Recursion-transform method and potential formulae of the $m \times n$ cobweb and fan networks*

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In this paper, we made a new breakthrough, which proposes a new recursion–transform (*RT*) method with potential parameters to evaluate the nodal potential in arbitrary resistor networks. For the first time, we found the exact potential formulae of arbitrary $m \times n$ cobweb and fan networks by the *RT* method, and the potential formulae of infinite and semi-infinite networks are derived. As applications, a series of interesting corollaries of potential formulae are given by using the general formula, the equivalent resistance formula is deduced by using the potential formula, and we find a new trigonometric identity by comparing two equivalence results with different forms.

Keywords: recursion-transform method, network model, potential formula, exact solution

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1. Introduction

Searching for potential formula and equivalent resistance formula of resistor networks is a classic research topic in electric circuit theory and physics. An important development in circuit theory is that Kirchhoff, a German scientist, established the node current law and the circuit voltage law in 1845.[1] From then on, resistor network models have been researched more than 170 years. Nowadays, the resistor networks are attracting more and more attention because they can be used in both electrical and non-electrical system models, and the simulation through modeling resistor networks has become a basic method to solve complicated problems in the fields of engineering and science.[1–43] For example, the computation of resistance is relevant to problems ranging from classical transport in disordered media,[2] electromigration phenomena in metallic lines,[3] first-passage processes,[4] lattice Greens functions,[5] random walks[6] to hybrid graph theory.[7] Modeling resistor networks are also useful for researching exact finite-size corrections in critical systems.[8,9] In addition, in physics and probability theory, the mean field theory can be used to study the behavior of large and complex stochastic models by studying simpler models such as finite disordered networks.[10–12] The mean field theory has also been widely used in statistical mechanics, condensed matter in complex systems, magnetism, structure transition, and other fields. In order to obtain an analytical solution, the resistance of a regular resistor network with a perfectly ordered structure is calculated. If the heterogeneity is weak, the mean-field approximation is valid.[10–12]

The potential theory is an important theory in natural science and engineering, for example, Laplace equation is a kind of second-order partial differential equation, Poisson equation is a kind of inhomogeneous Laplace equation,[13–19] and Laplace equation and Poisson equation are the simplest examples of elliptic partial differential equations. The general theory of solutions to Laplace equation is known as the potential theory. The solutions of Laplace equation, which are all harmonic functions, are important in many fields of science and engineering, notably in electromagnetism, astronomy, and fluid dynamics, as they can accurately describe the behaviors of electric, gravitational, and fluid potentials. In heat conduction study, the Laplace equation is the steady-state heat equation. The potential function has been applied not only in the electrical theory, but also in non-electric fields, such as potential fields obeying Laplace equation, although the physical properties of different potential fields are different but if they have similar boundary conditions. Due to the uniqueness of Laplace equation, a potential field of solution or the potential field model experiment of surveying and mapping of hot surface or flow chart can be shared with other potential fields after corresponding physical quantity of conversion. Thus, a large number of studies in natural science and engineering can be solved by the potential function equation. However, problems in real life are usually complicated because of different boundary conditions. A tiny change of boundary conditions affects the solutions. Therefore, accurately calculating the solution of Laplace equation under complex boundary conditions is difficult and approximate methods are usually used to obtain approximation solutions which meet actual needs, in which a common method called the finite difference method

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was established to study the approximate solutions of differential equations and integral differential equations.[13–19] The basic idea of the finite difference method is to use a finite number of discrete points of grid instead of a continuous definite region, these discrete points are called grid nodes, and that is to say, the finite difference method is implemented by means of grid models. Because network models have specific visual features, if we use the resistor network model to solve the finite difference problem, the electric potential of each node will represent the solution of function equation to be solved. If the potential analytical solution of each node in the resistor network model is obtained, we can precisely obtain the analytical solution of differential equations. However, to find exact solutions of nodal potential in resistor networks is important but difficult, the previous theory has not been work out the precise formula of electric potential of each node in resistor network models.

From the research history, we find that it is difficult to obtain the exact potential and resistance in an arbitrary $m \times n$ resistor network with an arbitrary boundary, because the boundary is like a wall or trap which affects the behavior of finite networks.[20–36] Therefore, several different effective methods used to calculate the equivalent resistances of resistor networks with different geometric constructions have been proposed. In 2000, Cserti[20] evaluated the two-point resistance of infinite lattices by using Green function, which cannot be applied to finite resistor networks. In 2004, Wu[22] proposed a different approach (called Laplacian matrix method) and derived explicit expressions of two-point resistance in arbitrary finite and infinite lattices in terms of the eigenvalues and eigenvectors of the Laplacian matrix relying on two matrices along two vertical directions. After that, the Laplacian approach has been extended to complex impedance networks[23] and resistor networks with a zero resistor boundary.[24,25] However, as the Laplacian method cannot be applied to a matrix with arbitrary elements, which is incapable to solve the eigenvalue problem, this approach cannot solve resistor network problems with arbitrary boundaries.

In recent years, a new recursion transform (*RT*) method was proposed by Tan,[26–28] the *RT* method, depending on one matrix along one direction, is different from the Laplacian method. With the development of *RT* technique, some problems of resistor networks have been resolved.[26–36] Although fruitful research results on the equivalent resistance of resistor networks have been obtained, the nodal potential of the resistor network has not been obtained. Therefore, in this paper, we make a new breakthrough for studying the nodal potential of resistor networks. The previous *RT* method[26–28] is based on current parameters, this paper further generalizes the *RT* method and promotes the *RT* method to suit to potential parameters. We find that the *RT* method can solve the analytical solutions of the potential of each node in resistor network models, thus our method can solve the Laplace equation with complex boundary conditions.

In addition, the resistor network model, which has been applied to a number of disciplines, is an interdisciplinary subject.[37–43] Therefore, the construction of and research on resistor network models can be useful in theory and application.

In this paper, we investigate the potential formula of arbitrary $m \times n$ cobweb and fan networks, which have not been resolved before. We focus on providing a new theory and technique useful for related scientific research.

This paper is organized as follows. In Section 2, we present potential formulae of the cobweb and fan resistor networks. In Section 3, we illustrate *RT* method by using the cobweb network model. In Section 4, some new applications of potential formulae in infinite and semi-infinite cases are given. Section 5 gives the discussion and summary.

2. The main result: potential formula

2.1. Potential formula of the cobweb network

We consider an arbitrary $m \times n$ resistor cobweb network with resistors r_0 and r in the respective longitude (radius) and latitude (circle) directions, where m and n are, respectively, the numbers of resistors along longitude and latitude directions as shown in Fig. 1. Assume that the center node O is the origin of the rectangular coordinate system and a longitude acts as Y axis. Denote the nodes of the network by coordinate $\{x, y\}$ and the potential of node $d(x, y)$ is $U_{m\times n}(x, y) = V_x^{(y)}$ as shown in Fig. 2.

