Stochastic bifurcations of generalized Duffing–van der Pol system with fractional derivative under colored noise*

Wei Li(李伟)[1,†], Mei-Ting Zhang(张美婷)[1], and Jun-Feng Zhao(赵俊锋)[2]

[1] School of Mathematics and Statistics, Xidian University, Xi'an 710071, China

[2] Applied Mathematics Department, School of Science, Northwestern Polytechnical University, Xi'an 710072, China

(Received 1 March 2017; revised manuscript received 8 May 2017; published online 18 July 2017)

The stochastic bifurcation of a generalized Duffing–van der Pol system with fractional derivative under color noise excitation is studied. Firstly, fractional derivative in a form of generalized integral with time-delay is approximated by a set of periodic functions. Based on this work, the stochastic averaging method is applied to obtain the FPK equation and the stationary probability density of the amplitude. After that, the critical parameter conditions of stochastic P-bifurcation are obtained based on the singularity theory. Different types of stationary probability densities of the amplitude are also obtained. The study finds that the change of noise intensity, fractional order, and correlation time will lead to the stochastic bifurcation.

Keywords: stochastic bifurcation, fractional derivative, color noise, stochastic averaging method

PACS: 05.45.–a, 05.10.Gg, 05.10.Ln, 05.40.Ca **DOI:** 10.1088/1674-1056/26/9/090501

1. Introduction

Stochastic bifurcation is one of the complicated nonlinear phenomena, different from the bifurcation in a deterministic dynamical system, which is characterized by the qualitative change with the change of system critical parameters. The name of stochastic bifurcation was proposed by the famous scholar Arnold[1] in 1998 in his book "*Random Dynamical System*"; he introduced a series of achievements on the research of stochastic bifurcation, and gave a clear definition of stochastic bifurcation from the point of mathematics. Although stochastic bifurcation is widely used in mathematics, physics, and engineering field nowadays, the theoretical analysis of stochastic bifurcation problems is still in the early stages of its development due to the lack of strict theorem and criterion. Many results obtained by now are mainly derived from numerical calculations. There are two kinds of stochastic bifurcations: dynamical bifurcation (D-bifurcation) and phenomenological bifurcation (P-bifurcation). The former focuses on finding a new invariant measure from a known one, which can be identified by the sign change of the maximum Lyapunov exponent.[2] The latter one can be observed by the shape changes including the peak numbers and positions of the stationary probability density function (PDF) of the system response.

In recent years, many valuable theoretical methods were developed to study stochastic bifurcation for dynamical systems with integer order derivatives under noises excitations. For example, references [3] and [4] investigated the double-peak probability density functions of a Duffing oscillator under narrow-band random excitations, combined deterministic and random excitations using the method of multiple scales and linearization; references [5]– [7] examined the stochastic P-bifurcation of a bistable Duffing–van der Pol system by using the Monte Carlo method; Hao and Wu[8–10] studied the stochastic P-bifurcation of a tri-stable van der Pol–Duffing oscillator subjected to multiplicative colored noise through the use of the singularity theory. Reference [11] investigated the P-bifurcations of a Duffing–Rayleigh vibro impact system under stochastic parametric excitation based on the equivalent nonlinear system method and the catastrophe theory.

Fractional calculus, as the generalization of the traditional integer order calculus, has more than 300 years of history. Unfortunately, fractional calculus has long been a purely theoretical problem in the field of mathematics and developed very slowly owing to the complication of definition. In the 1970s, Professor Mandelbrot of Yale University pointed out the existence of a large number of fractal dimensions in nature and engineering phenomenon. Since then, the fractional order calculus has been rapidly developed and applied to many fields, such as, biomedical, control system, signal processing, quantum mechanics, and especially viscoelastic materials. In order to describe the mechanical behavior of a viscoelastic material and its structures, it is necessary to establish the corresponding mathematical models. Many scholars consider using fractional derivative to model viscoelastic materials in engineering due to their properties of long-run memorability and hereditary. In recent years, stochastic dynamical systems with fractional

*Project supported by the National Natural Science Foundation of China (Grant No. 11302157), the Natural Science Basic Research Plan in Shaanxi Province of China (Grant No. 2015JM1028), the Fundamental Research Funds for the Central Universities, China (Grant No. JB160706), and Chinese–Serbian Science and Technology Cooperation for the Years 2015-2016 (Grant No. 3-19).

†Corresponding author. E-mail: liweilw@mail.xidian.edu.cn

© 2017 Chinese Physical Society and IOP Publishing Ltd

derivative terms involved have attracted much attention in the field of mechanics and mathematics. The works associated with stability,[12,13] stationary response,[12–14] reliability,[13] and chaos[15,16] have been explored gradually. However, according to my knowledge, stochastic bifurcation of such fractional order systems, especially the ones with multi-stable systems, is rarely studied in the existing literature. Moreover, there are two problems in the system after adding the fractional order. (i) How to deal with the time delay in the fractional derivative and what kind of bifurcation does the order of the fractional derivative lead to. (ii) Most research concentrates on considering Gaussian white-noise excitation because of its good statistical properties, such as equal intensity at different frequencies and constant power spectral density. However, white noise with infinite bandwidth is a purely realistic and theoretical noise, the real noise actually has a finite frequency band width and the power spectral density is a function not a constant. Unfortunately, for the real color noise, due to its complexity, the work related to fractional derivative under its excitation has not been reported yet.

