Retrieval of high-order susceptibilities of nonlinear metamaterials*

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Active metamaterials embedded with nonlinear elements are able to exhibit strong nonlinearity in microwave regime. However, existing *S*-parameter based parameter retrieval approaches developed for linear metamaterials do not apply in nonlinear cases. In this paper, a retrieval algorithm of high-order susceptibilities for nonlinear metamaterials is derived. Experimental demonstration shows that, by measuring the power level of each harmonic while sweeping the incident power, high-order susceptibilities of a thin-layer nonlinear metamaterial can be effectively retrieved. The proposed approach can be widely used in the research of active metamaterials.

Keywords: retrieval, high-order susceptibilities, nonlinear metamaterials

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1. Introduction

The nonlinearity of natural media in the microwave regime mainly comes from the relatively weak nonlinear electromagnetic (EM) polarization. Due to the small nonlinear susceptibilities of passive microwave media, the nonlinear phenomena, such as passive intermodulation,[1] can only be observed in high-power microwave devices. This limit has prevented natural media from practical nonlinear applications at microwave frequencies.

In 1999, Pendry *et al.* pointed out that metamaterials can be used to achieve strong nonlinearity.[2] Later, by introducing active nonlinear elements, such as microwave diodes, into passive, and subwavelength resonant structures, active nonlinear metamaterials were constructed.[3,4] In recent years, active metamaterials exhibiting strong nonlinearity even under micro-Watt microwave incidence have attracted substantial attention.[4–8] In the meantime, plenty of retrieval algorithms based on different mechanisms have been proposed to extract the effective constitutive parameters of the metamaterials.[18,19] Among these methods, the one based on inverse derivation from simulated or measured *S*-parameters is most often used.[9–12] Although these approaches suit for all passive isotropic metamaterials or biaxial metamaterials whose constitutive parameters do not contain off-diagonal elements, they do not apply for active metamaterials due to the strong nonlinearity introduced by the embedded active elements.

In this paper, we propose a retrieval algorithm for nonlinear metamaterials. By measuring the power level of each harmonic through a slab-like nonlinear metamaterial while sweeping the incident power, high-order susceptibilities of the nonlinear metamaterial can be retrieved. An experimental demonstration of this approach is given with a thin layer active metamaterial sample. The measured results are consistent with the theoretical analysis of the retrieval algorithm.

2. Retrieval algorithm for nonlinear metamaterials

According to the EM theory, for nonlinear media, the electric polarization $\bar{p}$ can be expressed by $\bar{p} = \bar{p}^{(1)} + \bar{p}^{\mathrm{NL}}$, where $\bar{p}^{(1)}$ and $\bar{p}^{\mathrm{NL}}$ are defined as the linear term and the nonlinear term of $\bar{p}$, respectively. Then the electric displacement $\bar{D}$ has the form

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{p} = \varepsilon_0 \bar{E} + \bar{p}^{(1)} + \bar{p}^{\mathrm{NL}} = \varepsilon_0 \bar{\bar{\varepsilon}}^{(1)} \cdot \bar{E} + \bar{p}^{\mathrm{NL}}, \quad (1)$$

where $\bar{\bar{\varepsilon}}^{(1)}$ denotes the linear tensor of the relative permittivity. The wave equation in nonlinear media becomes

$$\nabla^2 \bar{E} - \frac{\bar{\bar{\varepsilon}}^{(1)}}{c^2} \cdot \frac{\partial^2 \bar{E}}{\partial t^2} = \frac{1}{\varepsilon_0 c^2} \frac{\partial^2 \bar{p}^{\mathrm{NL}}}{\partial t^2}, \quad (2)$$

where c denotes the speed of light in free space.

In nonlinear media, the relationship between the electric polarization in a specific direction and the electric field of the incident wave can be depicted by a scalar equation simplified from Eq. (2), i.e.,

$$\nabla^2 E - \frac{\varepsilon^{(1)}}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{1}{\varepsilon_0 c^2} \frac{\partial^2 p^{\mathrm{NL}}}{\partial t^2}, \quad (3)$$

where the electric field and the electric polarization of the N-th harmonic are

$$E_N = E_{N,0} \mathrm{e}^{\mathrm{i}k_N z - \mathrm{i}\omega_N t} + E_{N,0} \mathrm{e}^{-\mathrm{i}k_N z + \mathrm{i}\omega_N t}, \quad (4)$$

$$p_N^{\mathrm{NL}} = p_{N,0} \mathrm{e}^{\mathrm{i}Nk_1 z - \mathrm{i}\omega_N t} + p_{N,0} \mathrm{e}^{-\mathrm{i}Nk_1 z + \mathrm{i}\omega_N t}, \quad (5)$$