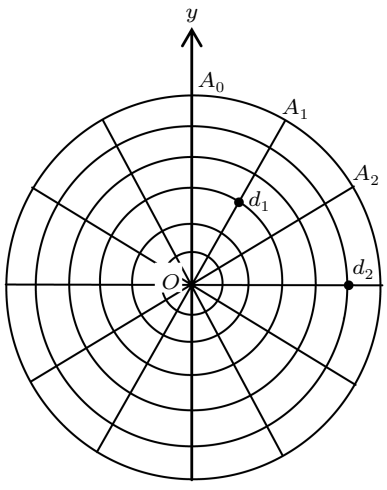


Fig. 1. An 6×12 resistor cobweb network, where 6 and 12 are, respectively, the numbers of resistors along longitude and latitude directions.

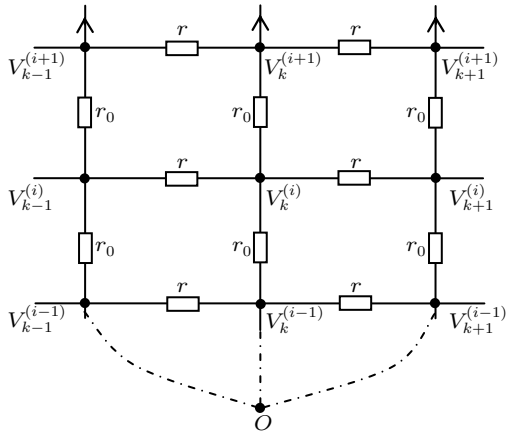


Fig. 2. Segment of resistor network with potential parameters.

By assuming that the current J goes from $d_1(x_1,y_1)$ to $d_2(x_2,y_2)$, selecting $d_0(0,0)$ as the reference point of the potential, and $U_{m\times n}(0,0)=0$, the potential of an arbitrary node in the $m\times n$ cobweb network can be written as

$$\frac{U_{m\times n}(x,y)}{J}=\frac{4r}{2m+1}\sum_{i=1}^{m}\left(\frac{g_{x_1,x}^{(i)}S_{y_1,i}-g_{x_2,x}^{(i)}S_{y_2,i}}{\lambda_i^n+\bar{\lambda}_i^n-2}\right)S_{y,i}, \quad (1)$$

where $\theta_i=(2i-1)\pi/(2m+1)$, $S_{y_k,i}=\sin(y_k\theta_i)$, and

$$g_{x_s,x}^{(i)}=F_{n-|x_s-x|}+F_{|x_s-x|}, \quad F_k^{(i)}=(\lambda_i^k-\bar{\lambda}_i^k)/(\lambda_i-\bar{\lambda}_i). \quad (2)$$

Define $h=r/r_0$ and

$$\lambda_i=1+h-h\cos\theta_i+\sqrt{(1+h-h\cos\theta_i)^2-1},$$
$$\bar{\lambda}_i=1+h-h\cos\theta_i-\sqrt{(1+h-h\cos\theta_i)^2-1}. \quad (3)$$

In addition, equation (1) can be expressed in another form as

$$\frac{U_{m\times n}(x,y)}{J}=\frac{4r}{2m+1}\sum_{i=1}^{m}\left(\frac{f_{x_1,x}^{(i)}S_{y_1,i}-f_{x_2,x}^{(i)}S_{y_2,i}}{\sinh(2L_i)\sinh(nL_i)}\right)S_{y,i}, \quad (4)$$

where $f_{x_s,x}^{(i)}=\cosh[(n-2|x_s-x|)L_i]$, and

$$e^{2L_i}\equiv\lambda_i, \quad \cosh(2L_i)\equiv 1+h-h\cos\theta_i. \quad (5)$$

2.2. Potential formula of the fan network

Consider an arbitrary $m\times n$ resistor fan network with resistors r_0 and r in the respective longitude and latitude directions, where m and n are, respectively, the numbers of resistors in longitude and latitude directions as shown in Fig. 3. All definitions of parameters are the same as these in the section above. The potential of an arbitrary node in the $m\times n$ fan network can be written as

$$\frac{U_{(0,0)}^{(x,y)}}{J}=\frac{2r_0}{2m+1}\sum_{i=1}^{m}\left(\frac{\beta_{x_1\vee x}^{(i)}S_{y_1,i}-\beta_{x_2\vee x}^{(i)}S_{y_2,i}}{(1-\cos\theta_i)F_{n+1}^{(i)}}\right)S_{y,i}, \quad (6)$$

where $\theta_i=(2i-1)\pi/(2m+1)$, $S_{y_k,i}=\sin(y_k\theta_i)$, $\Delta F_k^{(i)}=F_{k+1}^{(i)}-F_k^{(i)}$, and define

$$\beta_{x_s\vee x}^{(i)}=\begin{cases}\Delta F_{x_s}^{(i)}\Delta F_{n-x}^{(i)}, & \text{if } x\geq x_s,\\ \Delta F_x^{(i)}\Delta F_{n-x_s}^{(i)}, & \text{if } x\leq x_s.\end{cases} \quad (7)$$

In particular, when the output current J is at the center of $O(0,0)$ (the origin of coordinates) and the input current J is at the left edge of $d_1(0,y_1)$, the potential formula of the node $d(x,y)$ in the $m\times n$ fan network can be written as

$$\frac{U_{(0,0)}^{(x,y)}}{J}=\frac{4r}{2m+1}\sum_{i=1}^{m}\left(\frac{\cosh[(2n+1-2x)L_i]}{\sinh[2(n+1)L_i]\sinh(L_i)}\right)S_{y_1,i}S_{y,i}, \quad (8)$$

where $\theta_i=(2i-1)\pi/(2m+1)$ and L_i has been defined in Eq. (5).

Equations (1), (4), (6), and (8) are potential formulae of point $d(x,y)$ in arbitrary $m\times n$ cobweb and fan resistor networks, which are found for the first time. In the following sections, we derive them by using the RT method with potential parameters.

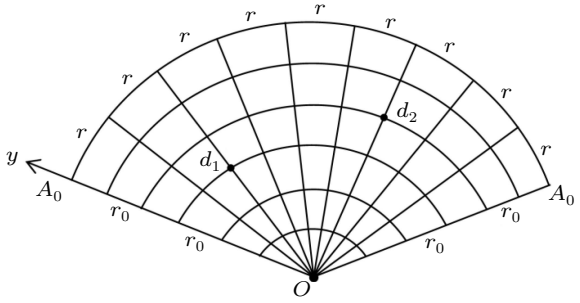


Fig. 3. A fan $m\times n$ resistor network, where m and n are, respectively, the numbers of resistors along longitude and latitude directions.

3. A general theory and approach

In this section, we first derive Eqs. (1) and (4), and then derive Eqs. (6) and (8) by the generalized RT method. The RT method divides the derivation into four steps: first, construct a matrix relation between the potentials in three successive transverse lines; second, derive a recursion relation between the potentials on the boundary; third, diagonalize the matrix relation to produce a recurrence relation involving only variables in the same transverse line; fourth, implement the inverse transformation of matrix to yield the node potential.