The rest of this paper is organized as follows. The stationary probability density of the generalized Duffing–van der Pol system under combination of additive and multiplicative color noise excitations is presented in Section 2. Then, Section 3 focuses on the stochastic P-bifurcation analysis of this system, two different cases about the P-bifurcation behavior are discussed. Finally, the paper ends with some conclusions in Section 4.

2. Stationary probability density

Consider a generalized Duffing–van der Pol system with a fractional derivative term subjected to color noise excitations

$$\begin{aligned}&\ddot{x}(t)+(\beta_1-\beta_2x^2+\beta_3x^4)\dot{x}(t)+\varepsilon \mathrm{D}^{\alpha}x(t)+\omega_0^2x\\ &=(b_1+b_2x)W(t),\quad 0<\alpha\leq 1,\end{aligned}\tag{1}$$

where ε and ω_0 are positive constants denoting small damping coefficient and natural frequency, respectively; β_1, β_2, β_3, b_1, and b_2 are constants with the same order of ε; $\mathrm{D}^{\alpha}x(t)$ is a fractional derivative damping defined in the sense of Caputo-type definition; and $W(t)$ is the color noise which satisfies

$$\langle W(t)\rangle=0,\ \ \langle W(t)W(t+h)\rangle=\frac{D}{\tau}\exp\left(\frac{h}{\tau}\right),$$

where D and τ denote the noise intensity and the correlation time, respectively; and $\langle\cdot\rangle$ is the expectation operator. The expression of $\mathrm{D}^{\alpha}x(t)$ in formula (1) is

$$\mathrm{D}^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\dot{x}(s)}{(t-s)^{\alpha}}\mathrm{d}s,\tag{2}$$

where $\Gamma(\cdot)$ is the gamma function.

Since ε and β_i $(i=1,2,3)$ are all small parameters, the system response is pseudo-periodic, therefore, we introduce the following generalized van der Pol transformation

$$\begin{cases}x(t)=a(t)\cos(\omega_0t+\theta(t)),\\ \dot{x}(t)=-a(t)\omega_0\sin(\omega_0t+\theta(t)),\\ \varphi=\omega_0t+\theta(t),\end{cases}\tag{3}$$

where $a(t)$ and $\theta(t)$ represent the amplitude process and the phase process of the system response, respectively, different from displacement $x(t)$ and velocity $\dot{x}(t)$, they are all slow-varying processes with respect to the time. Substituting Eq. (3) into Eq. (1) yields

$$\begin{aligned}\dot{a}=&-\frac{\sin\varphi}{\omega_0}\big[a\omega_0(\beta_1-\beta_2a^2\cos^2\varphi\\ &+\beta_3a^4\cos^4\varphi)\sin\varphi-\varepsilon\mathrm{D}^{\alpha}x(t)\big]\\ &-\frac{\sin\varphi}{\omega_0}(b_1+b_2a\cos\varphi)W(t),\end{aligned}\tag{4}$$

$$\begin{aligned}\dot{\theta}=&-\frac{\cos\varphi}{a\omega_0}\big[a\omega_0(\beta_1-\beta_2a^2\cos^2\varphi\\ &+\beta_3a^4\cos^4\varphi)\sin\varphi-\varepsilon\mathrm{D}^{\alpha}x(t)\big]\\ &-\frac{\cos\varphi}{a\omega_0}(b_1+b_2a\cos\varphi)W(t).\end{aligned}\tag{5}$$

Observing that the fractional derivative is a generalized complicated integral with time delay in the integration, we have to deal it with a common function in order to finish the following averaging procedure. For doing this, we propose a new method to approximate the fractional derivative. Firstly, rewrite the fractional derivative as a new integral, that is,

$$\mathrm{D}^{\alpha}x(t)=\int_0^{\infty}\psi(y,t)\mathrm{d}y,\tag{6}$$

where

$$\psi(y,t)=\frac{\sin\pi\alpha}{\pi\alpha}\int_0^t\mathrm{e}^{-(t-s)y^{1/\alpha}}\dot{x}(s)\mathrm{d}s,\tag{7}$$

which satisfies a partial differential equation

$$\frac{\partial\psi(y,t)}{\partial t}=-y^{1/\alpha}\psi(y,t)+\frac{\sin\pi\alpha}{\pi\alpha}\dot{x}(t).\tag{8}$$

Substitute Eq. (7) into the above equation and solve it on the basis of transformation (3), we obtain

$$\begin{aligned}\psi(y,t)=&\frac{\sin\pi\alpha}{\pi\alpha}a\frac{\omega_0^2\cos\varphi}{\omega_0^2+y^{2/\alpha}}\\ &-\frac{\sin\pi\alpha}{2}a\frac{\omega_0\sin^2\varphi y^{1/\alpha}}{\omega_0^2+y^{2/\alpha}}+c\mathrm{e}^{ty^{-1/\alpha}},\end{aligned}\tag{9}$$

where c is determined by the initial condition of $\psi(y,0)=0$. Furthermore, substitute Eq. (9) into Eq. (6) and finish the integral, then the approximated expression for the fractional derivative can be cast as

$$\mathrm{D}^{\alpha}x(t)\approx a\omega_0^{\alpha}\left(\cos\varphi\cos\frac{\pi\alpha}{2}+\sin\varphi\sin\frac{\pi\alpha}{2}\right),\quad 0<\alpha\leq 1.$$