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where $k_N = n_N\omega_N/c$, $n_N = [\varepsilon^{(1)}(\omega_N)]^{0.5}$, and $\omega_N = N\omega_1$ denote the wave vector, the effective refraction index, and the angular frequency corresponding to the N-th harmonic, respectively; $E_{N,0}$, $p_{N,0} = \varepsilon_0\chi^{(N)}E_{1,0}^N$, and $\chi^{(N)}$ denote the magnitude of the electric field, the magnitude of the electric polarization, and the nonlinear susceptibility corresponding to the N-th harmonic, respectively. The retrieval method proposed and verified in this paper is the algorithm used to retrieve the high-order $\chi^{(N)}$ from the simulated or measured data.

The schematic diagram of the proposed method is shown in Fig. 1. Same as the retrieval of a linear metamaterial, the nonlinear metamaterial sample used in the retrieval of a nonlinear metamaterial is also a thin slab which consists of single-layer sub-wavelength resonant structures. By sweeping the power of the normally incident plane wave, the characteristic constitutive parameters of the nonlinear metamaterial can be retrieved from the simulated or measured transmitted power of harmonics.

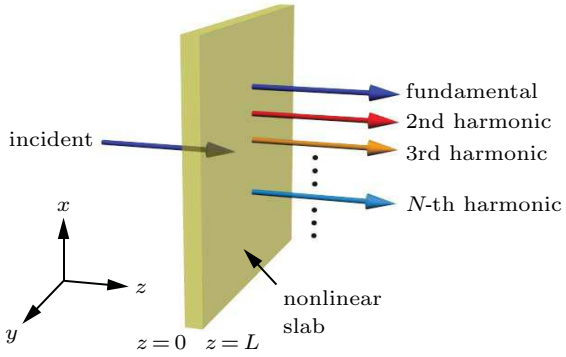


Fig. 1. (color online) Schematic diagram of the proposed retrieval method.

Firstly, we transfer Eq. (3) into a frequency-domain equation. For the N-th harmonic, we have

$$\nabla^2 E_N + \frac{\omega_N^2}{c^2}\varepsilon^{(1)}(\omega_N)E_N = -\frac{\omega_N^2}{\varepsilon_0 c^2}p_N^{\mathrm{NL}}. \tag{6}$$

Considering the wave incident along the z axis, ∇^2 can be reduced to $\mathrm{d}^2/\mathrm{d}z^2$, then

$$\left(\frac{\mathrm{d}^2E_{N,0}}{\mathrm{d}z^2} + 2\mathrm{i}k_N\frac{\mathrm{d}E_{N,0}}{\mathrm{d}z}\right)\mathrm{e}^{\mathrm{i}k_N z} = -\frac{\omega_N^2}{\varepsilon_0 c^2}p_{N,0}\mathrm{e}^{\mathrm{i}Nk_1 z}. \tag{7}$$

For plane wave incidence, the wave equation satisfies the slowly varying amplitude approximation[19]

$$\left|\frac{\mathrm{d}^2E_{N,0}}{\mathrm{d}z^2}\right| \ll \left|k_N\frac{\mathrm{d}E_{N,0}}{\mathrm{d}z}\right|. \tag{8}$$

Thus, equation (7) can be reduced to

$$2\mathrm{i}k_N\frac{\mathrm{d}E_{N,0}}{\mathrm{d}z}\mathrm{e}^{\mathrm{i}k_N z} = -\frac{\omega_N^2}{\varepsilon_0 c^2}p_{N,0}\mathrm{e}^{\mathrm{i}Nk_1 z}. \tag{9}$$

Then we have

$$\frac{\mathrm{d}E_{N,0}}{\mathrm{d}z} = \frac{\mathrm{i}\chi^{(N)}\omega_N^2}{2k_N c^2}E_{1,0}^N\mathrm{e}^{\mathrm{i}(Nk_1-k_N)z}. \tag{10}$$