3.1. Modeling the matrix equation

Consider that the center node O is the origin of the rectangular coordinate system and longitude OA_0 acts as Y axis as shown in Fig. 1. Denote the nodes of the network by coordinate $\{x,y\}$, denote the potential in all segments of the network as shown in Fig. 3, and assume that the potentials at node $d(x,y)$ is $U_{m\times n}(x,y)=V_x^{(y)}$ and at the center is $V_0^{(0)}=0$.

Firstly, we research the segment of resistor networks with potential parameters as shown in Fig. 3. We focus on four rectangular meshes and nine nodes, using Kirchhoff's current law ($\sum r_i^{-1}V_k = 0$) to the node along the longitude axis of kth, we can obtain the node potential formula as follows by ignoring the external current source:

$$\begin{aligned}
&V_{k+1}^{(1)} = (2+2h)V_k^{(1)} - V_{k-1}^{(1)} - hV_k^{(2)}, \quad i=1,\\
&V_{k+1}^{(i)} = (2+2h)V_k^{(i)} - V_{k-1}^{(i)} - hV_k^{(i-1)} - hV_k^{(i+1)}, \quad 1<i<m,\\
&V_{k+1}^{(m)} = (2+h)V_k^{(m)} - V_{k-1}^{(m)} - hV_k^{(m-1)}, \quad i=m,
\end{aligned} \tag{9}$$

where $h = r/r_0$. Equation (9) can be written in an matrix form by considering the injection of J at $d_1(x_1,y_1)$ and the exit of J at $d_2(x_2,y_2)$,

$$\boldsymbol{V}_{k+1} = \boldsymbol{A}_m\boldsymbol{V}_k - \boldsymbol{V}_{k-1} - r\boldsymbol{I}_k, \tag{10}$$

where $\boldsymbol{V}_k$ and $\boldsymbol{I}_k$ are, respectively, $m\times 1$ column matrices, and reads

$$\boldsymbol{V}_k = [V_k^{(1)}, V_k^{(2)}, \ldots, V_k^{(m)}]^{\mathrm{T}}, \tag{11}$$

$$I_k^{(i)} = J(\delta_{x_1,k} - \delta_{x_2,k}), \tag{12}$$

where $[\ ]^{\mathrm{T}}$ denotes matrix transpose and

$$\boldsymbol{A}_m = \begin{pmatrix}
(2+2h) & -h & 0 & 0 & 0\\
-h & (2+2h) & -h & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & -h & (2+2h) & -h\\
0 & 0 & 0 & -h & (2+h)
\end{pmatrix}. \tag{13}$$

Second, we consider the boundary conditions in Y axis of the network. Applying Kirchhoff's current law ($\sum r_i^{-1}V_k = 0$) to the nodes adjacent to Y axis, we obtain

$$\boldsymbol{V}_{n-1} + \boldsymbol{V}_1 = \boldsymbol{A}_m\boldsymbol{V}_0, \tag{14}$$

$$\boldsymbol{V}_0 = \boldsymbol{V}_n = \boldsymbol{A}_m\boldsymbol{V}_{n-1} - \boldsymbol{V}_{n-2}. \tag{15}$$

where matrix $\boldsymbol{A}_m$ has been given in Eq. (13) .

Equations (10)–(15) are all equations needed to calculate the potential formula. However, it is impossible for us to obtain the solution of the potential matrix directly by Eqs. (10)–(15). How to solve matrix equation (10) is the key to the problem. Therefore, we establish the following method of matrix transform to resolve this problem indirectly, the matrix transform method is our key technique.

3.2. Approach of matrix transform

Because equation (10) is a complex matrix equation whose solution cannot been obtianed directly, we rebuild a new difference equation and solve Eq. (10) indirectly by means of matrix transform in this subsection.

Firstly, we multiply Eq. (10) at the left side by an $m\times m$ undetermined square matrix $\boldsymbol{P}_m$,

$$\boldsymbol{P}_m\boldsymbol{V}_{k+1} = \boldsymbol{P}_m\boldsymbol{A}_m\boldsymbol{V}_k - \boldsymbol{P}_m\boldsymbol{V}_{k-1} - r\boldsymbol{P}_m\boldsymbol{I}_k. \tag{16}$$

Construct the transformation of diagonal matrix by the following two identities:

$$\det|\boldsymbol{A}_m - t\boldsymbol{E}_m| = 0, \quad \boldsymbol{P}_m\boldsymbol{A}_m = \boldsymbol{T}_m\boldsymbol{P}_m, \tag{17}$$

where $\boldsymbol{E}_m$ is an $m\times m$ identity matrix and $\boldsymbol{T}_m = \mathrm{diag}(t_1,t_2,\ldots,t_m)$. Since $\boldsymbol{A}_m$ is a special tridiagonal matrix which is Hermitian, it can be diagonalized by a similarity transformation. Solving Eq. (17), we obtain

$$t_i = 2(1+h) - 2h\cos\theta_i, \tag{18}$$

where $\theta_i = (2i-1)\pi/(2m+1)$ and the matrix of eigenvectors is (one may refer to Ref. [32])

$$\boldsymbol{P}_m = \begin{pmatrix}
\sin\theta_1 & \sin 2\theta_1 & \cdots & \sin m\theta_1\\
\sin\theta_2 & \sin 2\theta_2 & \cdots & \sin m\theta_2\\
\vdots & \vdots & \ddots & \vdots\\
\sin\theta_m & \sin 2\theta_m & \cdots & \sin m\theta_m
\end{pmatrix}, \tag{19}$$

with the inverse matrix (one may refer to Ref. [32])

$$\boldsymbol{P}_m^{-1} = \frac{4}{2m+1}\begin{pmatrix}
\sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_m\\
\sin 2\theta_1 & \sin 2\theta_2 & \cdots & \sin 2\theta_m\\
\vdots & \vdots & \ddots & \vdots\\
\sin m\theta_1 & \sin m\theta_m & \cdots & \sin m\theta_m
\end{pmatrix}. \tag{20}$$

For convenience, by Eqs. (16) and (17), letting

$$\boldsymbol{P}_m\boldsymbol{V}_k = \boldsymbol{X}_k \ \text{ or } \ \boldsymbol{V}_k = (\boldsymbol{P}_m)^{-1}\boldsymbol{X}_k, \tag{21}$$

where $\boldsymbol{X}_m$ is an $m\times 1$ column matrix and reading

$$\boldsymbol{X}_k = \left[X_k^{(1)}, X_k^{(2)}, \ldots, X_k^{(m)}\right]^{\mathrm{T}}, \tag{22}$$

thus, we obtain the following main equation after applying Eqs. (17) and (21) to Eq. (16):

$$X_{k+1}^{(i)} = t_iX_k^{(i)} - X_{k-1}^{(i)} - rJ\zeta_{y,i}, \tag{23}$$

where (simply read $\zeta_{y_1,i}$ as $\zeta_{1,i}$, and $\zeta_{y_2,i}$ as $\zeta_{2,i}$)

$$\zeta_{1,i} = \sin y_1\theta_i, \quad \zeta_{2,i} = -\sin y_2\theta_i. \tag{24}$$

We conduct the same matrix transform as used in Eq. (23). Applying $\boldsymbol{P}_m$ to Eqs. (14) and (15) on the left-hand side, we obtain

$$X_{n-1}^{(i)} + X_1^{(i)} = t_iX_0^{(i)}, \tag{25}$$

$$X_0^{(i)} = X_n^{(i)} = t_iX_{n-1}^{(i)} - X_{n-2}^{(i)}. \tag{26}$$

Equations (23)–(26) are all essential equations for solving the potential by solving the matrix of $X_k^{(i)}$. By using the above matrix transform, we can obtain variable $X_k^{(i)}$ with potential parameters.