By using the stochastic averaging method,[17] equations (4) and (5) can be transformed into the following Itô differential equations:

$$\begin{aligned}
\mathrm{d}a &= \left(-\frac{a\beta_1}{2}+\frac{a^3\beta_2}{8}-\frac{a^5\beta_3}{16}\right.\\
&\quad +\frac{a\varepsilon}{2}\omega_0^{\alpha-1}\sin\frac{\pi\alpha}{2}\\
&\quad \left.+\frac{b_1^2D}{2a\omega_0^2}\frac{1}{\tau^2\omega_0^2+1}+\frac{3b_2^2Da}{8\omega_0^2}\frac{1}{4\tau^2\omega_0^2+1}\right)\mathrm{d}t\\
&\quad +\sqrt{\frac{b_1^2D}{\omega_0^2}\frac{1}{\tau^2\omega_0^2+1}+\frac{b_2^2Da^2}{4\omega_0^2}\frac{1}{4\tau^2\omega_0^2+1}}\mathrm{d}B(t), \quad (10)\\
\mathrm{d}\theta &= \frac{\varepsilon}{2}\omega_0^{\alpha-1}\cos\frac{\pi\alpha}{2}\mathrm{d}t\\
&\quad +\sqrt{\frac{b_1^2D}{8a^2\omega_0^2}\frac{1}{\tau^2\omega_0^2+1}+\frac{b_2^2D}{8\omega_0^2}\frac{1}{4\tau^2\omega_0^2+1}}\mathrm{d}B(t), \quad (11)
\end{aligned}$$

where $B(t)$ is the standard Wiener process. Obviously, expression of $\mathrm{d}a$ is independent of θ, so the FPK equation of amplitude can be obtained by Eq. (10),

$$\frac{\partial p(a,t)}{\partial t} = \frac{\partial}{\partial a}[-m(a)p(a,t)]+\frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma^2(a)p(a,t)], \quad (12)$$

where

$$\begin{aligned}
m(a) &= -\frac{a\beta_1}{2}+\frac{a^3\beta_2}{8}-\frac{a^5\beta_3}{16}+\frac{a\varepsilon}{2}\omega_0^{\alpha-1}\sin\frac{\pi\alpha}{2}\\
&\quad +\frac{b_1^2D}{2a\omega_0^2}\frac{1}{\tau^2\omega_0^2+1}+\frac{3b_2^2Da}{8\omega_0^2}\frac{1}{4\tau^2\omega_0^2+1},\\
\sigma^2(a) &= \frac{b_1^2D}{\omega_0^2}\frac{1}{\tau^2\omega_0^2+1}+\frac{b_2^2Da^2}{4\omega_0^2}\frac{1}{4\tau^2\omega_0^2+1}. \quad (13)
\end{aligned}$$

If $\partial p(a,t)/\partial a = 0$, we can obtain the stationary probability density of amplitude

$$p(a) = \frac{C}{\sigma^2(a)}\exp\left[\int\frac{2m(a)}{\sigma^2(a)}\mathrm{d}a\right], \quad (14)$$

where C is a normalized constant, which satisfies

$$C = \left[\int_0^\infty\left(\frac{1}{\sigma^2(a)}\exp\left[\int\frac{2m(a)}{\sigma^2(a)}\mathrm{d}a\right]\right)\mathrm{d}a\right]^-.$$

Substituting Eq. (13) into Eq. (14) and finishing the integral, we can obtain the exact expression of the stationary amplitude response of system (1)

$$\begin{aligned}
p(a) &= \frac{16C\omega_0^2(4\tau^2\omega_0^2+1)(\tau^2\omega_0^2+1)}{D(a^2\tau^2\omega_0^2+16\tau^2\omega_0^2+a^2+4)}\\
&\quad \times\exp\left(\frac{(y_1+y_2+y_3)y_4+y_5y_6}{8Db_2^6(\tau^2\omega_0^2+1)^2}\right), \quad (15)
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= 16b_2^4\varepsilon(4\omega_0^{\alpha+7}\tau^6+9\omega_0^{\alpha+5}\tau^4+\omega_0^{\alpha+1})\sin(\pi\alpha/2),\\
y_2 &= 8Db_2^6(\tau^2\omega_0^2+1)^2\\
&\quad -64\omega_0^2\beta_1(\tau^2\omega_0^2+1)^2(\tau^2\omega_0^2+1/4)b_2^4,\\
y_3 &= -256b_1^2\omega_0^2(\tau^2\omega_0^2+1)(\tau^2\omega_0^2+1/4)^2\beta_2b_2^2\\
&\quad -2048b_1^4\omega_0^2\beta_3(\tau^2\omega_0^2+1/4)^3,\\
y_4 &= \ln(\tau^2(a^2b_2^2+16b_1^2)\omega_0^2+a^2b_2^2+4b_1^2),\\
y_5 &= 8b_2^2(\tau^2\omega_0^2+1),\\
y_6 &= Db_2^4(\tau^2\omega_0^2+1)\ln(a)+2\omega_0^2a^2(\tau^2\omega_0^2+1/4)\\
&\quad \times((\tau^2\omega_0^2+1)(-\beta_3a^2/4+\beta_2)b_2^2+8b_1^2\beta_3(\tau^2\omega_0^2+1/4)).
\end{aligned}$$

3. Stochastic P-bifurcation

The stochastic P-bifurcation can be identified according to the change of the number of extreme points in the probability density function. According to the singularity theory,[18] the change of the extreme value of the probability density function needs to meet the following two conditions:

$$\frac{\partial p(a)}{\partial a}=0, \quad \frac{\partial^2 p(a)}{\partial a^2}=0. \quad (16)$$

For the sake of convenience, we can write $p(a)$ as follows:

$$\begin{aligned}
p(a) &= CR(a,D,\tau,\varepsilon,\alpha,\beta_1,\beta_2,\beta_3,b_1,b_2)\\
&\quad \times\exp(Q(a,D,\tau,\varepsilon,\alpha,\beta_1,\beta_2,\beta_3,b_1,b_2)). \quad (17)
\end{aligned}$$

According to formulas (16) and (17), the following conditions can be obtained:

$$H=\{R'+RQ'=0, R''+2R'Q'+RQ''+RQ'^2=0\}, \quad (18)$$

where H represents the condition of the change of peak values of the PDF curve.

