Leader-following consensus of discrete-time fractional-order multi-agent systems

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Leader-following consensus of fractional order multi-agent systems is investigated. The agents are considered as discrete-time fractional order integrators or fractional order double-integrators. Moreover, the interaction between the agents is described with an undirected communication graph with a fixed topology. It is shown that the leader-following consensus problem for the considered agents could be converted to the asymptotic stability analysis of a discrete-time fractional order system. Based on this idea, sufficient conditions to reach the leader-following consensus in terms of the controller parameters are extracted. This leads to an appropriate region in the controller parameters space. Numerical simulations are provided to show the performance of the proposed leader-following consensus approach.

Keywords: multi-agent systems, fractional order systems, leader-following consensus, discrete-time fractional order systems

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1. Introduction

The application of multi-agent systems in sensor networks,[1] autonomous vehicles,[2] and mobile robots[3] has attracted the researchers to develop new methods for cooperative control of multi-agent systems.[4] These methods have been proposed to attain different goals such as consensus,[5] formation control,[6] and flocking.[7] Agreement of all the agents in a multi-agent system is called consensus. In a leader-following consensus, the control signals of the agents are appropriately selected such that their state trajectories follow the leader state.[5] This could be achieved by obtaining information from the leader and other agents.

Various works have been reported about leaderless consensus and leader-following consensus of multi-agent systems. Hierarchical cooperative control of multi-agent systems with switching directed topologies has been verified in Ref. [8]. In Ref. [9], a novel finite-time stability criterion for a linear discrete-time stochastic system with applications to consensus of multi-agent systems has been presented. Distributed node-to-node consensus of multi-agent systems with stochastic sampling has been investigated in Ref. [10]. Distributed observer-based stabilization of nonlinear multi-agent systems with sampled-data control[11] and observer-based consensus tracking of nonlinear agents in hybrid varying directed topology[12] have been studied too. These works include the first order agents (integrator), the second order agents (double-integrator), or high-order agents (with arbitrary number of states). However, in practical applications, consensus of multi-agent systems with the first-order or second-order agents is preferred. In Ref. [13], finite-time consensus of second-order multi-agent systems using the auxiliary system approach has been verified. Leader-following consensus of second-order multi-agent systems with fixed and time-varying delays has been studied in Refs. [14] and [15]. In Ref. [16], a leader-following consensus control for first-order multi-agent systems in the presence of measurement noise and communication delays has been proposed. Consensus of second-order discrete-time multi-agent systems with fixed topology has been addressed in Ref. [17]. In Ref. [18], robust finite-time leader-following consensus for second-order multi-agent systems with nonlinear dynamics has been discussed. Finite-time consensus of second-order leader-following multi-agent systems without velocity measurements has been verified in Ref. [19]. In Ref. [20], consensus of second-order multi-agent systems in the presence of velocity constraints has been verified. Moreover, consensus of second-order multi-agent systems with external disturbances has been considered in the literature.[21–23]

Fractional order systems have been utilized to describe dynamics of some physical phenomena (like diffusion and viscoelasticity) with non-integer operators.[24] Dynamical systems could be better described by fractional order systems. For example, the dynamical properties of neural networks could be better described with fractional complex-valued neural networks.[25] Consensus of fractional order multi-agent systems (FOMAS) has been considered in the literature. In Ref. [26], necessary and sufficient conditions for consensus of delayed fractional-order systems have been extracted using the generalized Nyquist stability criterion. The same approach has been employed in Ref. [27] to reach consensus for fractional order systems with simultaneously non-uniform input and communication delays. In Ref. [28], sufficient con-

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ditions for consensus of uncertain FOMAS in terms of linear matrix inequalities (LMI) have been extracted. In Ref. [29], leader-following consensus of FOMAS under fixed topology has been studied. Consensus of FOMAS with communication delays has been verified in Ref. [30]. In Ref. [31], the multiconsensus problem of uncertain FOMAS has been verified. Robust consensus problem of FOMAS with positive real uncertainty has been solved using the information of second-order neighbors in Ref. [32]. In Ref. [33], the leader-following consensus of FOMAS using adaptive pinning control has been investigated. Robust consensus of FOMAS with external disturbances over a directed fixed topology has been studied in Ref. [34]. In Ref. [35], the consensus problem of FOMAS with a reference state has been investigated. Consensus of FOMAS with linear models via observer-type protocol has been addressed in Ref. [36]. In Ref. [37], necessary and sufficient conditions to reach consensus for FOMAS with double-integrator were derived. Consensus of linear and nonlinear FOMAS with input time delay has been investigated in Ref. [38]. In Ref. [39], the concept of consensus region for FOMAS has been defined and a multistep control protocol has been proposed for the mentioned region. Moreover, to improve the robustness of FOMAS against communication time delay, the delayed state fractional order derivative has been introduced.[40]