3.3. General solutions of the matrix equations

Next, we derive the exact solution of X_k by Eqs. (23)–(26). Supposing that λ_i and $\bar{\lambda}_i$ are the roots of the characteristic equation for X_k in Eq. (23), we obtain the characteristic roots (3) according to Eq. (23).

From Eq. (23), we obtain the piecewise solutions after considering the injection of current J at $d_1(x_1, y_1)$ and the exit of current J at $d_2(x_2, y_2)$,

$$X_k^{(i)} = X_1^{(i)} F_k^{(i)} - X_0^{(i)} F_{k-1}^{(i)}, \quad 0 \le k \le x_1, \tag{27}$$

$$X_{x_1+1}^{(i)} = t_i X_{x_1}^{(i)} - X_{x_1-1}^{(i)} - rJ\zeta_{1,i}, \tag{28}$$

$$X_k^{(i)} = X_{x_1+1}^{(i)} F_{k-x_1}^{(i)} - X_{x_1}^{(i)} F_{k-x_1-1}^{(i)}, \quad x_1 \le k \le x_2, \tag{29}$$

$$X_{x_2+1}^{(i)} = t_i X_{x_2}^{(i)} - X_{x_2-1}^{(i)} - rJ\zeta_{2,i}, \tag{30}$$

$$X_k^{(i)} = X_{x_2+1}^{(i)} F_{k-x_2}^{(i)} - X_{x_2}^{(i)} F_{k-x_2-1}^{(i)}, \quad x_2 \le \mathrm{k} \le n, \tag{31}$$

where $F_k^{(i)} = (\lambda_i^k - \bar{\lambda}_i^k)/(\lambda_i - \bar{\lambda}_i)$ has been defined in Eq. (2).

Solving matrix equations (25)–(31), we obtain the general solution of $X_k^{(i)}$ with potential parameters $(0 \le k \le n)$,

$$X_x^{(i)} = rJ \frac{(F_{n-|x_1-x|}^{(i)} + F_{|x_1-x|}^{(i)})\zeta_{1,i} + (F_{n-|x_2-x|}^{(i)} + F_{|x_2-x|}^{(i)})\zeta_{2,i}}{\lambda_i^n + \bar{\lambda}_i^n - 2}, \tag{32}$$

where $\zeta_{1,i}$ and $\zeta_{2,i}$ have been given in Eq. (24) and $F_k^{(i)}$ has been defined in Eq. (2).

3.4. Matrix inverse transform

In this section, we derive the potential formula by matrix inverse transformation. Using Eq. (21), $\boldsymbol{V}_k = (\boldsymbol{P}_m)^{-1}\boldsymbol{X}_k$, and expanding the inverse matrix, we have

$$\begin{pmatrix} V_k^{(1)} \\ V_k^{(2)} \\ \vdots \\ V_k^{(m)} \end{pmatrix} = \frac{4}{2m+1} \begin{pmatrix} \sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_m \\ \sin 2\theta_1 & \sin 2\theta_2 & \cdots & \sin 2\theta_m \\ \vdots & \vdots & \ddots & \vdots \\ \sin m\theta_1 & \sin m\theta_m & \cdots & \sin m\theta_m \end{pmatrix} \begin{pmatrix} X_k^{(1)} \\ X_k^{(2)} \\ \vdots \\ X_k^{(m)} \end{pmatrix}. \tag{33}$$

According to Eq. (33), we have

$$V_k^{(y)} = \frac{4}{2m+1} \sum_{i=1}^{m} X_k^{(i)} \sin(y\theta_i), \tag{34}$$

where $X_k^{(i)}$ has been given in Eq. (32). Substituting Eq. (32) into Eq. (34), we obtain

$$V_k^{(y)} = J \frac{4r}{2m+1} \sum_{i=1}^{m} \frac{g_{x_1,x}^{(i)}\zeta_{1,i} + g_{x_2,x}^{(i)}\zeta_{2,i}}{\lambda_i^n + \bar{\lambda}_i^n - 2} \sin(y\theta_i). \tag{35}$$

Substituting Eq. (24) into Eq. (35), we therefore obtain Eq. (1).

In addition, when $\lambda_i = \mathrm{e}^{2L_i}$ (one may refer to Ref. [30]), substituting to $F_k^{(i)} = (\lambda_i^n - \bar{\lambda}_i^n)/(\lambda_i - \bar{\lambda}_i)$ yields

$$F_{\Delta x}^{(i)} = \frac{\mathrm{e}^{2\Delta x L_i} - \mathrm{e}^{-2\Delta x L_i}}{\mathrm{e}^{2L_i} - \mathrm{e}^{-2L_i}} = \frac{2\sinh(\Delta x L_i)\cosh(\Delta x L_i)}{\sinh(2L_i)}, \tag{36}$$

$$F_{n-\Delta x}^{(i)} + F_{\Delta x}^{(i)} = \frac{2\sinh(nL_i)\cosh[(n-2\Delta x)L_i]}{\sinh(2L_i)}. \tag{37}$$

Since $\lambda_i = \mathrm{e}^{2L_i}$ and $\bar{\lambda}_i = \mathrm{e}^{-2L_i}$, we have

$$\lambda_i^n + \bar{\lambda}_i^n - 2 = (\lambda_i^{n/2} - \bar{\lambda}_i^{n/2})^2 = 4\sinh^2(nL_i). \tag{38}$$

Substituting Eqs. (36)–(38) into Eq. (1), we therefore obtain Eq. (4).

3.5. Deriving quations of the fan network

Similar to that of the cobweb network, we calculate the potential of fan network by an major matrix equation and two boundary condition equations. The major matrix equation of the fan network is just exactly the same as Eqs. (9)–(13), and its main transformation matrix is just Eq. (23), while the boundary condition equations of the fan network are different from those of the cobweb network, they are

$$\boldsymbol{V}_1 = (\boldsymbol{A}_m - \boldsymbol{E})\boldsymbol{V}_0, \tag{39}$$

$$\boldsymbol{V}_{n-1} = (\boldsymbol{A}_m - \boldsymbol{E})\boldsymbol{V}_n. \tag{40}$$

We conduct the same matrix transformation as that in Eq. (16). Applying $\boldsymbol{P}_m$ to Eqs. (39) and (40) on the left-hand side, we obtain

$$X_1^{(i)} = (t_i - 1)X_0^{(i)}, \tag{41}$$

$$X_{n-1}^{(i)} = (t_i - 1)X_n^{(i)}. \tag{42}$$

where $X_k^{(i)}$ has been defined in Eq. (21).