Normally, in nonlinear media, the efficiency of power transfer from fundamental frequency to high-order harmonics is quite low. Therefore, $E_{1,0}$ can be treated as a constant during the propagation of the incident wave in the z direction. After propagating through the thin layer of metamaterial whose effective thickness is L, the electric field of the N-th harmonic, $E_{N,0}(L)$, is obtained through the integration of Eq. (10) from $z = 0$ to $z = L$, which is

$$E_{N,0}(L) = \frac{\mathrm{i}\omega_N^2E_{1,0}^N}{2k_N c^2}\chi^{(N)}L\frac{\mathrm{e}^{\mathrm{i}(Nk_1-k_N)L}-1}{\mathrm{i}(Nk_1-k_N)}. \tag{11}$$

The power intensity of the N-th harmonic, I_N, is calculated from the corresponding time-average Poynting vector, which is $I_N = 2n_N\varepsilon_0 c\,|E_{N,0}|^2$. Therefore we have

$$I_N = \frac{\omega_N^2 I_1^N}{2^{N+1}n_1^N n_N\varepsilon_0^{N-1}c^{N+1}}(\chi^{(N)}L)^2\left|\frac{\mathrm{e}^{\mathrm{i}(Nk_1-k_N)L}-1}{Nk_1-k_N}\right|^2, \tag{12}$$

where $I_1 = 2n_1\varepsilon_0 c\,|E_{1,0}|^2$ is the power density of the incident wave. Let $\Delta k = Nk_1 - k_N$, we have

$$\begin{aligned}\left|\frac{\mathrm{e}^{\mathrm{i}(Nk_1-k_N)L}-1}{Nk_1-k_N}\right|^2 &= \left|\frac{\mathrm{e}^{\mathrm{i}\Delta kL}-1}{\Delta k}\right|^2\\ &= \frac{(\mathrm{e}^{\mathrm{i}\Delta kL}-1)(\mathrm{e}^{-\mathrm{i}\Delta kL}-1)}{\Delta k^2}\\ &= L^2\frac{\sin^2(\Delta kL/2)}{(\Delta kL/2)^2}.\end{aligned} \tag{13}$$

In Eq. (13), when $\Delta k = Nk_1 - k_N = 0$, which means nonlinear phase matching is satisfied, I_N can reach its maximum. Then the relationship between the power intensity of harmonics and the fundamental wave can be expressed as

$$I_N = \frac{\omega_N^2 I_1^N}{2^{N+1}n_1^N n_N\varepsilon_0^{N-1}c^{N+1}}\left(\chi^{(N)}L\right)^2. \tag{14}$$

According to the effective medium theory, the effective thickness of the thin layer of the nonlinear metamaterial in Fig. 1, which is normally considered to be the periodicity of the metamaterial, satisfies

$$L \ll \min(\lambda_1,\lambda_N) = 2\pi/\max(k_1,k_N). \tag{15}$$

In the condition of weak phase mismatch, $|\Delta k| < \max(k_1,k_N)$. Then we have $L \ll 1/\Delta k$, which means $\Delta kL \to 0$. Thus equation (14) is approximately satisfied.

In Eq. (14), the power intensity of the N-th harmonic increases quadratically with the increase of $\chi^{(N)}L$. If reflection exists when the fundamental wave incidents on the thin layer of the nonlinear metamaterial, then the transmitted power intensity can be used as I_1 in Eq. (14).

Transferring Eq. (14) into the logarithmic mode, we have

$$\begin{aligned}\log_{10}I_N - N\log_{10}I_1 &= \log_{10}\omega_N^2 + \log_{10}\left(\chi^{(N)}L\right)^2\\ &\quad - \log_{10}\left(2^{N+1}n_1^N n_N\varepsilon_0^{N-1}c^{N+1}\right).\end{aligned} \tag{16}$$

Equations (14) and (16) are the basic formulas for the retrieval of the $N-$th order nonlinear susceptibility. When $\chi^{(N)}L$ is determined, the left side of Eq. (16) becomes constant, which means I_N and I_1 in their logarithmic scale ($\log_{10} I_N$, $\log_{10} I_1$) form a straight line in the plot, as shown in Fig. 2(a). The slope of the line is N. However, due to the ignorance of the depletion of the fundamental wave and the effect of the nonlinear mismatch, the simulated or measured data may deviate from the straight line whose slope is N. In practice, a linear fitting method is used in the data processing to obtain the real relationship between I_N and I_1, as shown in Fig. 2(b). The slope of the fitted line, N', is usually close to N. Replacing the constant N in the term $N\log_{10} I_1$ of Eq. (16) with N', the value of $\chi^{(N)}L$ can be retrieved, so as to realize the retrieval of the characteristic constitutive parameters of the nonlinear metamaterial.