All previous works about consensus of FOMAS have been devoted to continuous-time fractional order systems. As we know, the leader-following consensus of multi-agent systems describing with discrete-time fractional difference equations has not been studied in the literature. The stability analysis of the systems described by fractional order difference state space equations is more complicated comparing with that of the systems described by fractional order differential equations. In this paper, leader-following consensus of discrete-time FOMAS is investigated. The dynamics of the agents and leader are considered as discrete-time pseudo first-order or second-order fractional systems. Moreover, a fixed topology with undirected communication graph is considered. The consensus problem of discrete-time FOMAS is converted to the stability analysis of a discrete-time fractional order system. The results obtained in Refs. [41] and [42] are employed for the stability analysis of the closed loop system. For discrete-time pseudo first-order agents, the necessary and sufficient condition in terms of the controller gain to reach the leader-following consensus is derived. For agents described by discrete-time pseudo second-order systems, sufficient conditions in terms of the controller gains are obtained. This leads to a special two-dimensional region in the controller parameters space. Numerical examples are given to show the effectiveness of the proposed method.

This paper is arranged as follows. Section 2 reviews the necessary concepts of the fractional calculus. The mathematical preliminaries about graph theory and multi-agent systems are given in Section 3. The details about the leader-following consensus of discrete-time FOMAS with pseudo first or second-order agents are presented in Section 4. Numerical simulations and concluding remarks are given in Sections 5 and 6, respectively.

2. Review on fractional calculus concepts

Although there are different definitions for the fractional order derivative, in this paper the Grunwald–Letnikov definition, the discrete version of the fractional order derivative, is utilized.[24] According to the Grunwald–Letnikov definition, the fractional order derivative of a function $f(t)$, called ${}_aD_t^\alpha f(t)$, is given by[24]

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left\lfloor \frac{t-a}{h} \right\rfloor} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{1} $$

where α is the fractional order. Considering $a=0$ and $t=kh$ yields the following approximated version of Eq. (1):

$$ D_t^\alpha f(t) \approx \Delta_h^\alpha f(kh), \tag{2} $$

where

$$ \Delta_h^\alpha f(kh) = h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f((k-j)h). \tag{3} $$

Now, consider a continuous-time commensurate fractional order system with the following state space equations:

$$ D_t^\alpha x(t) = \boldsymbol{F}x(t) + \boldsymbol{G}u(t), \tag{4} $$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$ are the state vector and the input signal, respectively. $\boldsymbol{F} \in \mathbf{R}^{n\times n}$ and $\boldsymbol{G} \in \mathbf{R}^{n\times m}$ are the state space matrices and α is the commensurate order belonging to (0, 1]. According to relations (2) and (3), the discrete version of state space equation (4) could be obtained as[24]

$$ \Delta_T^\alpha x(k+1) = \boldsymbol{F}\boldsymbol{x}(k) + \boldsymbol{G}\boldsymbol{u}(k), \tag{5} $$

where T is the sampling period and $\Delta_T^\alpha x(k+1)$ is given by[24]

$$ \Delta_T^\alpha x(k+1) = T^{-\alpha} \sum_{j=0}^{k+1} (-1)^j \binom{\alpha}{j} x(k+1-j). \tag{6} $$

Theorem 1 The discrete-time fractional order system (5) is asymptotically stable if and only if the following conditions are satisfied:[41,42]

$$ \varphi_i^F \in \left[\frac{\alpha\pi}{2}, 2\pi - \frac{\alpha\pi}{2}\right] \wedge \left|\lambda_i^F\right| < |\omega_i|, \tag{7} $$

where λ_i^F is the i-th eigenvalue of $\boldsymbol{F}$, φ_i^F is the argument of λ_i^F, and $|\omega_i|$ is given by

$$ |\omega_i| = \left(\frac{2}{T}\left|\sin\frac{\varphi_i^F - \alpha\pi/2}{2-\alpha}\right|\right)^\alpha, \quad i = 1,\ldots,n. \tag{8} $$

This means that system (5) is stable if and only if all λ_i^F lie inside the specified region plotted in Fig. 1.

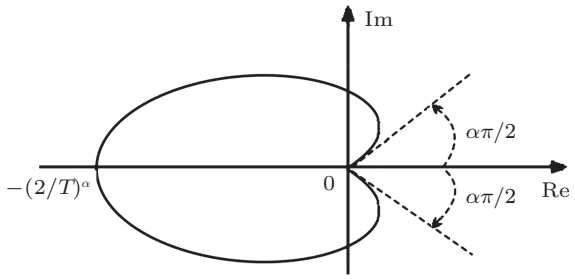


Fig. 1. The stability region for discrete-time fractional-order system (5).