Equations (23), (41), and (42) are all essential equations for solving the potential of the fan network. Similar to the derivation of the cobweb network, by Eqs. (23), (41), and (42), we work out after some algebra and reduction the general solution of $X_k^{(i)}$ with potential parameters $(0 \le k \le n)$,

$$X_x^{(i)} = \frac{\beta_{x_1 \vee x}^{(i)}\zeta_{1,i} + \beta_{x_2 \vee x}^{(i)}\zeta_{2,i}}{(t_i - 2)F_{n+1}^{(i)}} rJ, \tag{43}$$

where t_i has been given in Eq. (18) and $\beta_{x_s \vee x}^{(i)}$ has been defined in Eq. (7).

Next, we derive the potential formula by matrix inverse transformation. Using Eq. (21), we have $\boldsymbol{V}_k = (\boldsymbol{P}_m)^{-1}\boldsymbol{X}_k$, together with Eqs. (20) and (43), we therefore obtain

$$V_k^{(y)} = J \frac{4r}{2m+1} \sum_{i=1}^{m} \frac{\beta_{x_1,x}^{(i)}\zeta_{1,i} + \beta_{x_2,x}^{(i)}\zeta_{2,i}}{(t_i - 2)F_{n+1}^{(i)}} \sin(y\theta_i). \tag{44}$$

Substituting Eq. (24) into Eq. (44), we therefore obtain Eq. (6).

In addition, similar to the approach to prove Eq. (4), we can easy prove Eq. (8) by using $\lambda_i = \mathrm{e}^{2L_i}$ and $\bar{\lambda}_i = \mathrm{e}^{-2L_i}$.

For the first time, we obtained two general potential formulae of arbitrary $m \times n$ cobweb and fan networks by the *RT* method, which provides a new theoretical method for related scientific research.

4. Applications of the potential formula

In this section, we mainly give a series of specific applications of the potential formula of the cobweb network under different conditions. In the applications below, we always assume that $U_{m\times n}(0,0)=0$, $U^{(x,y)}_{(0,0)}=U_{m\times n}(x,y)-U_{m\times n}(0,0)$, and $U(\infty,\infty)=0$.

4.1. Specific potential in the finite network

Application 1 Consider an arbitrary $m\times n$ cobweb resistor network as shown in Fig. 1. Assume that the output current J locates at the center of $O(0,0)$ (the origin of coordinates), the input current J locates at an arbitrary node $d_1(x_1,y_1)$, the potential formula of the node $d(x,y)$ in an arbitrary $m\times n$ cobweb network can be written as

$$\frac{U^{(x,y)}_{(0,0)}}{J}=\frac{4r}{2m+1}\sum_{i=1}^{m}\frac{F^{(i)}_{n-|x_1-x|}+F^{(i)}_{|x_1-x|}}{\lambda_i^n+\bar{\lambda}_i^n-2}\sin(y_1\theta_i)\sin(y\theta_i). \tag{45}$$

Derivation of Eq. (45) is as follows.

When the output current locates at the center of $O(0,0)$, $y_2=0$ and the input current locates at the node of $d_1(x_1,y_1)$, we obtain

$$S_{y_2,i}=S_{0,i}=\sin(y_2\theta_i)=0,\quad S_{y_1,i}=\sin(y_1\theta_i). \tag{46}$$

Substituting Eq. (46) into Eq. (1), we immediately obtain Eq. (45).

Application 2 When the input current and output current locate in the same longitude, the current J goes from node $d_1(x_1,y_1)$ to node $d_2(x_1,y_2)$. From Eq. (1) with $x_2=x_1$, the potential formula of node $d(x,y)$ in the $m\times n$ cobweb network can be written as

$$\frac{U^{(x,y)}_{(0,0)}}{J}=\frac{4r}{2m+1}\sum_{i=1}^{m}\left(\frac{(S_{y_1,i}-S_{y_2,i})S_{y,i}}{\lambda_i^n+\bar{\lambda}_i^n-2}\right)g^{(i)}_{x_1,x}, \tag{47}$$

where $g^{(i)}_{x_k,x}$ has been defined in Eq. (2).

In particular, when the node $d(x,y)=d(x_1,y)$ locates in the same longitude axis with $d_1=(x_1,y_1)$ and $d_2=(x_1,y_2)$, the potential formula (47) reduces to

$$\frac{U^{(x_1,y)}_{(0,0)}}{J}=\frac{4r}{2m+1}\sum_{i=1}^{m}\left(\frac{(S_{y_1,i}-S_{y_2,i})S_{y,i}}{\lambda_i^n+\bar{\lambda}_i^n-2}\right)F^{(i)}_{n}. \tag{48}$$

Application 3 When the current J goes from $d_1(x_1,y_1)$ to $d_2(x_2,y_1)$ and the two current nodes locate in the same latitude, from Eq. (1) with $y_2=y_1$, the potential formula of an arbitrary node $d(x,y)$ in the $m\times n$ cobweb network can be written as

$$\frac{U^{(x,y)}_{(0,0)}}{J}=\frac{4r}{2m+1}\sum_{i=1}^{m}\left(\frac{g^{(i)}_{x_1,x}-g^{(i)}_{x_2,x}}{\lambda_i^n+\bar{\lambda}_i^n-2}\right)S_{y_1,i}S_{y,i}. \tag{49}$$

Application 4 In Fig. 1, assume that the current J goes from $d_1(x_1,y_1)$ to the two output nodes of $d_2(x_2,y_2)$ and $d_3(x_2,y_3)$, where two output currents are respectively $J/2$, this situation is regarded as two independent current sources. From Eq. (1), the potential formula of an arbitrary node $d(x,y)$ in the $m\times n$ cobweb network can be written as

$$\frac{U^{(x,y)}_{(0,0)}}{J}=\frac{2r}{2m+1}\sum_{i=1}^{m}\left(\frac{2g^{(i)}_{x_1,x}S_{y_1,i}-g^{(i)}_{x_2,x}S_{y_2,i}-g^{(i)}_{x_3,x}S_{y_3,i}}{\lambda_i^n+\bar{\lambda}_i^n-2}\right)S_{y,i}, \tag{50}$$

where $S_{k,i}=\sin(y_k\theta_i)$ has been given in Eq. (1) and $g^{(i)}_{x_k,x}$ has been defined in Eq. (2).