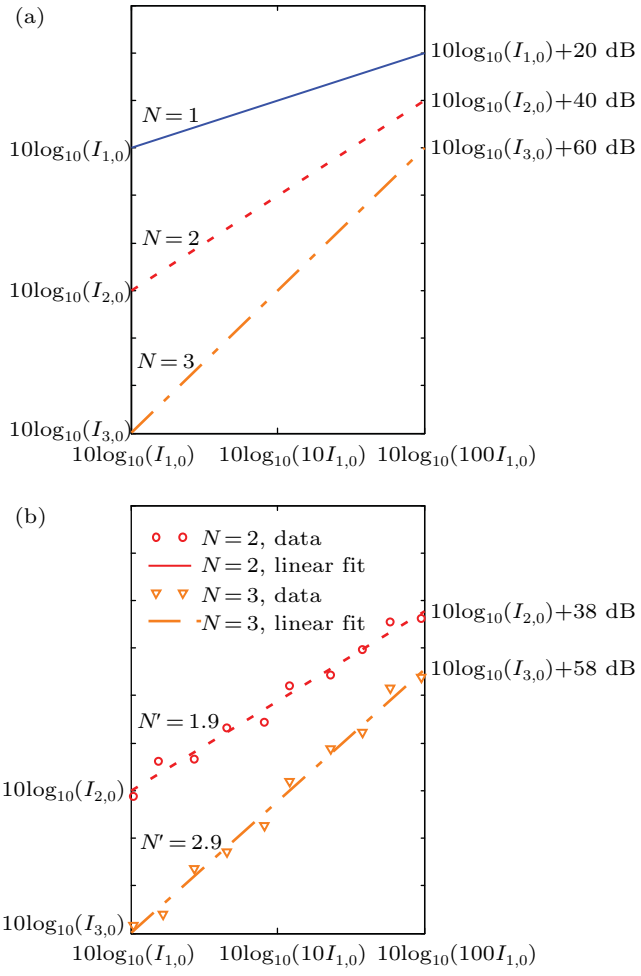


Fig. 2. (color online) Logarithmic relationships between the power intensity of the fundamental wave and harmonics in the retrieval of nonlinear characteristic constitutive parameters. (a) Schematic diagram of theoretical analysis. (b) Schematic diagram of linear fitting.

In practice, data measured by microwave instruments are usually power, so we finally change Eqs. (14) and (16) into a version of power for convenience,

$$\frac{P_N}{A_N} = \frac{\omega_N^2 (P_1/A_1)^N}{2^{N+1} n_1^N n_N \varepsilon_0^{N-1} c^{N+1}} \left(\chi^{(N)} L\right)^2, \tag{17}$$

$$\begin{aligned}\log_{10} P_N - N\log_{10} P_1 = {} & \log_{10}\omega_N^2 + \log_{10}\left(\chi^{(N)}L\right)^2 \\ & - \log_{10}\left(2^{N+1} n_1^N n_N \varepsilon_0^{N-1} c^{N+1}\right) \\ & + \log_{10} A_N - N\log_{10} A_1,\end{aligned} \tag{18}$$

where P_N denotes the power of the N-th harmonic, and A_N is the effective aperture of the receiving antenna at the frequency of the N-th harmonic. Based on Eq. (17) or (18), $\chi^{(N)}$ can be retrieved from the measured P_N and P_1.

3. Experimental demonstration and results

By using the above retrieval approach, the nonlinear susceptibilities of a thin layer of nonlinear metamaterial are retrieved from the measured power of the second and the third harmonics for validation. The metamaterial sample used for the validation is the one reported in Ref. [7], whose unit cell consists of a double-layer I-shaped metallic resonator and two embedded biased Schottky microwave diodes (Infineon's BAT15-03W), as shown in Fig. 3(a). Periodicity of such unit cells printed on a 1-mm-thick FR4 substrate is 6 mm. The forward bias direct current (DC) voltage of each diode is 0.225 V.