3. Graph theory preliminaries

The graph theorem is utilized to describe the topology of multi-agent systems. In this section, some fundamental definitions and notions about the graph theory are illustrated. A graph denoted by $G = (S,E)$ is a set of nodes and edges. $S = \{s_1, s_2, \ldots, s_N\}$ is the set of nodes (the number of nodes is denoted by N), $E \subseteq S \times S$ is the set of edges, and members of E are denoted by $e_{ij} = (s_i, s_j)$. The set of neighbors of node s_i is defined by $N_i = \{s_j \in S : (s_j, s_i) \in E\}$. The adjacency matrix of graph G is an $N \times N$ matrix with non-negative elements denoted by $\boldsymbol{A} = [a_{ij}]$. The diagonal elements of $\boldsymbol{A}$ are assumed to be zero. Moreover, if $(s_j, s_i) \in E, i \neq j$, then $a_{ij} > 0$, otherwise $a_{ij} = 0$. If $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$, then the graph is called undirected. Otherwise, the graph is called directed or a digraph. The Laplacian matrix of the graph is denoted by $\boldsymbol{L} = [l_{ij}]$ with $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. A directed graph is called strongly connected if there is a directed path between any pair of distinct nodes. For undirected graphs, the word "strongly" is omitted. A graph has a spanning tree if at least one node has a path to all of the other nodes. This means that strongly connected (or connected) graphs have at least one spanning tree. The Laplacian matrix of the directed graphs has at least one zero eigenvalue with the corresponding eigenvector $\mathbf{1} = (1, 1, \ldots, 1)^{\mathrm{T}}$.

Lemma 1 The Laplacian matrix of undirected connected graphs has one zero eigenvalue with eigenvector $\mathbf{1}$ and all of its nonzero eigenvalues are real and positive.[4]

Moreover, the leader is denoted by node s_0. The connection weight between agent i and the leader is denoted by $\boldsymbol{D} = \mathrm{diag}\{d_1, \ldots, d_N\}$. If agent i is connected to the leader, then $d_i = 1$, otherwise $d_i = 0$.

Lemma 2 For connected undirected graphs (with at least one spanning tree), if there is at least one path between the leader and all the agents, then all the eigenvalues of matrix $\boldsymbol{H} = \boldsymbol{L} + \boldsymbol{D}$ are non-zero and positive.[5]

4. Problem statement

In this section, leader-following consensus of discrete-time fractional-order multi-agent systems is studied. The dynamics of agents is described with fractional order integrators (pseudo first-order) or fractional order double integrators (pseudo second-order). The same dynamics is considered for the leaders. The details are given in the following.

4.1. Leader-following consensus of discrete-time pseudo first-order agents

In this section, the dynamics of the agents is considered as

$$\Delta_T^{\alpha} x_i(k+1) = u_i(k), \quad i = 1, \ldots, N, \tag{9}$$

where $x_i(k)$ and $u_i(k)$ are the position and the control signal of the i-th agent at time k, respectively, and $\alpha \in (0, 1]$. This is the discrete version of a fractional order integrator. This dynamics is called pseudo first-order system.

The dynamics of the leader is considered as

$$\Delta_T^{\alpha} x_0(k+1) = 0, \tag{10}$$

where $x_0(k)$ is the position of the leader at time k.

Definition 1 The leader-following consensus of systems (9) and (10) will be achieved if for each agent $i, i = 1, \ldots, N$, the appropriate control signal $u_i(k)$ (depending on x_j, $j \in N_i$) exists such that[5]

$$\lim_{k \to \infty} \|x_i(k) - x_0(k)\| = 0, \quad i = 1, \ldots, N \tag{11}$$

for any initial condition $x_i(0)$, $i = 1, \ldots, N$.

Assumption 1 In this paper, the interaction between the agents is described by an undirected connected graph with a fixed topology.

If the position error between agent i and the leader is denoted by $\tilde{x}_i(k) = x_i(k) - x_0(k)$, then according to Eqs. (9) and (10), we have

$$\Delta_T^{\alpha} \tilde{x}_i(k+1) = u_i(k), \quad i = 1, \ldots, N. \tag{12}$$

The leader-following consensus is achieved if and only if $\lim_{k \to \infty} \|\tilde{x}_i(k)\| = 0, i = 1, \ldots, N$. Now, the control signal of each agent is considered as

$$\begin{aligned} u_i(k) \;=\; & k_1 \sum_{j \in N_i} a_{ij}(x_j(k) - x_i(k)) + k_1 d_i(x_0(k) - x_i(k)), \\ & i = 1, \ldots, N, \end{aligned} \tag{13}$$

where k_1 is the feedback control gain.

Remark 1 In this paper, the leader is considered without the control signal. If the control signal is considered for the leader, i.e., $\Delta_T^{\alpha} x_0(k+1) = u_0(k)$, where $u_0(k)$ is the control signal of the leader, the control signal (13) should be replaced with $u_i(k) = u_0(k) + k_1 \sum_{j \in N_i} a_{ij}(x_j(k) - x_i(k)) + k_1 d_i(x_0(k) - x_i(k)), i = 1, \ldots, N$.

Relations (13) and (12) give

$$\Delta_T^{\alpha} \tilde{x}_i(k+1) \;=\; k_1 \sum_{j \in N_i} a_{ij}(x_j(k) - x_i(k)) + k_1 d_i(x_0(k) - x_i(k)),$$

$$i = 1, \ldots, N. \tag{14}$$

Relation (14) could be rewritten as

$$\begin{aligned} \Delta_T^\alpha \tilde{x}_i(k+1) &= k_1 \sum_{j \in N_i} a_{ij}(\tilde{x}_j(k) - \tilde{x}_i(k)) - k_1 d_i \tilde{x}_i(k), \\ & i = 1, \ldots, N. \end{aligned} \tag{15}$$

According to the definition of matrices $\boldsymbol{D}$, $\boldsymbol{L}$, and $\boldsymbol{H}$, equation (15) could be rewritten as

$$\Delta_T^\alpha \tilde{\boldsymbol{x}}(k+1) = -k_1 \boldsymbol{H} \tilde{\boldsymbol{x}}(k), \tag{16}$$

where $\tilde{\boldsymbol{x}}(k) = [\tilde{x}_1(k), \ldots, \tilde{x}_N(k)]^{\mathrm{T}}$.