Application 5 When the input current has k nodes, assume that the output current J locates at the center node $O(0,0)$ and the input currents have k nodes of $d_k(x_k,y_1)$ $(k=1,2,\ldots,k)$ in the same latitude, which is J/k. The potential at node $d(x,y)$ is composed of the sum of k potentials. From Eq. (45), we have the potential formula

$$\frac{U^{(x,y)}_{(0,0)}}{J}=\frac{4r}{2m+1}\sum_{i=1}^{m}\frac{\sin(y_1\theta_i)\sin(y\theta_i)}{\lambda_i^n+\bar{\lambda}_i^n-2}\left(\frac{1}{k}\sum_{k=1}^{k}g^{(i)}_{x_k,x}\right). \tag{51}$$

Application 6 When the input current has n nodes, assume that the output current J locates at the center node $O(0,0)$ and the current inputting to node $d_k(x_k,y_1)$ $(k=1,2,\ldots,n)$ in the same latitude is J/n. From Eq. (51), we have the potential formula

$$\frac{U^{(x,y)}_{(0,0)}}{J}=\frac{2r_0}{n(2m+1)}\sum_{i=1}^{m}\frac{\sin(y_1\theta_i)}{1-\cos\theta_i}\sin(y\theta_i). \tag{52}$$

Derivation of Eq. (52) is as follows.

When there are n independent current sources, each node potential is the summations of n potentials since the potential is a scalar. Using Eq. (51) with $k=n$, we compute the sums of $\sum_{k=1}^{n}g^{(i)}_{x_k,x}$. From Eq. (2), we have

$$\begin{aligned}\sum_{k=1}^{n}g^{(i)}_{x_k,x}&=\sum_{k=1}^{n}(F^{(i)}_{n-|x_k-x|}+F^{(i)}_{|x_k-x|})\\&=\frac{\lambda_i^n+\bar{\lambda}_i^n-2}{t_i-2}=\frac{\lambda_i^n+\bar{\lambda}_i^n-2}{2h(1-\cos\theta_i)}.\end{aligned} \tag{53}$$

Substituting Eq. (53) into Eq. (51) with $k=n$, we therefore obtain Eq. (52).

Equation (52) gives an interesting result as it has nothing to do with x, but only related to y. This equation shows that the nodes in the same latitude have the same potential, which is completely in line with the facts.

4.2. The potential in the infinite network

Application 7 Assume that figure 1 shows an infinite network and the electric current J goes from node $d_1(x_1,y_1)$ to node $d_2(x_2,y_2)$. By supposing the potential at node $d_0(0,0)$ is

$U(0,0)=U(\infty)=0$, the potential formula of node $d(x,y)$ can be written as

$$\frac{U_\infty(x,y)}{J}=\frac{r}{2\pi}\int_0^\pi \frac{\bar{\lambda}_\theta^{|x_1-x|}\cos(y_1-y)\theta-\bar{\lambda}_\theta^{|x_2-x|}\cos(y_2-y)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta, \tag{54}$$

where $\bar{\lambda}_\theta=1+h-h\cos\theta-\sqrt{(1+h-h\cos\theta)^2-1}$.

Derivation of Eq. (54) is as follows.

In order to derive Eq. (54), the following integral theory is necessary. If $\phi_k=(2k-1)\pi/(2m+1)$, we have $\Delta\phi_k=\phi_{k+1}-\phi_k=2\pi/(2m+1)$. In the limit of $m\to\infty$, we have

$$\lim_{m\to\infty}\frac{1}{2m+1}\sum_{k=1}^{m}g(\phi_k)=\frac{1}{2\pi}\int_0^\pi g(\phi)\mathrm{d}\phi. \tag{55}$$

First, we consider the case of $n\to\infty$ with finite m. By using Eq. (2), we obtain

$$\lim_{n\to\infty}\frac{g^{(i)}_{k,x_s}}{\lambda_i^n+\bar{\lambda}_i^n-2}=\lim_{n\to\infty}\frac{F^{(i)}_{n-|k-x_s|}+F^{(i)}_{|k-x_s|}}{\lambda_i^n+\bar{\lambda}_i^n-2}=\frac{\bar{\lambda}_i^{|k-x_s|}}{\lambda_i-\bar{\lambda}_i}. \tag{56}$$

Taking $m,n\to\infty$ and applying Eqs. (55) and (56) to Eq. (1), we therefore obtain Eq. (54).

4.3. Expressing the potential by resistance

Application 8 Assume that figure 1 shows an infinite network and the electric current J goes from node $d_1(x_1,y_1)$ to node $d_2(x_2,y_2)$. By supposing $U(0,0)=U(\infty)=0$, the potential formula of node $d(x,y)$ can be written as

$$U(x,y)=\frac{J}{2}\left(R_{dd_2}-R_{dd_1}\right), \tag{57}$$

where R_{dd_k} represents the resistance between two nodes $d(x,y)$ and $d_k(x_k,y_k)$ in an infinite network, which is (one may refer to Ref. [27])

$$R_{dd_k}=\frac{r}{\pi}\int_0^\pi\frac{1-\bar{\lambda}_\theta^{|x_k-x|}\cos(y_k-y)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta. \tag{58}$$

4.4. The potential formula in the semi-infinite network

Application 9 Assume that figure 1 shows a semi-infinite cobweb network, where m is infinite, but the grid number n in the latitude line is finite, and (x,y) and (x_1,y_1) are finite. When the output current is at node $O(0,0)$ and the input current is at node $d_1(x_1,y_1)$, the potential of node $d(x,y)$ in the semi-infinite cobweb network can be written as

$$\frac{U^{(x,y)}_{(0,0)}}{J}=\frac{r}{2\pi}\int_0^\pi\frac{F_{n-|x_1-x|}+F_{|x_1-x|}}{\lambda_\theta^n+\bar{\lambda}_\theta^n-2}\sin(y_1\theta)\sin(y\theta)\mathrm{d}\theta, \tag{59}$$

where $\bar{\lambda}_\theta=1+h-h\cos\theta-\sqrt{(1+h-h\cos\theta)^2-1}$.

In particular, when $m,n\to\infty$, y_1,y and x_1-x are finite, taking $n\to\infty$ in Eq. (59), we obtain the potential of node $d(x,y)$ in the semi-infinite cobweb network as

$$\frac{U^{(x,y)}_{(0,0)}}{J}=\frac{r}{2\pi}\int_0^\pi\sin(y_1\theta)\sin(y\theta)\bar{\lambda}_\theta^{|x_1-x|}\mathrm{d}\theta. \tag{60}$$

4.5. Application of the potential of an infinite network

Application 10 Assume that figure 1 shows an infinite network, the input current J locates at $d_0(x_0,y_0)$, and the output current locates at infinity. By supposing that the potential at node $d_0(x_0,y_0)$ is U_0, the node potential of an infinite cobweb network can be written as

$$U_\infty(x,y)=U_0-\frac{Jr}{2\pi}\int_0^\pi\frac{1-\bar{\lambda}_\theta^{|x-x_0|}\cos(y_0-y)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta, \tag{61}$$

where $\bar{\lambda}_\theta=1+h-h\cos\theta-\sqrt{(1+h-h\cos\theta)^2-1}$. While equation (61) can be written in another form

$$U(x,y)=U_0-\frac{J}{2}R_{dd_0}. \tag{62}$$

Derivation of Eq. (61) is as follows.

