Revisit to frozen-in property of vorticity*

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Considering some simple topological properties of vorticity vector, the frozen-in property of vorticity herein is revisited. A vortex line, as is analogous to velocity vector along a streamline, is defined as such a coincident material (curve) line that connects many material fluid elements, on which the local vorticity vector for each fluid element is also tangent to the vortex line. The vortex line evolves in the same manner as the material line that it is initially associated with. The vortex line and the material line are both oriented to the same directions, and evolve with the proportional magnitude, just like being 'frozen' or 'glued' to the material elements of the fluid under the barotropic assumption. To relax the limits of incompressible and barotropic atmosphere, the frozen-in property is further derived and proved in the baroclinic case. Then two effective usages are given as examples. One is the derivation of potential vorticity conservation from the frozen-in property in both barotropic and baroclinic atmospheres, as a theory application, and the other is used to illuminate the vorticity generation and growth in ideal cases and real severe weather process, e.g., in squall line, tornado, and other severe convection weather with vortex. There is no necessity to derive vorticity equation, and this method is very intuitive to explain vorticity development qualitatively, especially for fast analysis for forecasters. Certainly, by investigating the evolution of vortex line, it is possible to locate the associated line element vector and its development on the basis of the frozen-in property of vorticity. Because it is simple and visualized, it manifests broad application prospects.

Keywords: vortex, frozen-in property, material line element, baroclinic fluid

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1. Introduction

Vortex is one of the important systems to trigger heavy precipitation and severe convective weather in summer in China. For example, as a typical mesoscale vortex, the southwest vortex (SWV) greatly influences precipitation in Sichuan province or even along the Yangtze River Basin (YRB) as it moves out of Sichuan and propagates towards YRB, bringing serious flood disasters.[1,2] On a small scale, bookend vortices with a dimension of tens of kilometers are a pair of midlevel counter-rotating vortices that usually appear in squall line.[3–6] And the generation of near-ground vorticity plays a key role in tornadogenesis.[7]

Traditionally, vorticity equation is used to describe the generation and development of vortex.[8–18] We must say that it is a very effective method to quantify the evolution of vorticity. However, is there another way to simply give some explanation to and perform fast analysis on the vortex growth for forecasters by perceptual intuition? With the aid of some simple topological properties of vorticity vector, such as the frozen-in property herein, answering the above question may be possible.

What is the frozen-in property of vorticity? As is well known, for every fluid element along streamline, its velocity vector points towards the direction tangent to the streamline. Similarly, a vortex line is defined as such a coincident material (curve) line (blue solid line in Fig. 1(a)) that connects many material fluid elements (little cylindrical volumes), on which the local vorticity for each fluid element (indicated as red arrow) is also tangent to the vortex line (Fig. 1(a)), analogous to velocity along a streamline. The ensemble of vortex lines piercing a closed curve forms a vortex tube (the large twisted tube).

The vortex line will deform following the motion of fluid, which is controlled by the motion equations. The vortex line has a notable property, its evolution keeps consistent with the material line that it was initially associated with for the inviscid barotropic fluid under the assumption of no external forcing. That is to say, vortex line always contains the same material elements, or more imaginatively the vorticity is said to be 'frozen' or 'glued' to the material elements of the fluid.[7] In other words, vorticity has the 'frozen-in' property in inviscid, incompressible, barotropic atmosphere.

If the incompressible and barotropic conditions are relaxed, does the frozen-in property of vorticity exist in baro-

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clinic fluid yet? If the answer is yes, another question is coming, how to prove this point? As an effective supplement of diagnosing the vorticity equation, is there another manner, e.g., the frozen-in property of vorticity, that could be utilized to simply give intuitive explanation in theoretical application and perform fast analysis of real cases on the vortex growth for forecasters?