Thus, the leader-following consensus of systems (9) and (10) depends on the asymptotic stability of discrete-time fractional order system (16). The following theorem gives the necessary and sufficient condition for the asymptotic stability of system (16).

Theorem 2 Consider that Assumption 1 is satisfied. Now, the necessary and sufficient condition for leader-following consensus of the multi-agent systems described by relations (9) and (10) is

$$0 < k_1 < \frac{2^\alpha}{T^\alpha \mu_{\max}}, \tag{17}$$

where $\mu_{\max}$ is the maximum eigenvalue of matrix $\boldsymbol{H}$.

Proof The leader-following consensus is realized if and only if system (16) is asymptotically stable. The eigenvalues of the state matrix of system (16) are calculated as

$$\lambda_i = -k_1 \mu_i, \quad i = 1, \ldots, N, \tag{18}$$

where $\mu_i, i = 1, \ldots, N$ are the eigenvalues of matrix $\boldsymbol{H}$. According to Theorem 1, system (16) is asymptotically stable if and only if its eigenvalues fulfill the conditions (7). On the other hand, according to Assumption 1 and Lemma 2, all μ_i are real and positive. Now, the argument of the eigenvalues $\lambda_i, i = 1, \ldots, N$ could be obtained as (considering $k_1 > 0$)

$$\varphi_i = \arg(\lambda_i) = \arg(-k_1 \mu_i) = \pi. \tag{19}$$

It is obvious that $\pi \in [\alpha\pi/2, 2\pi - \alpha\pi/2]$. Thus, the argument condition in Eq. (7) is satisfied. It is enough to verify the condition on absolute value of $\lambda_i, i = 1, \ldots, N$. According to Eqs. (8) and (18),

$$|\omega_i| = \left(\frac{2}{T} \left| \sin \frac{\pi - \alpha\pi/2}{2 - \alpha} \right| \right)^\alpha = \frac{2^\alpha}{T^\alpha}, \quad i = 1, \ldots, N.$$

Thus, according to Eq. (7), we have

$$|\lambda_i| = k_1 \mu_i < \frac{2^\alpha}{T^\alpha}, \quad i = 1, \ldots, N. \tag{20}$$

Therefore, $k_1 < 2^\alpha / \mu_i T^\alpha, i = 1, \ldots, N$. This means that $0 < k_1 < 2^\alpha / T^\alpha \mu_{\max}$. Thus, system (16) is asymptotically stable (or equivalently $\lim\limits_{k \to \infty} \|\tilde{x}_i(k)\| = 0, i = 1, \ldots, N$) if and only if condition (17) is realized. This completes the proof.

4.2. Leader-following consensus of discrete-time pseudo second-order agents

Consider the pseudo second-order agents with the following discrete-time dynamics:

$$\begin{cases} \Delta_T^\alpha x_i(k+1) = v_i(k), \\ \Delta_T^\alpha v_i(k+1) = u_i(k), \end{cases} \quad i = 1, \ldots, N, \tag{21}$$

where $x_i(k)$ and $v_i(k)$ are the position and the velocity of the i-th agent, respectively.

The state space equations of the leader are given by

$$\begin{cases} \Delta_T^\alpha x_0(k+1) = v_0(k), \\ \Delta_T^\alpha v_0(k+1) = 0, \end{cases} \tag{22}$$

where $x_0(k)$ and $v_0(k)$ are the leader position and velocity, respectively.

Definition 2 The leader-following consensus of systems (21) and (22) will be attained if for each agent i, $i = 1, \ldots, N$, there is an appropriate control signal $u_i(k)$ (depending on $x_j, v_j, j \in N_i$) such that[5]

$$\begin{aligned} &\lim_{k \to \infty} \|x_i(k) - x_0(k)\| = \lim_{k \to \infty} \|v_i(k) - v_0(k)\| = 0, \\ &i = 1, \ldots, N \end{aligned} \tag{23}$$

for any initial condition $x_i(0), i = 1, \ldots, N$.

Let us define the position error as $\tilde{x}_i(k) = x_i(k) - x_0(k)$ and the velocity error as $\tilde{v}_i(k) = v_i(k) - v_0(k)$. Now, according to Eqs. (21) and (22), the following error dynamics will be obtained as

$$\begin{cases} \Delta_T^\alpha \tilde{x}_i(k+1) = \tilde{v}_i(k), \\ \Delta_T^\alpha \tilde{v}_i(k+1) = u_i(k), \end{cases} \quad i = 1, \ldots, N. \tag{24}$$

It is obvious that leader-following consensus of multi-agent systems (21) and (22) will be achieved if and only if $\lim\limits_{k \to \infty} \|\tilde{x}_i(k)\| = \lim\limits_{k \to \infty} \|\tilde{v}_i(k)\| = 0, i = 1, \ldots, N$.