Assuming that the input current J locates at $d_0(x_0,y_0)=d_1(x_1,y_1)$, from Eq. (54), we have

$$U_0=U(x_0,y_0)=\frac{rJ}{2\pi}\int_0^\pi\frac{1-\bar{\lambda}_\theta^{|x_2-x_0|}\cos(y_2-y_0)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta, \tag{63}$$

and then, the potential difference between two nodes $d(x,y)$ and $d_0(x_0,y_0)$ is

$$\begin{aligned}U-U_0=&\frac{rJ}{2\pi}\int_0^\pi\frac{\bar{\lambda}_\theta^{|x_0-x|}\cos(y_0-y)\theta-\bar{\lambda}_\theta^{|x_2-x|}\cos(y_2-y)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta\\&-\frac{rJ}{2\pi}\int_0^\pi\frac{1-\bar{\lambda}_\theta^{|x_2-x_0|}\cos(y_2-y_0)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta.\end{aligned} \tag{64}$$

Since $d_2(x_2,y_2)$ locates at infinity, then $x_2,y_2\to\infty$, because $0<\bar{\lambda}_\theta^{|x_2-x|}<1$ and x is finite, we have $\bar{\lambda}_\theta^{|x_2-x|}\to 0$, $\bar{\lambda}_\theta^{|x_2-x_0|}\to 0$, thus, taking Eq. (64) to limit, we therefore obtain Eq. (61). In addition, applying Eq. (58) to Eq. (61), we immediately obtain Eq. (62). Please note that equations (61) and (62) are only suitable when the output current is at infinity.

Application 11 Assume that figure 1 shows an infinite network, the input and output currents locate in the same latitude, and the current J goes from $d_1(x_1,y_1)$ to $d_2(x_2,y_1)$, suggest the potential at infinity is zero. According to Eq. (54), the potential of node $d(x,y)$ can be written as

$$\frac{U_\infty(x,y)}{J}=\frac{r}{2\pi}\int_0^\pi\frac{(\bar{\lambda}_\theta^{|x_1-x|}-\bar{\lambda}_\theta^{|x_2-x|})\cos(y_1-y)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta, \tag{65}$$

The coordinate scope of Eq. (65) is suitable for all coordinates.

Application 12 Assume that figure 1 shows an infinite network, the input and output currents locate in the same longitude, and the current J goes from $d_1(x_1,y_1)$ to $d_2(x_1,y_2)$. Suppose the potential at infinity is zero. According to Eq. (54) the potential of node $d(x,y)$ can be written as

$$\frac{U_\infty(x,y)}{J}=\frac{r}{2\pi}\int_0^\pi\frac{\bar{\lambda}_\theta^{|x_1-x|}[\cos(y_1-y)\theta-\cos(y_2-y)\theta]}{\sqrt{(1+h-h\cos\theta)^2-1}}\mathrm{d}\theta, \tag{66}$$

Application 13 Consider a semi-infinite $\infty \times n$ cobweb network, assume that the output current at the center node $O(0,0)$ is J and the current input to the nodes $d_k(x_k,y_1)$ $(k = 1,2,\ldots,n)$ in the same latitude is J/n. Applying Eq. (55) to Eq. (52), we obtain the potential formula of the semi-infinite network

$$\frac{U_{(0,0)}^{(x,y)}}{J} = \frac{r_0}{\pi n}\int_0^{\pi} \frac{\sin(y_1\theta)}{1-\cos\theta}\sin(y\theta)\,\mathrm{d}\theta. \tag{67}$$

As we all know, it is difficult to compute the potential based on the multiple nodes of the current input and current output in science. However, the research methods in this paper can resolve this problem.

4.6. Resistance formula of an $m \times n$ cobweb network

Application 14 Consider an arbitrary $m \times n$ resistor cobweb network with resistors r and r_0 in the respective latitude and longitude, where m and n are, respectively, the numbers of grids along longitude and latitude directions as shown in Fig. 1. The equivalent resistance between two arbitrary nodes $d_1(x_1,y_1)$ and $d_2(x_2,y_2)$ in an arbitrary $m \times n$ cobweb network can be written as

$$R_{m\times n}(d_1,d_2) = \frac{4r}{2m+1}\sum_{i=1}^{m}\frac{F_n^{(i)}(S_{1,i}^2+S_{2,i}^2)-2(F_{\Delta x}^{(i)}+F_{n-\Delta x}^{(i)})S_{1,i}S_{2,i}}{\lambda_i^n+\bar{\lambda}_i^n-2}, \tag{68}$$

where $\Delta x = |x_2-x_1|$, $S_{k,i} = \sin(y_k\theta_i)$, and $F_k^{(i)}$ and λ_i have been, respectively, defined in Eqs. (2) and (3).

The derivation of Eq. (68) is as follows.

Suggesting that the potentials at nodes of $d_1(x_1,y_1)$ and $d_2(x_2,y_2)$ are $U_{(0,0)}^{(x_1,y_1)}$ and $U_{(0,0)}^{(x_2,y_2)}$ and by Ohm's law, we have

$$R_{m\times n}(d_1,d_2) = \frac{1}{J}(U_{(0,0)}^{(x_1,y_1)} - U_{(0,0)}^{(x_2,y_2)}). \tag{69}$$

Substituting $(x,y) = (x_1,y_1)$ and $(x,y) = (x_2,y_2)$ into Eq. (1), we obtain the values of $U_{(0,0)}^{(x_1,y_1)}$ and $U_{(0,0)}^{(x_2,y_2)}$, substituting them into Eq. (69), we therefore obtain Eq. (68).

Application 15 Similar to the derivation of Eq. (68), using Eq. (4), we obtain another resistance formula expressed by the hyperbolic functions as

$$R_{m\times n}(d_1,d_2) = \frac{2r}{2m+1}\times\sum_{i=1}^{m}\frac{\cosh(nL_i)(S_{1,i}^2+S_{2,i}^2)-2\cosh[(n-2\Delta x)L_i]S_{1,i}S_{2,i}}{\sinh(2L_i)\sinh(nL_i)}, \tag{70}$$

where $L_i = \ln\lambda_i/2$, $S_{k,i} = \sin(y_k\theta_i)$, $\Delta x = |x_2-x_1|$, and λ_i has been defined in Eq. (3).

Please note that the resistance of an arbitrary $m \times n$ resistor cobweb network has been studied in Refs. [24], [28], and [31], by comparison, we find that all resistance equatons are completely equivalent although different calculation methods were used.