Generally, multiplicative noise excitation is more complicated than additive noise excitation during the analysis of the system response, because the existence of multiplicative noise excitation usually destroys the superposition principle. Thus, we discuss the stochastic P-bifurcation under additive and multiplicative color noise, respectively.

3.1. Additive color noise case

Additive noise means that the system is only affected by the external random factors, in this case, $b_1\neq 0$, $b_2=0$. Substituting $b_1=1$ and $b_2=0$ into Eq. (15), we obtain the following equations:

$$\begin{aligned}
&R(a,D,\tau,\varepsilon,\alpha,\beta_1,\beta_2,\beta_3,b_1,b_2)\\
&= \frac{\omega_0^2(\tau^2\omega_0^2+1)a}{D},\\
&Q(a,D,\tau,\varepsilon,\alpha,\beta_1,\beta_2,\beta_3,b_1,b_2)\\
&= \frac{\omega_0^2(\tau^2\omega_0^2+1)}{D}\\
&\quad \times\left(-\frac{a^2\beta_1}{2}+\frac{a^4\beta_2}{16}-\frac{a^6\beta_3}{48}+\frac{a^2\varepsilon}{2}\omega_0^{\alpha-1}\sin\frac{\pi\alpha}{2}\right). \quad (19)
\end{aligned}$$

Substituting Eq. (19) into Eq. (18), we obtain the critical parametric conditions of stochastic P-bifurcation with respect to noise intensity D

$$D = \frac{\omega_0^2(\tau^2\omega_0^2+1)}{8}\left(a^6\beta_3 - 2a^4\beta_2 - 8a^2 \times \left(\varepsilon\omega_0^{\alpha-1}\sin\frac{\pi\alpha}{2} + \beta_1\right)\right), \quad (20)$$

where amplitude a satisfies

$$3a^6\beta_3 - 4a^4\beta_2 - 8a^2\left(\varepsilon\omega_0^{\alpha-1}\sin\frac{\pi\alpha}{2} - \beta_1\right) = 0. \quad (21)$$

Given parameter values $\beta_1 = 0.401$, $\beta_2 = 1$, $\beta_3 = 2$, $\varepsilon = 0.5$, $\alpha = 0.5$, and $\omega_0 = 1$, we can obtain the relationship between parameter D and τ. Figure 1 displays the transition set for P-bifurcation on the unfolding parameter plane (τ, D) according to Eqs. (20) and (21). The peak numbers of the stationary PDF in different regions are also different. In Fig. 1, the curve L divides the plane into I and II regions, and the regions I and II represent unimodal and bimodal distributions, respectively. Similarly, given parameter values $\beta_1 = 0.2$, $\beta_2 = 1$, $\beta_3 = 2$, $\varepsilon = 0.5$, $\tau = 0.3$, and $\omega_0 = 1$, we can obtain the relationship between noise intensity D and fractional order α. Figure 2 displays two different regions derived from Eqs. (20) and (21) on the unfolding parameter plane (α, D). The peak numbers of the stationary PDF of the system amplitude will change from one to two if the parametric values change from region I to II. From Fig. 2, we conclude that the peak number of the stationary PDF is always one if $\alpha = 1$.

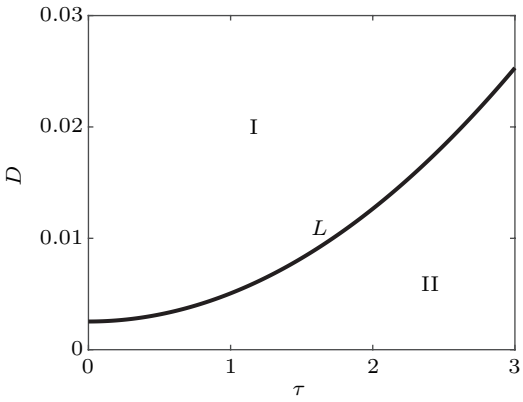


Fig. 1. The bifurcation diagram of parametric plane (τ, D) under additive color noise.

Taking one point at random with the same value $\tau = 0.3$ in area I and area II in Fig. 1 respectively, and drawing the corresponding stationary PDF, we obtain Fig. 3. Obviously, the peak number exactly changes with critical parameter D from 1 to 2, these curves prove the efficiency and correction of our analysis. Similarly, if we choose one point with the same value $D = 0.005$ in area I and area II in Fig. 1 respectively, and draw the corresponding stationary PDF, we obtain Fig. 4.

