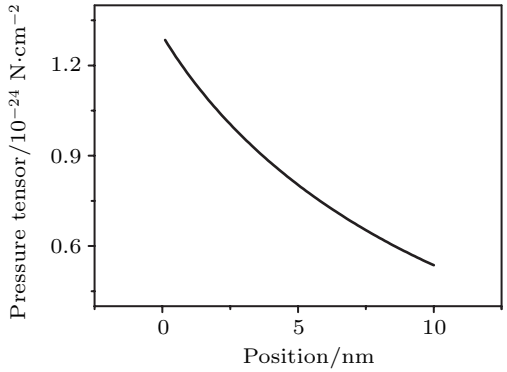


Fig. 5. (color online) The xx-component of the pressure tensor versus position. The momentum relaxation time is $\tau = 0.01$ ps, and the electric field $E = 1$ mV/nm.

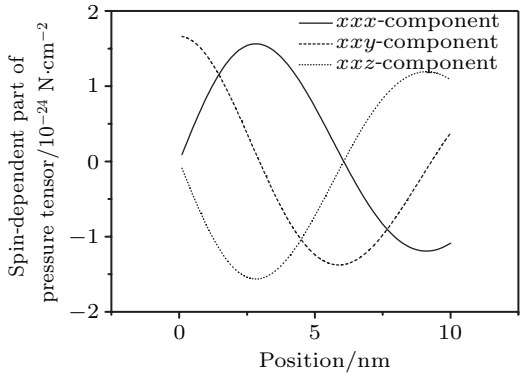


Fig. 6. (color online) Spin-dependent part of pressure tensor versus position. The momentum relaxation time is $\tau = 0.01$ ps, and the electric field $E = 1$ mV/nm.

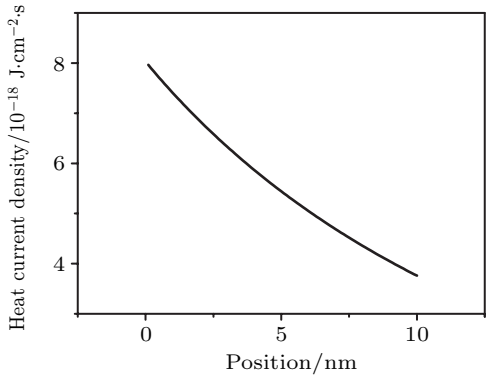


Fig. 7. (color online) Heat current versus position. The momentum relaxation time is $\tau = 0.01$ ps, and the electric field $E = 1$ mV/nm.

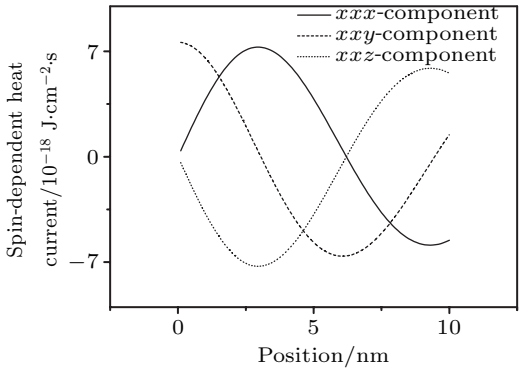


Fig. 8. (color online) Spin-dependent heat current versus position. The momentum relaxation time is $\tau = 0.01$ ps, and the electric field $E = 1$ mV/nm.

It should be highlighted that the balance equations in our manuscript are not closed as the usual hydrodynamic equations, and the equations depend on the scalar and vector distributions. However, it is useful to study the physical quantities in the hydrodynamic equations by the statistical method that is spin-dependent. In the derivation of the quantum Boltzmann equation using nonequilibrium Green function theory, Mahan[26] proved that if the lesser (greater) self-energy can be expressed as the product of the lesser (greater) Green function and another single-particle Green function, then the continuity equation for the charge density and current is closed. Moreover, the contribution from the scattering terms tends to zero. However, we cannot assure the validity of this assumption in the spin-dependent case, so generally speaking, we should keep the scattering terms in our balance equations which are therefore not closed. The scattering terms in the spin-dependent case[27,28] are complicated to calculate and the relaxation time approximation is usually adopted to simplify them. The development of other approximations for these scattering terms are important.

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