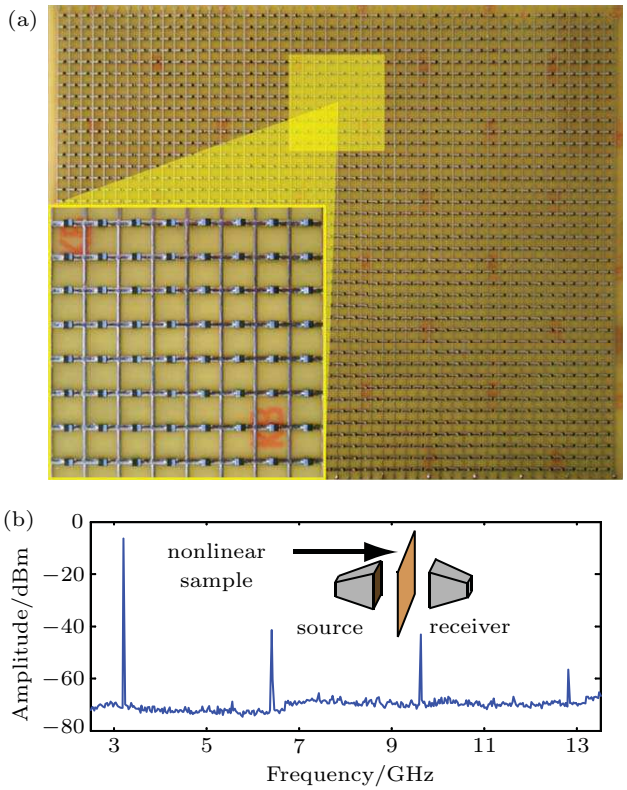


Fig. 3. (color onlilne) (a) Photo of a thin layer of metamaterial. (b) Spectrum of the transmitted wave when a monochromatic wave at 3.2 GHz normally incidents on the thin layer of metamaterial. The inset shows the experimental setup.

The experimental setup is shown in the inset of Fig. 3(b). Two double-ridged horn antennas are used as the transmitter and the receiver. The receiver is connected with a spectrum analyzer (Agilent 4407B). When a monochromatic wave at

3.2 GHz normally incidents on the thin layer of metamaterial, the measured power of the fundamental, the second, the third, and the fourth harmonics are shown in Fig. 3(b). The power of the incident wave used in Fig. 3(b) is 5.7 dBm. The periodicity of the metamaterial (6 mm) is much smaller than the wavelength of the incident wave (100 mm). As a result, the metamaterial can be treated as an effective nonlinear medium in the frequency range up to the third harmonic.

Sweeping the incident power from −14 dBm to 7 dBm, the measured power of the transmitted fundamental, second, and third harmonics are shown in Fig. 4(a). Due to the existing reflection of the incident wave, based on the above analysis, we use the power of the transmitted fundamental wave instead of the power of the incident wave in Fig. 4(a). The dashed black line indicates the noise level of the spectrum analyzer. The solid red line and dashed-dotted orange line indicate the fitted result of the transmitted second and third harmonics from the measured data, respectively. The linear relationship between the power of the fundamental wave and the power of harmonics above the noise level is shown in Fig. 4(b).