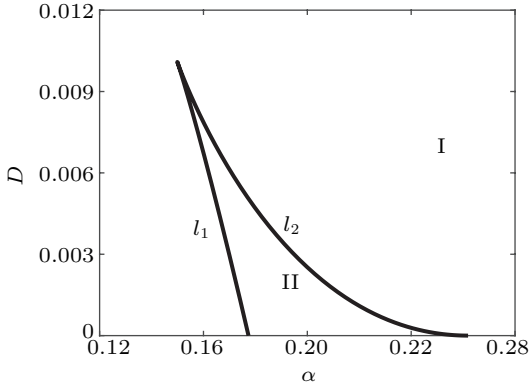


Fig. 2. The bifurcation diagram of parametric plane (α, D) under additive color noise.

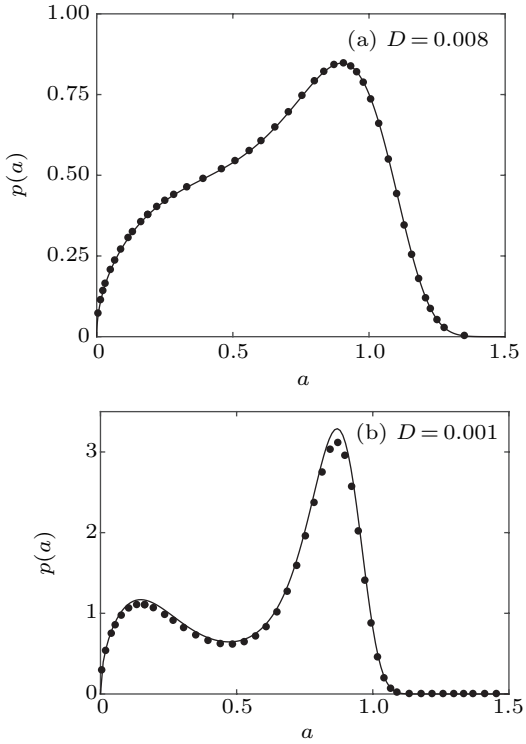


Fig. 3. Stationary PDF curves at different critical parameter values of D: (a) unimodal shape, (b) bimodal shape.

Likewise, if $D = 0.001$, the stationary probability density function of the system amplitude alters at different values of fractional order α according to Fig. 2, see Fig. 5.

Figures 3–5 show the transition process of the stationary probability density function of the amplitude with the change of the parameters only considering the influence of external factors on the system. Figure 3 indicates that the increase of the noise intensity will cause the change of the peak number, i.e., the transition between unimodal and bimodal distributions, which further indicates that the change of the noise intensity can lead to the stochastic P-bifurcation. Similarly, it is shown in Fig. 4 that along with the change of the correlation time, the stationary probability density function of the amplitude will also cause the transition between unimodal and bimodal distributions, and we can conclude that correlation time τ induces the stochastic bifurcation. Figure 5 shows

that along with the change of the fractional order α, the peak number of the stationary PDF will change from one to two.

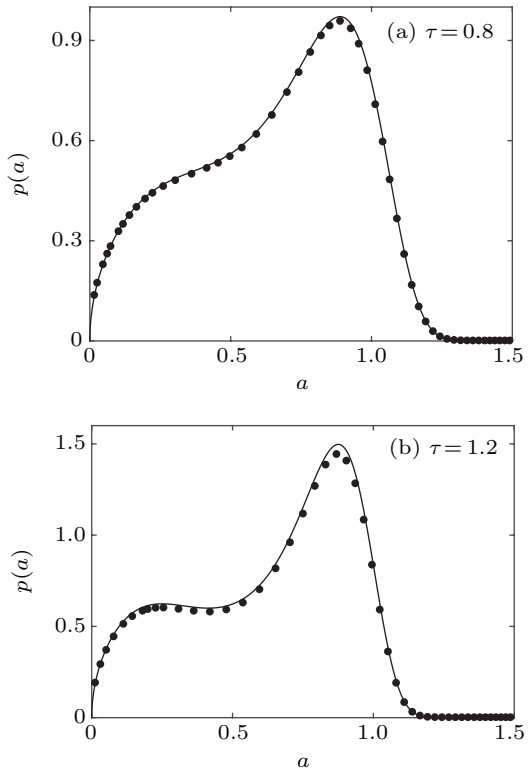


Fig. 4. Stationary PDF curves at different critical parameter values of τ: (a) unimodal shape, (b) bimodal shape.

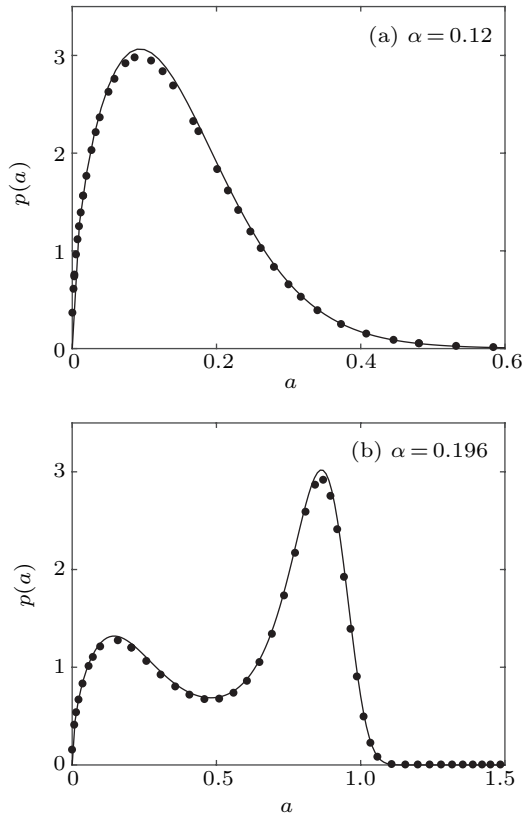


Fig. 5. Stationary PDF curves at different critical parameter values of α: (a) unimodal shape, (b) bimodal shape.