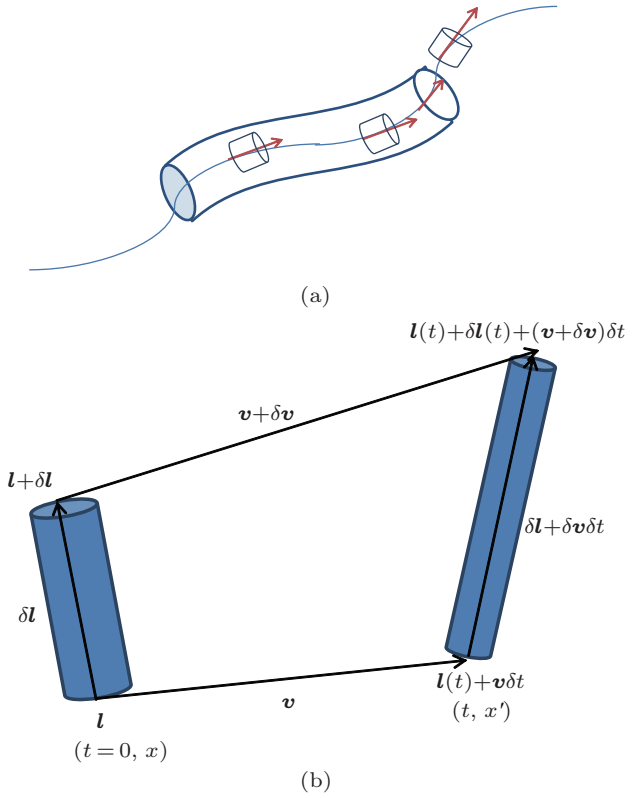


Fig. 1. (color online) (a) Vortex tube (large twisted tube), vortex line (blue solid line), material line elements (little cylindrical volumes), and the local vorticity vector (red arrows) along the vortex line. (b) Infinitesimal material line element $\delta\boldsymbol{l}$ evolving from $(t=0,x)$ to (t,x').

After reviewing the frozen-in property of vorticity in barotropic fluid in Section 2, it will be further derived and proved in baroclinic case, to relax the limit of incompressible and barotropic atmosphere in Section 3. Then two applications are given as examples from both viewpoints of theory and diagnostic analyses. One is the derivation of generalized PV (GPV) conservation from the frozen-in property in both barotropic and baroclinic atmospheres, and the other is used to investigate the vorticity generation and growth in ideal cases and real severe weather processes, e.g., in squall line, tornado, and other severe convection weather with vortex. After the frozen-in property of vorticity is applied in Section 4, summary is given in the last section.

2. Review the frozen-in property of vorticity in barotropic fluid

The 'frozen-in' property of vorticity exists indeed[7] in barotropic fluid, which has been proven under inviscid, incompressible conditions. In order to better understand the problem, it is necessary to review and re-express the frozen-in property of vorticity in more detail. Consider an infinitesimal material line element $\delta\boldsymbol{l}$ which connects $\boldsymbol{l}$ with $\boldsymbol{l}+\delta\boldsymbol{l}$ (Fig. 1(b)). The change rate of $\delta\boldsymbol{l}$ with time is

$$\frac{\mathrm{d}\delta\boldsymbol{l}}{\mathrm{d}t}=\frac{1}{\delta t}[\delta\boldsymbol{l}(t+\delta t)-\delta\boldsymbol{l}(t)]. \tag{1}$$

Using Taylor formula and the definition of velocity, there is the relation below (refer to Fig. 1(b)).

$$\begin{aligned}\delta\boldsymbol{l}(t+\delta t) &= \boldsymbol{l}(t)+\delta\boldsymbol{l}(t)+(\boldsymbol{v}+\delta\boldsymbol{v})\delta t-(\boldsymbol{l}(t)+\boldsymbol{v}\delta t)\\ &= \delta\boldsymbol{l}+\delta\boldsymbol{v}\delta t.\end{aligned} \tag{2}$$

Substituting Eq. (2) into Eq. (1) gives

$$\frac{\mathrm{d}\delta\boldsymbol{l}}{\mathrm{d}t}=\delta\boldsymbol{v}. \tag{3}$$

Because of the relation $\delta\boldsymbol{v}=(\delta\boldsymbol{l}\cdot\nabla)\boldsymbol{v}$, equation (3) becomes

$$\frac{\mathrm{d}\delta\boldsymbol{l}}{\mathrm{d}t}=(\delta\boldsymbol{l}\cdot\nabla)\boldsymbol{v}. \tag{4}$$

In the incompressible, frictionless, barotropic fluid, vorticity equation is expressed as

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}=(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}. \tag{5}$$