The control law for the agents is considered as

$$\begin{aligned} u_i(k) &= \sum_{j \in N_i} a_{ij}[k_1(x_j(k) - x_i(k)) + k_2(v_j(k) - v_i(k))] \\ &\quad + d_i[k_1(x_0(k) - x_i(k)) + k_2(v_0(k) - v_i(k))], \\ &\quad i = 1, \ldots, N, \end{aligned} \tag{25}$$

where k_1 and k_2 are the the feedback control gains.

Incorporating relations (24) and (25) yields

$$\begin{aligned} \Delta_T^\alpha \tilde{x}_i(k+1) &= \tilde{v}_i(k), \\ \Delta_T^\alpha \tilde{v}_i(k+1) &= \sum_{j \in N_i} a_{ij}[k_1(\tilde{x}_j(k) - \tilde{x}_i(k)) + k_2(\tilde{v}_j(k) \\ &\quad - \tilde{v}_i(k))] - d_i[k_1 \tilde{x}_i(k) + k_2 \tilde{v}_i(k)]. \end{aligned} \tag{26}$$

According to Eq. (26) and the definition of Laplacian matrix, we have

$$\begin{bmatrix} \Delta_T^\alpha \tilde{\boldsymbol{x}}(k+1) \\ \Delta_T^\alpha \tilde{\boldsymbol{v}}(k+1) \end{bmatrix} = \boldsymbol{\Gamma} \begin{bmatrix} \tilde{\boldsymbol{x}}(k) \\ \tilde{\boldsymbol{v}}(k) \end{bmatrix}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} 0 & \boldsymbol{I}_N \\ -k_1 \boldsymbol{H} & -k_2 \boldsymbol{H} \end{bmatrix}, \tag{27}$$

where $\tilde{\boldsymbol{x}}(k)=[\tilde{x}_1(k),\ldots,\tilde{x}_N(k)]^{\mathrm{T}}$, $\tilde{\boldsymbol{v}}(k)=[\tilde{v}_1(k),\ldots,\tilde{v}_N(k)]^{\mathrm{T}}$, and $\boldsymbol{I}_N$ is an $N\times N$ identity matrix.

In the following, sufficient conditions for reaching the leader-following consensus of multi-agent systems (21) and (22) are derived.

Leader-following consensus of multi-agent systems (21) and (22) depends on the asymptotic stability of system (27). If the eigenvalue of matrix $\boldsymbol{\Gamma}$ in Eq. (27) is denoted by λ, then we have

$$\begin{aligned}\det(\lambda I_{2N}-\boldsymbol{\Gamma}) &= \det\begin{bmatrix}\lambda\boldsymbol{I}_N & -\boldsymbol{I}_N\\ k_1\boldsymbol{H} & \lambda\boldsymbol{I}_N+k_2\boldsymbol{H}\end{bmatrix}\\ &= \det(\lambda\boldsymbol{I}_N)\det\left(\lambda\boldsymbol{I}_N+k_2\boldsymbol{H}+\frac{k_1\boldsymbol{H}}{\lambda}\right)\\ &= \det(\lambda^2\boldsymbol{I}_N+(k_2\lambda+k_1)\boldsymbol{H})\\ &= \prod_{i=1}^{N}(\lambda^2+k_2\mu_i\lambda+k_1\mu_i)\\ &= 0. \end{aligned}\tag{28}$$

Solving equation (28) gives

$$\begin{cases}\lambda_{i1}=\dfrac{-k_2\mu_i+\sqrt{k_2^2\mu_i^2-4k_1\mu_i}}{2},\\ \lambda_{i2}=\dfrac{-k_2\mu_i-\sqrt{k_2^2\mu_i^2-4k_1\mu_i}}{2},\end{cases}\quad i=1,\ldots,N.\tag{29}$$

Consider that λ_{i1} and λ_{i2} could be complex numbers. Thus, conditions (7) may give complicated relations in terms of k_1 and k_2. To attain explicit inequalities in terms of k_1 and k_2, the parameters are chosen such that the resultant eigenvalues are real numbers. This means

$$k_2^2\mu_i^2-4k_1\mu_i\geq 0,\quad i=1,\ldots,N.\tag{30}$$

Or, equivalently

$$k_1\leq\frac{\mu_{\min}}{4}k_2^2.\tag{31}$$

Consider that condition (32) is conservative (because eigenvalues may have complex values while the conditions in Eq. (7) could be satisfied too). This is the reason why only sufficient conditions for leader-following consensus of multi-agent systems (21) and (22) will be obtained.