This shows that the change of the fractional order can also cause the stochastic P-bifurcation.

In this section, we utilize the Monte Carlo simulation scheme proposed by Chen in Ref. [19] to prove the efficiency and correction of the approaches we used. The solid lines in Figs. 3–5 are drawn for the analytical results from Eqs. (20) and (21), and all the dotted lines are derived from the Monte Carlo simulation to the original system (1). It is seen that all the results are in good agreement with each other.

3.2. Multiplicative color noise case

Multiplicative noise implies that the system is only affected by its natural factors, in this case, $b_1 = 0$ and $b_2 \neq 0$. Substituting $b_1 = 0$ and $b_2 = 1$ into Eq. (15), we obtain the following equations:

$$\begin{aligned} &R(a,D,\tau,\varepsilon,\alpha,\beta_1,\beta_2,\beta_3,b_1,b_2) \\ &= \frac{4\omega_0^2(4\tau^2\omega_0^2+1)}{D} \\ &\quad \times a^{-\frac{4\omega_0^2\beta_1(4\tau^2\omega_0^2+1)}{D}+\varepsilon\sin\frac{\pi\alpha}{2}\frac{4\omega_0^{\alpha+1}(4\tau^2\omega_0^2+1)}{D}+1}, \\ &Q(a,D,\tau,\varepsilon,\alpha,\beta_1,\beta_2,\beta_3,b_1,b_2) \\ &= \omega_0^2(4\tau^2\omega_0^2+1)a^2\left[\frac{\beta_2}{2D}-\frac{\beta_3}{8D}a^2\right]. \end{aligned} \tag{22}$$

Substituting Eq. (22) into Eq. (18), then we derive the critical parametric conditions of stochastic P-bifurcation

$$\begin{aligned} D = 2\left(\tau^2\omega_0^2+\frac{1}{4}\right)\Bigg(&-8\varepsilon\sin\frac{\pi\alpha}{2}\omega_0^{\alpha+1} \\ &+\omega_0^2\left(a^4\beta_3-2a^2\beta_2+8\beta_1\right)\Bigg), \end{aligned} \tag{23}$$

the stationary probability density of amplitude a satisfies

$$a^4\beta_3-2a^2\beta_2-8\varepsilon\sin\frac{\pi\alpha}{2}\omega_0^{\alpha-1}+8\beta_1=0. \tag{24}$$

Fix parametric values $\beta_2 = 1.5$, $\beta_3 = 2.6$, $\varepsilon = 0.5$, $\alpha = 0.5$, $\tau = 0.3$, and $\omega_0 = 1$, the relationship between parameter D and β_1 can be obtained. Figure 6 displays three different regions by Eqs. (23) and (24) on the unfolding parameter plane (β_1, D). In Fig. 6, the straight lines l_1 and l_2 are the boundary among three different regions I, II, and III. The regions I, II, and III represent unimodal, crater-shape, and non-peak distributions, respectively. Similarly, given parameter values $\beta_1 = 0.2$, $\beta_2 = 1.5$, $\beta_3 = 2.6$, $\varepsilon = 0.5$, $\tau = 0.3$, and $\omega_0 = 1$, we can obtain the relationship between parameter D and α. Figure 7 displays three different regions by Eqs. (23) and (24) on the unfolding parameter plane (α, D), and the shape of the stationary PDF will change from unimodal to crater-shape if $\alpha = 1$.

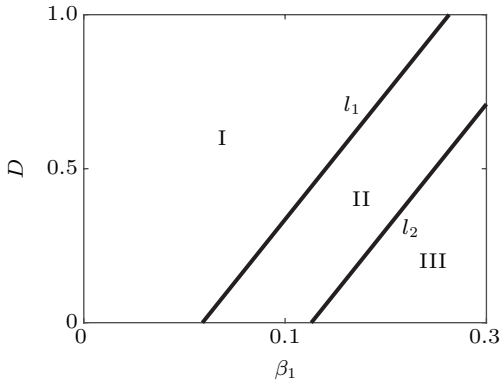


Fig. 6. The bifurcation diagram of parametric plane (β_1, D) under multiplicative color noise.

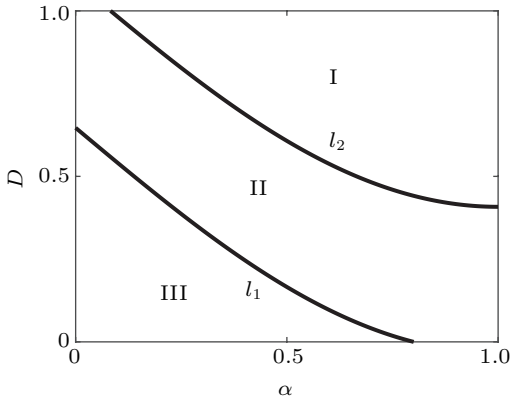


Fig. 7. The bifurcation diagram of parametric plane (α, D) under multiplicative color noise.

Taking one point at random with the same value $\beta_1 = 0.2$ in areas I, II, and III from Fig. 6 respectively, and drawing the corresponding stationary PDF, we obtain Fig. 8. Apparently, with the decrease of the noise intensity D, the mode of the stationary PDF curve turns from one peak to one peak and one valley then to non-peak if the parametric values change from the region I to II then to III, they are typical stochastic P-bifurcation behaviors. Likewise, if $D = 0.45$, the stationary probability density function of the system amplitude alters at different values of fractional order α according to Fig. 7, see Fig. 9.