Comparing Eq. (4) with Eq. (5), it is found that two equations present completely the same framework in form. Clearly the vorticity evolves in the same manner as a material line element. To illustrate this point in more detail, an infinitesimal material line element is assumed to be parallel to the vorticity at some initial time $t=0$ and some initial location x, i.e.,

$$\delta\boldsymbol{l}(x,t=0)=A\boldsymbol{\xi}(x,t=0), \tag{6}$$

where A is a constant. At some subsequent time t, fluid element arrives at a new location x'. The magnitude of vorticity remains proportional to the length of the material line element, and both are oriented to the same direction. The expression (6) evolves into the following equation at time t:

$$\delta\boldsymbol{l}(x',t)=A\boldsymbol{\xi}(x',t). \tag{7}$$

How is this point proved? Note that $\delta\boldsymbol{l}=A\boldsymbol{\xi}$ at $t=0$ implies $\delta\boldsymbol{l}\times\boldsymbol{\xi}=0$, if the time derivative of $\delta\boldsymbol{l}\times\xi$ equals zero, which could be verified, and $\delta\boldsymbol{l}$ and $\boldsymbol{\xi}$ will continue to keep parallel, which is derived.

For line element, there is

$$\frac{\mathrm{d}(\xi\times\delta\boldsymbol{l})}{\mathrm{d}t}=\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}\times\delta\boldsymbol{l}-\frac{\mathrm{d}\delta\boldsymbol{l}}{\mathrm{d}t}\times\boldsymbol{\xi}. \tag{8}$$

Combining with Eqs. (4)–(6), then equation (8) will be rewritten as

$$\frac{\mathrm{d}(\xi\times\delta\boldsymbol{l})}{\mathrm{d}t}=(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}\times\delta\boldsymbol{l}-(\delta\boldsymbol{l}\cdot\nabla)\boldsymbol{v}\times\boldsymbol{\xi}$$

$$= (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}\times A\boldsymbol{\xi} - (A\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}\times\boldsymbol{\xi}$$
$$= 0. \tag{9}$$

Therefore, the tendency of $\boldsymbol{\xi}\times\delta\boldsymbol{l}$ is zero and the vortex line continues to be a material line. In order to better understand the above-mentioned property of vorticity, we can make the following analogy.

The potential temperature (PT) of an air particle is conserved. In other words, the PT of an air particle remains invariant as the air particle moves. Vorticity also has this property: the vorticity always evolves in the same way as material line element. Both develop in the same orientation, and grow in the manner of proportional magnitude. It seems that vorticity is glued inside the fluid element. This kind of property of vorticity enables vorticity line itself to be a tracer-like material line, to be utilized to track the location of fluid element.

The above derivation is for the frictionless, incompressible, and barotropic case. Does the vorticity still present frozen-in property for compressible fluid under the barotropic condition? For inviscid barotropic fluid, the vorticity equation and continuity equation are expressed as

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{v}, \tag{10}$$
$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \nabla\cdot\boldsymbol{v} = 0. \tag{11}$$

Substitute Eq. (11) into Eq. (10), then we will have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\xi}}{\rho}\right) = \left(\frac{\boldsymbol{\xi}}{\rho}\cdot\nabla\right)\boldsymbol{v}. \tag{12}$$

As can be seen, equation (12) is completely consistent with Eq. (5) in form except that $\boldsymbol{\xi}$ is replaced by $\boldsymbol{\xi}/\rho$. Therefore the frozen-in property is still retained in inviscid, compressible, barotropic fluid. Thus the assumption of incompressible condition (which implies the presence of a density term in the denominator, (i.e., if $\nabla\cdot\boldsymbol{v}=0$)) is not necessary.