According to Lemma 2, all μ_i are positive. Now, if $k_1>0$ and $k_2>0$ are considered, both λ_{i1} and λ_{i2} are negative. This means

$$\arg(\lambda_{i1})=\arg(\lambda_{i2})=\pi,\quad i=1,\ldots,N.\tag{32}$$

Thus, the argument condition in Eq. (7) is fulfilled (consider that $\pi\in[\alpha\pi/2,2\pi-\alpha\pi/2]$). Now, only the absolute value condition given in Eq. (7) should be verified. According to Eq. (8),

$$|\omega_i|=\left(\frac{2}{T}\left|\sin\frac{\pi-\alpha\pi/2}{2-\alpha}\right|\right)^{\alpha}=\frac{2^\alpha}{T^\alpha},\quad i=1,\ldots,N.$$

It is obvious that $|\lambda_{i1}|<|\lambda_{i2}|$. Thus, satisfaction of Eq. (8) leads to

$$|\lambda_{i2}|=\frac{k_2\mu_i+\sqrt{k_2^2\mu_i^2-4k_1\mu_i}}{2}<\frac{2^\alpha}{T^\alpha},\quad i=1,\ldots,N.\tag{33}$$

Relation (33) gives

$$\sqrt{k_2^2\mu_i^2-4k_1\mu_i}<\frac{2^{\alpha+1}}{T^\alpha}-k_2\mu_i,\quad i=1,\ldots,N.\tag{34}$$

The right side of relation (34) should be positive. Thus, we have

$$k_2<\frac{2^{\alpha+1}}{T^\alpha\mu_i},\quad i=1,\ldots,N.\tag{35}$$

According to relation (35), we have

$$k_2<\frac{2^{\alpha+1}}{T^\alpha\mu_{\max}}.\tag{36}$$

Relation (34) gives

$$k_1>\frac{2^\alpha}{T^\alpha}\left(k_2-\frac{2^\alpha}{\mu_iT^\alpha}\right),\quad i=1,\ldots,N.\tag{37}$$

Relation (37) leads to

$$k_1>\frac{2^\alpha}{T^\alpha}\left(k_2-\frac{2^\alpha}{\mu_{\max}T^\alpha}\right).\tag{38}$$

According to relations (31) and (38), we have

$$\frac{\mu_{\min}}{4}k_2^2>\frac{2^\alpha}{T^\alpha}\left(k_2-\frac{2^\alpha}{\mu_{\max}T^\alpha}\right),\tag{39}$$

or

$$\frac{\mu_{\min}}{4}k_2^2-\frac{2^\alpha}{T^\alpha}k_2+\frac{2^{2\alpha}}{\mu_{\max}T^{2\alpha}}>0.\tag{40}$$

It could be easily verified that the discriminant of the quadratic equation in the left side of inequality (40) (or $(2^{2\alpha}/T^{2\alpha})(1-\mu_{\min}/\mu_{\max})$) is positive. Thus, we have

$$k_2>\frac{2^{\alpha+1}}{T^\alpha\mu_{\min}}\left(1+\sqrt{1-\frac{\mu_{\min}}{\mu_{\max}}}\right)$$

or

$$k_2<\frac{2^{\alpha+1}}{T^\alpha\mu_{\min}}\left(1-\sqrt{1-\frac{\mu_{\min}}{\mu_{\max}}}\right).\tag{41}$$

Consider that

$$\frac{2^{\alpha+1}}{T^\alpha\mu_{\max}}=\frac{2^{\alpha+1}}{T^\alpha\mu_{\min}}\left(1+\sqrt{1-\frac{\mu_{\min}}{\mu_{\max}}}\right)\left(1-\sqrt{1-\frac{\mu_{\min}}{\mu_{\max}}}\right).$$

Therefore, we have

$$\begin{aligned}&\frac{2^{\alpha+1}}{T^\alpha\mu_{\min}}\left(1-\sqrt{1-\frac{\mu_{\min}}{\mu_{\max}}}\right)<\frac{2^{\alpha+1}}{T^\alpha\mu_{\max}}\\ &<\frac{2^{\alpha+1}}{T^\alpha\mu_{\min}}\left(1+\sqrt{1-\frac{\mu_{\min}}{\mu_{\max}}}\right).\end{aligned}\tag{42}$$

According to relation (42), the inequalities (36) and (41) will be realized if

$$k_2 < \frac{2^{\alpha+1}}{T^{\alpha}\mu_{\min}}\left(1-\sqrt{1-\frac{\mu_{\min}}{\mu_{\max}}}\right). \tag{43}$$

Finally, the following theorem could be obtained.

Theorem 3 The sufficient conditions to achieve the leader-following consensus for multi-agent systems (21) and (22) under Assumption 1 using control signal (25) are

$$\begin{cases} k_1 > 0, \ \ k_2 > 0, \\ \dfrac{2^{\alpha}}{T^{\alpha}}\left(k_2 - \dfrac{2^{\alpha}}{T^{\alpha}\mu_{\max}}\right) < k_1 \leq \dfrac{\mu_{\min}}{4}k_2^2, \\ k_2 < \dfrac{2^{\alpha+1}}{T^{\alpha}\mu_{\min}}\left(1-\sqrt{1-\dfrac{\mu_{\min}}{\mu_{\max}}}\right), \end{cases} \tag{44}$$

where $\mu_{\min}$ and $\mu_{\max}$ are the minimum and maximum eigenvalues of $\boldsymbol{H}$.