Figure 9 chooses the fractional order as the bifurcation parameter. Figures 9(a)–9(c) represent the stationary PDFs when the parameters are taken from three different regions in Fig. 7, respectively. It is shown that along with the decrease of the fractional order, the stationary probability density function of the amplitude will also cause the transition between unimodal, crater-shape, and non-peak distributions. This shows that the change of the fractional order can also cause the stochastic P-bifurcation.

In the same way, the Monte Carlo simulation to the original system (1) testifies the correction of our analysis above.

The stochastic P-bifurcation problem of $\tau \neq 0$ is discussed below. Given parameter values $\beta_2 = 1.5$, $\beta_3 = 2.6$, $\varepsilon = 0.5$, $\alpha = 0.5$, and $\omega_0 = 1$, we obtain the following equations:

$$\begin{cases} D = 4(4\tau^2 + 1)\beta_1 - 5.65685425\tau^2, \\ D = 4(4\tau^2 + 1)\beta_1 - 7.387623481\tau^2. \end{cases} \tag{25}$$

The above two equations can be seen as two parallel lines with slope $k = 4(4\tau^2 + 1)$. If $\tau = 0.3$, the relationship between parameters D and β_1 are given, see Fig. 10. Figure 11 shows the effect of τ changes on the P-bifurcation.

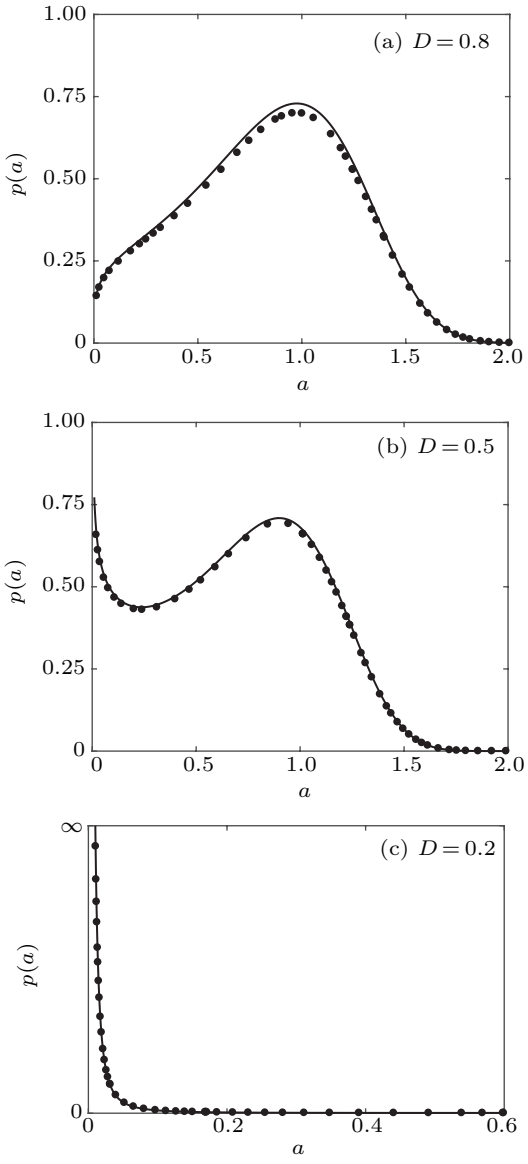


Fig. 8. Stationary PDF curves at different critical parameter values of D: (a) unimodal shape, (b) crater-shape, (c) non-peak shape.

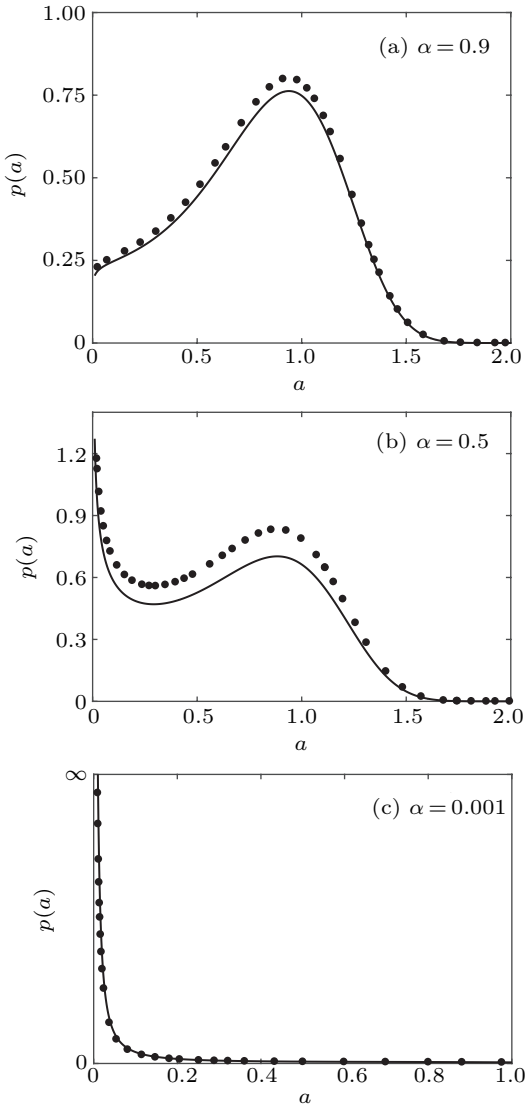


Fig. 9. Stationary PDF curves at different fractional order of α: (a) unimodal shape, (b) crater-shape, (c) non-peak shape.