3. Derivation of the frozen-in property of vorticity in the baroclinic atmosphere

In Section 2, the frozen-in property of vorticity barotropic fluid is reviewed and re-derived in detail. How about the scenario in the baroclinic case? For baroclinic fluid, motion equation is expressed as

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v} + \frac{1}{\rho}\nabla p + g\boldsymbol{k} = 0. \tag{13}$$

Utilizing the entropy ($S = c_p\ln\theta + C$, where c_p is the specific heat at constant pressure and C is a constant) and enthalpy ($H = c_p\nabla T$) of an air particle, we have

$$\nabla S = \frac{c_p}{\theta}\nabla\theta = \frac{c_p}{T}\nabla T - \frac{R}{p}\nabla p = \frac{c_p}{T}\nabla T - \frac{1}{T\rho}\nabla p. \tag{14}$$

Thus,

$$\frac{1}{\rho}\nabla p = c_p\nabla T - T\nabla S = \nabla H - T\nabla S. \tag{15}$$

Substituting Eq. (15) into Eq. (13), then taking curl operator on Eq. (13), we obtain

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v} - \boldsymbol{\xi}\nabla\cdot\boldsymbol{v} + \nabla T\times\nabla S. \tag{16}$$

Combining with continuity equation (11), equation (16) is re-expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\xi}}{\rho}\right) = \left(\frac{\boldsymbol{\xi}}{\rho}\cdot\nabla\right)\boldsymbol{v} + \frac{1}{\rho}\nabla T\times\nabla S. \tag{17}$$

And term $[\mathrm{d}(\nabla\eta\times\nabla S/\rho)/\mathrm{d}t]$ is expanded into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\nabla\eta\times\nabla S\right) = -\frac{1}{\rho^2}(\nabla\eta\times\nabla S)\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\nabla\eta\times\nabla S + \frac{1}{\rho}\nabla\eta\times\frac{\mathrm{d}}{\mathrm{d}t}\nabla S, \tag{18}$$

with $T = \mathrm{d}\eta/\mathrm{d}t$.

To avoid the confusion of dual vector product, the following equation

$$\frac{\mathrm{d}\nabla\psi}{\mathrm{d}t} = \nabla\frac{\mathrm{d}\psi}{\mathrm{d}t} - \left(\frac{\partial\boldsymbol{v}}{\partial x}\cdot\nabla\psi\right)\boldsymbol{i} - \left(\frac{\partial\boldsymbol{v}}{\partial y}\cdot\nabla\psi\right)\boldsymbol{j} - \left(\frac{\partial\boldsymbol{v}}{\partial z}\cdot\nabla\psi\right)\boldsymbol{k} \tag{19}$$

is expanded into single-component form, which is also adopted in the following derivations.

We have the equations as follows:

$$\frac{\mathrm{d}\nabla S}{\mathrm{d}t} = \nabla\frac{\mathrm{d}S}{\mathrm{d}t} - \left(\frac{\partial\boldsymbol{v}}{\partial x}\cdot\nabla S\right)\boldsymbol{i} - \left(\frac{\partial\boldsymbol{v}}{\partial y}\cdot\nabla S\right)\boldsymbol{j} - \left(\frac{\partial\boldsymbol{v}}{\partial z}\cdot\nabla S\right)\boldsymbol{k}, \tag{20}$$
$$\frac{\mathrm{d}\nabla\eta}{\mathrm{d}t} = \nabla\frac{\mathrm{d}\eta}{\mathrm{d}t} - \left(\frac{\partial\boldsymbol{v}}{\partial x}\cdot\nabla\eta\right)\boldsymbol{i} - \left(\frac{\partial\boldsymbol{v}}{\partial y}\cdot\nabla\eta\right)\boldsymbol{j} - \left(\frac{\partial\boldsymbol{v}}{\partial z}\cdot\nabla\eta\right)\boldsymbol{k}. \tag{21}$$

Also from the continuity equation (11), there is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\boldsymbol{v}. \tag{22}$$

And we use the conservation property of S ($\mathrm{d}S/\mathrm{d}t = 0$), equation (18) becomes the following form