4.7. Resistance formula of an $m \times n$ fan network

Application 16 Consider an arbitrary $m \times n$ resistor fan network with resistors r and r_0 in the respective latitudes and longitudes, where m and n are, respectively, the numbers of grids along longitude and latitude directions as shown in Fig. 2. Assume that the center node O is the origin of the rectangular coordinate system and the left edge acts as Y axis. Denote nodes of the network by coordinate $\{x,y\}$. The equivalent resistance between two arbitrary nodes $d_1(x_1,y_1)$ and $d_2(x_2,y_2)$ in an arbitrary $m \times n$ fan network can be written as

$$R_{m\times n}(d_1,d_2) = \frac{2r_0}{2m+1}\times\sum_{i=1}^{m}\frac{\beta_{x_1,x_1}^{(i)}S_{y_1,i}^2-2\beta_{x_1,x_2}^{(i)}S_{y_1,i}S_{y_2,i}+\beta_{x_2,x_2}^{(i)}S_{y_2,i}^2}{(1-\cos\theta_i)F_{n+1}^{(i)}}, \tag{71}$$

where $x_1 \le x_2$, $S_{k,i} = \sin(y_k\theta_i)$, $\beta_{x_s,x}^{(i)} = \Delta F_{x_s}^{(i)}\Delta F_{n-x}^{(i)}$, and $F_k^{(i)}$, λ_i have been, respectively, defined in Eqs. (2) and (3).

In particular, when $d_1(x_1,y_1) = (0,0)$ is at the center O, from Eq. (71), we have the resistance formula

$$R_{m\times n}(d_1,d_2) = \frac{2r_0}{2m+1}\sum_{i=1}^{m}\frac{\Delta F_{x_2}^{(i)}\Delta F_{n-x_2}^{(i)}}{F_{n+1}^{(i)}}\left(\frac{\sin^2(y_2\theta_i)}{1-\cos\theta_i}\right), \tag{72}$$

Please note that the resistance of an arbitrary $m \times n$ resistor fan network has been researched in Refs. [27] and [31], by comparison, we find that all resistance formulae are completely equivalent although different calculation methods were used.

4.8. Comparation and product identities

Reference [24] gave a resistance formula between two arbitrary nodes in $m \times n$ cobweb resistor networks by means of the Laplacian matrix method, which is in the form of double sums. The equation is

$$R_{m\times n}(d_1,d_2) = \frac{2r}{N(2M+1)}\times\sum_{m=1}^{M}\sum_{n=0}^{N-1}\frac{S_{1,m}^2+S_{2,m}^2-2S_{1,m}S_{2,m}\cos(x\varphi_n)}{(1-\cos\varphi_n)+h(1-\cos\theta_m)}, \tag{73}$$

where $x = x_2-x_1$, $S_{k,m} = \sin(y_k\theta_m)$ and $\varphi_n = 2n\pi/N$, $\theta_m = (2m-1)\pi/(2M+1)$.

This paper gives Eq. (68) by the RT method, obviously, the two results with different forms are necessarily equivalence because they are from the same network with the same coordinates. Comparing Eq. (68) with Eq. (73), we immediately obtain a new identity below.

Proposition Define $F_k^{(i)} = (\lambda_i^k-\bar{\lambda}_i^k)/(\lambda_i-\bar{\lambda}_i)$, $\theta_i = (2i-1)\pi/(2M+1)$, $\varphi_n = 2n\pi/N$, $S_{k,i} = \sin(y_k\theta_i)$. When M, N and x,y_k are natural numbers, and $0 \le x \le N-1$, $0 \le y_1$, $y_2 \le M$, we have the following identity:

$$\frac{1}{2N}\sum_{i=1}^{M}\sum_{n=0}^{N-1}\frac{S_{1,i}^2+S_{2,i}^2-2S_{1,i}S_{2,i}\cos(x\varphi_n)}{(1-\cos\varphi_n)+h(1-\cos\theta_i)}$$

$$= \sum_{i=1}^{M} \frac{F_N^{(i)}(S_{1,i}^2 + S_{2,i}^2) - 2(F_x^{(i)} + F_{N-x}^{(i)})S_{1,i}S_{2,i}}{\lambda_i^N + \bar{\lambda}_i^N - 2}, \quad (74)$$

where λ_i and $\bar{\lambda}_i$ have been given in Eq. (3).

Equation (74) is interesting as it simplifies double sums to a single sum. In particular, when x, m, n and y_1, y_2 are specific values, we have the following simple trigonometric identities.

Case 1 When $y_1 = y_2$, equation (74) reduces to

$$\frac{1}{2N}\sum_{i=1}^{M}\sum_{n=0}^{N-1} \frac{S_i^2[1-\cos(x\varphi_n)]}{(1-\cos\varphi_n)+h(1-\cos\theta_i)} = \sum_{i=1}^{M} \frac{S_i^2(F_N^{(i)} - F_x^{(i)} - F_{N-x}^{(i)})}{\lambda_i^N + \bar{\lambda}_i^N - 2}, \quad (75)$$

where $S_i = \sin(y_1\theta_i)$, $\theta_i = (2i-1)\pi/(2M+1)$, and $\varphi_n = 2n\pi/N$.

Case 2 When $M = 1$, $\varphi_n = 2n\pi/N$, equation (74) reduces to

$$\frac{1}{N}\sum_{n=0}^{N-1} \frac{S_1^2 + S_2^2 - 2S_1S_2\cos(x\varphi_n)}{2(1-\cos\varphi_n)+h} = \frac{F_N^{(1)}(S_1^2+S_2^2) - 2(F_x^{(1)} + F_{N-x}^{(1)})S_1S_2}{\lambda_1^N + \bar{\lambda}_1^N - 2}, \quad (76)$$

where $S_1 = \sin(y_1\pi/3)$, $S_2 = \sin(y_2\pi/3)$, $y_1 = \{0,1\}$, $y_2 = \{0,1\}$, and

$$\lambda_1 = 1 + \frac{1}{2}h + \sqrt{\left(1+\frac{1}{2}h\right)^2 - 1}, \quad \bar{\lambda}_1 = 1 + \frac{1}{2}h - \sqrt{\left(1+\frac{1}{2}h\right)^2 - 1}.$$

These identities are interesting because they come from physics rather than mathematics. Please note that the above identities are found for the first time by this paper.

5. Discussion and summary

In this paper, we obtained a series of exact potential formulae in arbitrary $m \times n$ cobweb and fan networks by developing the *RT* method. In some extent, we obtained the analytical solutions of Laplace equations in a variety of boundary conditions by the *RT* method, because Laplace equation can be simulated by resistor network models.

Take the cobweb network as an example, the *RT* method divides the derivation into four steps: first, construct the matrix relation of the currents (or voltages) in three successive transverse lines; second, build boundary condition equations of the currents (or voltages); third, diagonalize the matrix relation to produce a recurrence relation involving only variables in the same transverse line. Basically, the method reduces the problem from two dimensions to one dimension; fourth, implement the inverse transformation to matrix to calculate the node potential (or branch currents).

As applications, we obtained general potential formulae (1), (4), (6), and (8) in arbitrary $m \times n$ cobweb and fan resistor networks. The equivalent resistance formulae were deduced by using potential formulae (68), (70), and (72). As applications of Eq. (1), some interesting results were produced, and the resistance formulae (68), (70), and (72) were produced. The *RT* method and potential formulae (1), (4), (6), and (8) are useful in theory and practical applications.

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