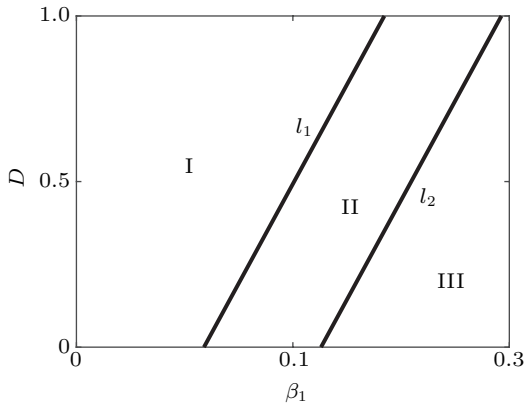


Fig. 10. The bifurcation diagram of $\tau = 0.5$.

From Fig. 11, the slope of the straight line increases with the increase of the correlation time τ. The increase of correlation time τ will reduce the parameter area I and expand the parameter area III in Fig. 10. This shows that the change of the correlation time can also cause stochastic P-bifurcation.

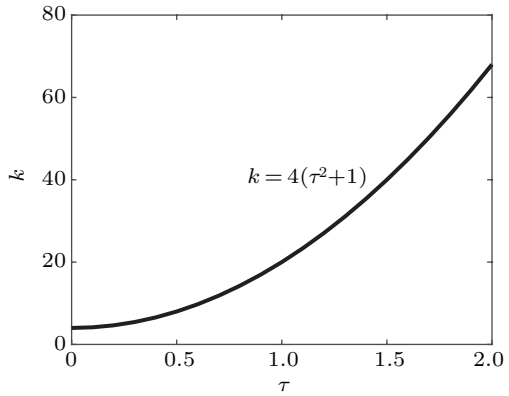


Fig. 11. The effect of τ changes on the P-bifurcation.

4. Conclusion

The stochastic P-bifurcations of a generalized Duffing–van der Pol system with fractional derivative under color noise are explored. Firstly, we propose a new method to approximate the fractional derivative as a series of periodic functions. Based on this work, stochastic P-bifurcation conditions are obtained by using the stochastic averaging method and the singularity theory. The behavior of bifurcations in two different cases of additive excitation and multiplicative excitation are discussed by observing the stationary probability density function of the system. The following conclusions are obtained.

1) The noise intensity is an important parameter to change statistical characteristics of the system amplitude, its change can cause transition phenomenon of stationary probability density function, i.e., stochastic P-bifurcation.

2) The order of the fractional derivative can also apparently cause stochastic P-bifurcation.

3) The change of the correlation time can lead to the stochastic P-bifurcation.

4) All the analytical results are in good agreement with those from Monte Carlo simulation to the original system, which proves the efficiency and correction of our proposed methods.

References

[1] Arnold L 1998 *Random dynamical systems* (Berlin: Springer)
[2] Yang J H, Cai X M and Liu X B 2010 *Commun. Nonlinear Sci. Numer. Simulat.* **15** 3498
[3] Wang X D, Rong H W, Meng G, Xu W and Fang T 2005 *Acta Phys. Sin.* **54** 2557 (in Chinese)
[4] Rong H W, Wang X D, Meng G, Xu W and Fang T 2006 *Chin. J. Appl. Mech.* **27** 1373 (in Chinese)
[5] Zakharova A, Vadivasova T, Anishchenko V, Koseska A and Kurths J 2010 *Phys. Rev. E* **81** 011106
[6] Xu Y, Gu R C, Zhang H Q, Xu W and Duan J Q 2011 *Phys. Rev. E* **83** 056215
[7] Gu R C, Xu Y, Hao M L and Yang Z Q 2011 *Acta Phys. Sin.* **60** 060513 (in Chinese)

[8] Hao Y and Wu Z Q 2013 *Chin. J. Theor. Appl. Mech.* **45** 257 (in Chinese)
[9] Wu Z Q and Hao Y 2013 *Sci. Sin.: Phys. Mech. Astron.* **43** 524 (in Chinese)
[10] Wu Z Q and Hao Y 2015 *Acta Phys. Sin.* **64** 060501 (in Chinese)
[11] Xu W, Yang G D and Yue X L 2016 *Acta Phys. Sin.* **65** 210501 (in Chinese)
[12] Huang Z L and Jin X L 2009 *J. Sound Vib.* **319** 1121
[13] Chen L C and Zhu W Q 2011 *Probablistic Eng. Mech.* **26** 208
[14] Hu F, Chen L C and Zhu W Q 2012 *Int. J. Non-Linear Mech.* **47** 1081
[15] Sheu L J, Chen H K, Chen J H and Tam L M 2007 *Chaos Solitons Fractals* **32** 1459
[16] Chen J H and Chen W C 2008 *Chaos Solitons Fractals* **35** 188
[17] Zhu W Q 1992 *Random Vibration* (Beijing: Science Press) (in Chinese)
[18] Ling F H 1987 *Catastrophe Theory and its Applications* (Shanghai: Shang Hai Jiao Tong University Press) p. 4 (in Chinese)
[19] Chen L C, Zhao T L, Li W and Zhao J 2016 *Nonlinear Dyn.* **82** 529