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\nabla\eta\times\nabla S\right)\\ &= -\frac{1}{\rho^2}(\nabla\eta\times\nabla S)(-\rho\nabla\cdot\boldsymbol{v})\\ &\quad + \frac{1}{\rho}\left(\nabla\frac{\mathrm{d}\eta}{\mathrm{d}t} - \left(\frac{\partial\boldsymbol{v}}{\partial x}\cdot\nabla\eta\right)\boldsymbol{i} - \left(\frac{\partial\boldsymbol{v}}{\partial y}\cdot\nabla\eta\right)\boldsymbol{j}\right.\\ &\quad \left. - \left(\frac{\partial\boldsymbol{v}}{\partial z}\cdot\nabla\eta\right)\boldsymbol{k}\right)\times\nabla S + \frac{1}{\rho}\nabla\eta\times\left(\nabla\frac{\mathrm{d}S}{\mathrm{d}t}\right.\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{\partial \boldsymbol{v}}{\partial x}\cdot\nabla S\right)\boldsymbol{i} - \left(\frac{\partial \boldsymbol{v}}{\partial y}\cdot\nabla S\right)\boldsymbol{j} - \left(\frac{\partial \boldsymbol{v}}{\partial z}\cdot\nabla S\right)\boldsymbol{k}\Big) \\
&= \frac{1}{\rho}(\nabla\eta\times\nabla S)(\nabla\cdot\boldsymbol{v}) + \frac{1}{\rho}(\nabla T\times\nabla S) \\
&\quad + \frac{1}{\rho}\nabla S\times\left[\left(\frac{\partial \boldsymbol{v}}{\partial x}\cdot\nabla\eta\right)\boldsymbol{i} + \left(\frac{\partial \boldsymbol{v}}{\partial y}\cdot\nabla\eta\right)\boldsymbol{j}\right. \\
&\quad \left. + \left(\frac{\partial \boldsymbol{v}}{\partial z}\cdot\nabla\eta\right)k\right] \\
&\quad - \frac{1}{\rho}\nabla\eta\times\left[\left(\frac{\partial \boldsymbol{v}}{\partial x}\cdot\nabla S\right)\boldsymbol{i} + \left(\frac{\partial \boldsymbol{v}}{\partial y}\cdot\nabla S\right)\boldsymbol{j}\right. \\
&\quad \left. + \left(\frac{\partial \boldsymbol{v}}{\partial z}\cdot\nabla S\right)\boldsymbol{k}\right]. \qquad (23)
\end{aligned}
$$

Consider a single Cartesian component of the above equation, e.g., in the x direction marked as superscript x,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\nabla\eta\times\nabla S\right)^x \\
&= \frac{1}{\rho}(\nabla\eta\times\nabla S)^x(\nabla\cdot\boldsymbol{v}) + \frac{1}{\rho}(\nabla T\times\nabla S)^x \\
&\quad + \frac{1}{\rho}\left[(\nabla S)^y\left(\frac{\partial \boldsymbol{v}}{\partial z}\cdot\nabla\eta\right) - (\nabla S)^z\left(\frac{\partial \boldsymbol{v}}{\partial y}\cdot\nabla\eta\right)\right] \\
&\quad + \frac{1}{\rho}\left[(\nabla\eta)^z\left(\frac{\partial \boldsymbol{v}}{\partial y}\cdot\nabla S\right) - (\nabla\eta)^y\left(\frac{\partial \boldsymbol{v}}{\partial z}\cdot\nabla S\right)\right] \\
&= \frac{1}{\rho}(\nabla T\times\nabla S)^x + \frac{1}{\rho}[(\nabla\eta\times\nabla S)\cdot\nabla]u. \qquad (24)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\nabla\eta\times\nabla S\right)^y \\
&= \frac{1}{\rho}(\nabla T\times\nabla S)^y + \frac{1}{\rho}[(\nabla\eta\times\nabla S)\cdot\nabla]v \qquad (25)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\nabla\eta\times\nabla S\right)^z \\
&= \frac{1}{\rho}(\nabla T\times\nabla S)^z + \frac{1}{\rho}[(\nabla\eta\times\nabla S)\cdot\nabla]w. \qquad (26)
\end{aligned}
$$

From Eqs. (23)–(26), there is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\nabla\eta\times\nabla S\right) \\
&= \frac{1}{\rho}(\nabla T\times\nabla S) + \frac{1}{\rho}[(\nabla\eta\times\nabla S)\cdot\nabla]\boldsymbol{v}. \qquad (27)
\end{aligned}
$$

Subtracting Eq. (27) from Eq. (17) we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\rho}(\boldsymbol{\xi} - \nabla\eta\times\nabla S)\right] = \frac{1}{\rho}[(\boldsymbol{\xi} - \nabla\eta\times\nabla S)\cdot\nabla]\boldsymbol{v}. \qquad (28)
$$

Define baroclinic vorticity as $\boldsymbol{\xi}_\eta = (\boldsymbol{\xi} - \nabla\eta\times\nabla S)$, then equation (28) will turn into

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\xi}_\eta}{\rho}\right) = \left(\frac{\boldsymbol{\xi}_\eta}{\rho}\cdot\nabla\right)\boldsymbol{v}. \qquad (29)
$$

It is obvious that equation (29) is also completely consistent with Eq. (12) in form. Thus, it is deduced that the frozen-in property still preserves in baroclinic fluid and might be verified by the time tendency of $\delta\boldsymbol{l}\times A\boldsymbol{\xi}_\eta$ to zero.