The shaded region in the k_1–k_2 plane shown in Fig. 2 is the acceptable region for the controller parameter to attain the leader-following consensus for the discrete-time pseudo second-order agents.

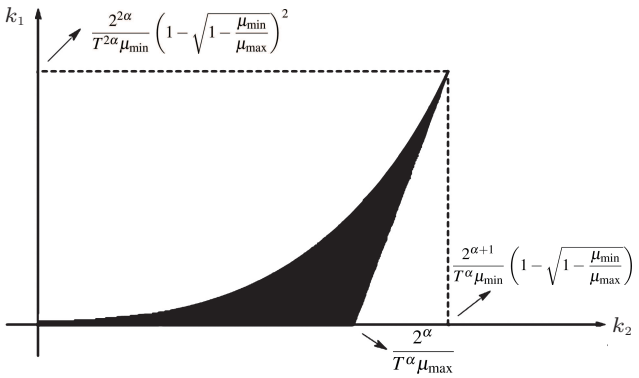


Fig. 2. The acceptable region in terms of controller parameters.

Remark 2 To design discrete-time controllers, two methods are utilized. The first one is the indirect method in which a continuous-time controller is designed for the continuous-time plant. Then, the obtained controller is discretized via discretization methods. In this method, the continuous-time controller should be designed for the series connection of the plant and the zero-order hold (ZOH) block. The ZOH block is described with a non-rational transfer function that may be approximated with a rational transfer function. Thus, the precision of the indirect method is low. The second method is the direct one in which a discrete-time controller is directly designed for the discretized plant. This method is more precise comparing with the indirect method. However, designing a discrete-time controller usually is more complicated than designing a continuous-time one. In this paper, the direct method is utilized. Thus, the agents are considered as discrete-time fractional order systems. The previously published papers in this area considered the continuous time fractional-order agents (or equally the indirect discretization method).[26–40]

5. Numerical examples

In this section, two illustrative examples are presented to show the efficiency of the proposed leader-following consensus approach. In the first example, the agents are considered as fractional order integrators, while in the second one, fractional order double integrators are considered as the agents.

Example 1 Consider a fractional-order multi-agent system consisting of a leader and five discrete-time pseudo first-order agents. The topology is described with an undirected graph shown in Fig. 3. The Laplacian matrix $\boldsymbol{L}$ and matrix $\boldsymbol{D}$ are obtained as

$$\boldsymbol{L} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{45}$$

According to Eq. (45), $\mu_{\min} = 0.26$ and $\mu_{\max} = 3.73$ are obtained. $\alpha = 0.85$ and $T = 0.01$ are considered. According to Eq. (17), $k_1 < 24.2$. The initial conditions of the agents are considered as $x_0(0) = 25, x_1(0) = 2, x_2(0) = 6, x_3(0) = 10, x_4(0) = 15, x_5(0) = 20$.

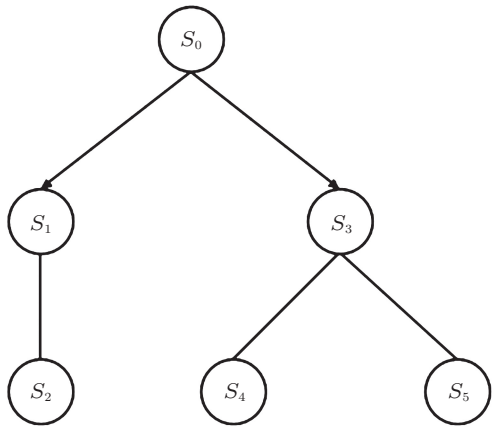


Fig. 3. The topology structure of the agents.

The state trajectories of the agents and the leader for different values of $k_1(8, 16, 22)$ are shown in Figs. 4–6. As can be seen from these figures, the agents follow the leader position. Increasing k_1 (or getting closer to the instability boundary) reduces the consensus time and increases the transient response oscillations. Thus, $k_1 = 16$ is a reasonable choice.

Example 2 Consider a multi-agent system with the same topology as shown in Fig. 3. The agents and the leader are considered as fractional order double integrators with $\alpha = 0.95$ and $T = 0.01$. The acceptable region of the controller parameters in the k_1–k_2 plane is shown in Fig. 7. As can be

seen from Fig. 7, $k_1 = 35$ and $k_2 = 25$ are considered as the controller parameters. The initial positions and velocities of the agents and the leader are chosen as $x_0(0) = 100, x_1(0) = 10, x_2(0) = 20, x_3(0) = 30, x_4(0) = 40, x_5(0) = 50$ and $v_0(0) = 20, v_1(0) = 15, v_2(0) = 25, v_3(0) = 35, v_4(0) = 45, v_5(0) = 55$, respectively. The position and velocity of the agents and the leader are shown in Fig. 8. It is obvious that leader-following consensus in both position and velocity is achieved.