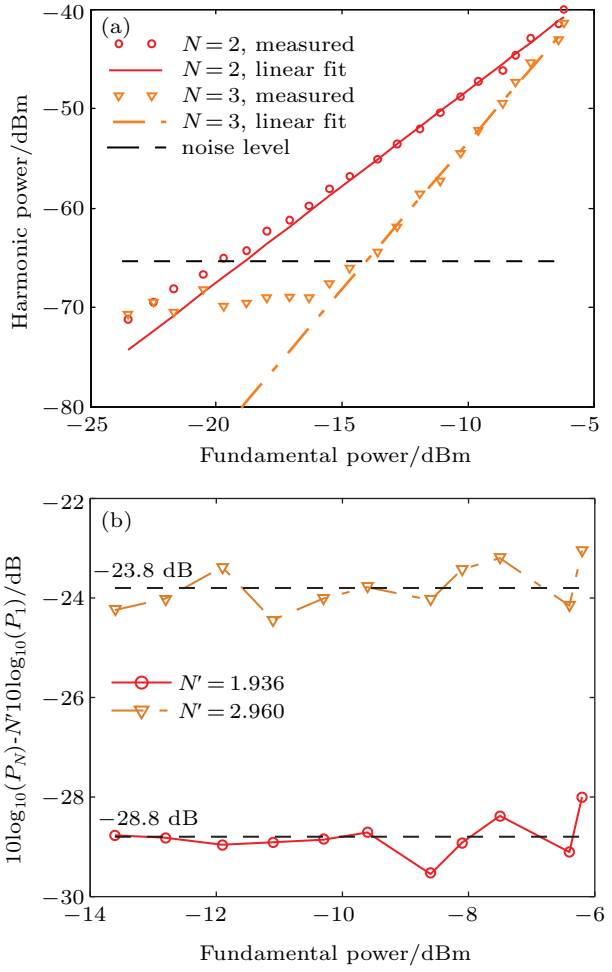


Fig. 4. (color onlilne) (a) Measured data and the linear fitting results. (b) Linear relationships of the measured data.

In Fig. 4(a), the slopes of the fitted lines, N', show the linear relationships between $\log_{10} P_N$ and $\log_{10} P_1$. The values of N' corresponding to the second and the third harmonics are 1.936 and 2.960, respectively, very close to the theoretical values of N (2 and 3). The fitted results indicate that the measured data match well with the above theoretical analysis derived based on a thin slab of effective nonlinear medium. Replacing the constant N on the left side of Eq. (18) with N', the values of $10\log_{10} P_N - N'10\log_{10} P_1$ are shown in Fig. 4(b). The dashed black lines in Fig. 4(b) show two constant values, −28.8 dB and −23.8 dB, obtained from the two linear fitting curves in Fig. 4(a). All values calculated from the measured data fluctuate around these two values.

Then, the effective apertures of the receiving antenna on the right side of Eq. (18) are calculated using

$$A_N = \frac{\lambda_N^2}{4\pi} D_N = \pi (c/\omega_N)^2 D_N, \tag{19}$$

where D_N denotes the directivity of the receiving antenna at the frequency of the N-th harmonic. The directivities of the two-ridge horn antenna we used at the frequencies of the fundamental, second, and third harmonics are 10.65 dB, 12.8 dB, and 12.5 dB, respectively. The calculated effective apertures are 7.45×10^{-3} m^2, 2.24×10^{-3} m^2, and 9.7×10^{-4} m^2, respectively.

Substituting $10\log_{10} P_N - N'10\log_{10} P_1$ and A_N into Eq. (18), the normalized effective nonlinear susceptibilities of the thin layer of metamaterial, $\chi^{(N)}L$, can be estimated. Assuming the effective refractive index n_N equals 4.6, which is the refractive index of the substrate of the thin layer of metamaterial, we have $\chi^{(2)}L = 6.17 \times 10^{-4}$ m^2/V and $\chi^{(3)}L = 3.24 \times 10^{-4}$ (m^3/V^2). The orders of magnitude of $\chi^{(N)}$ can be estimated using the physical thickness of the metamaterial sample (1 mm) instead of the effective thickness L. Thus we have $\chi^{(2)} = 0.617$ m/V and $\chi^{(3)} = 0.324$ m^2/V^2. We should note that, when the physical thickness of a single-layer nonlinear metamaterial is very small compared with the wavelength of the considered harmonic, the effective thickness, L, is not the same as its physical thickness. In this case, the thin layer of metamaterial can be treated as a thin film, and $\chi^{(N)}L$ can be used as the effective nonlinear susceptibility.[20] The obvious nonlinear effect measured above is obtained under the illumination of a micro-Watt microwave incidence. Compared with the passive optical media exhibiting strong nonlinear effect under the illumination of high-power intensity lasers, such as GaAs whose $\chi^{(2)} = 7.4 \times 10^{-10}$ m/V and $\chi^{(3)} = 1.4 \times 10^{-18}$ m^2/V^2, the orders of the retrieved effective nonlinear susceptibilities are reasonable.[20]

4. Conclusion

A retrieval algorithm aiming for retrieving the nonlinear parameters of active metamaterials is derived. We show by theoretical analysis and experimental measurement that the high-order nonlinear susceptibilities of active metamaterials

can be effectively obtained using this approach. We envision this approach could contribute to quantitative exploration of exotic nonlinear phenomena of active nonlinear metamaterials.

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