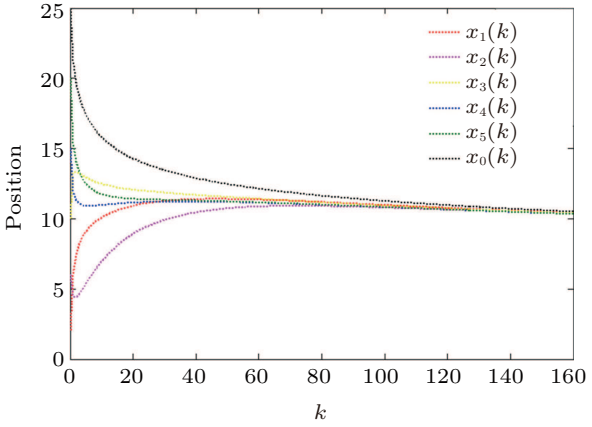


Fig. 4. (color online) The state trajectories of the agents and the leader for $k_1 = 8$ in Example 1.

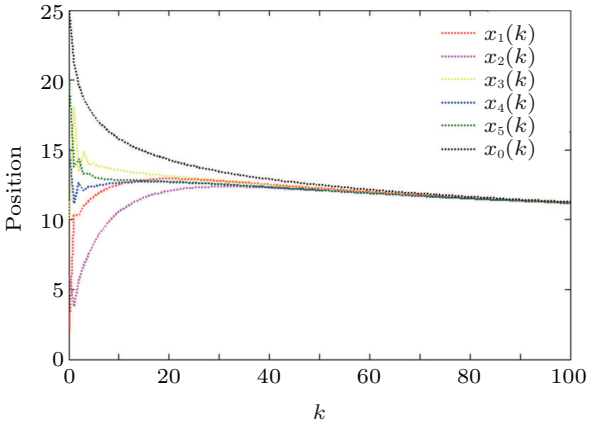


Fig. 5. (color online) The state trajectories of the agents and the leader for $k_1 = 16$ in Example 1.

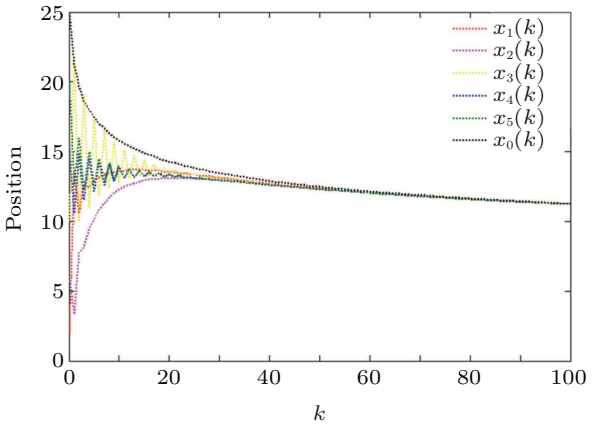


Fig. 6. (color online) The state trajectories of the agents and the leader for $k_1 = 22$ in Example 1.

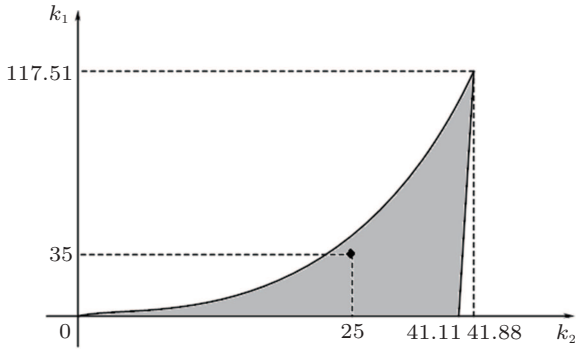


Fig. 7. The acceptable region of the controller parameters in k_1–k_2 plane for Example 2.

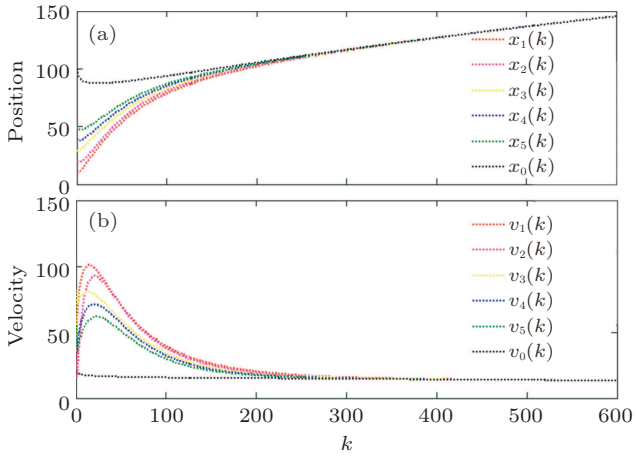


Fig. 8. (color online) The state trajectories of the agents and the leader in Example 2: (a) position, (b) velocity.

6. Conclusion

In this paper, the leader-following consensus of special case of discrete-time fractional order multi-agent systems with undirected fixed topology is verified. The leader and agents are considered as discrete-time pseudo first-order and second-order systems. The range of admissible feedback control gains to achieve leader-following consensus is obtained. There are some open problems in this area. For example, the topology of the agents could be considered as a directed graph. The effect of uncertainty in the fractional order and the communication delay between the agents should be studied too. Investigating the group consensus of discrete-time fractional-order multi-agents systems could be considered as another future research topic.

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