In detail, equation (29) has the same form as Eq. (5) except that $\boldsymbol{\xi}$ is replaced by $\boldsymbol{\xi}_\eta/\rho$. Since vorticity evolves in the same way as a material line element based on Eqs. (4) and (5), the baroclinic vorticity $\boldsymbol{\xi}_\eta/\rho$ in compressible fluid should also develop in the same manner as a material line element. To verify it, at some initial time and location $(x, t=0)$, an infinitesimal material line element is set to be parallel to the baroclinic vorticity, i.e.,

$$
\delta\boldsymbol{l}_\eta(x, t=0) = A\tilde{\boldsymbol{\xi}}_\eta(x, t=0), \qquad (30)
$$

where $\tilde{\boldsymbol{\xi}}_\eta = \boldsymbol{\xi}_\eta/\rho$, $\delta\boldsymbol{l}_\eta$ is an infinitesimal material line element in baroclinic fluid. Thus the relation $\delta\boldsymbol{l}_\eta\times\tilde{\boldsymbol{\xi}}_\eta = 0$ is established. At time t, fluid element goes to a new location x'. The time tendency of $\delta\boldsymbol{l}_\eta\times\tilde{\boldsymbol{\xi}}_\eta$ becomes

$$
\frac{\mathrm{d}(\tilde{\boldsymbol{\xi}}_\eta\times\delta\boldsymbol{l}_\eta)}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{\boldsymbol{\xi}}_\eta}{\mathrm{d}t}\times\delta\boldsymbol{l}_\eta - \frac{\mathrm{d}\delta\boldsymbol{l}_\eta}{\mathrm{d}t}\times\tilde{\boldsymbol{\xi}}_\eta. \qquad (31)
$$

Combine with Eqs. (4), (29), (30), then equation (31) will be rewritten as

$$
\begin{aligned}
\frac{\mathrm{d}(\tilde{\boldsymbol{\xi}}_\eta\times\delta\boldsymbol{l}_\eta)}{\mathrm{d}t} &= (\tilde{\boldsymbol{\xi}}_\eta\cdot\nabla)\boldsymbol{v}\times\delta\boldsymbol{l}_\eta - (\delta\boldsymbol{l}_\eta\cdot\nabla)\boldsymbol{v}\times\tilde{\boldsymbol{\xi}}_\eta \\
&= (\tilde{\boldsymbol{\xi}}_\eta\cdot\nabla)\boldsymbol{v}\times A\tilde{\boldsymbol{\xi}}_\eta - (A\tilde{\boldsymbol{\xi}}_\eta\cdot\nabla)\boldsymbol{v}\times\tilde{\boldsymbol{\xi}}_\eta \\
&= 0. \qquad (32)
\end{aligned}
$$

Therefore, at time t, $\delta\boldsymbol{l}_\eta(\boldsymbol{x}', t) = A\tilde{\boldsymbol{\xi}}_\eta(\boldsymbol{x}', t)$ still holds, which means that the magnitude of baroclinic vorticity is also proportional to the length of the material line element, and both are oriented to the same direction. In other words, the vortex line continues to be a material line in baroclinic fluid. The frozen-in property of vorticity also holds for baroclinic fluid regardless of the compressible assumption.

4. Applications

Two applications are given as examples from both viewpoints of theoretical and diagnostic analyses. One is the derivation of generalized PV (GPV) conservation from the frozen-in property in both barotropic and baroclinic atmospheres. The other is used to analyze the vorticity generation and growth in tornado and squall line by ideal schematic chart and real severe weather process with vortex such as squall line and mountainous heavy precipitation.

4.1. Example 1

This example is used to prove the GPV conservation.

4.1.1. Barotropic case

For any tracer χ, which is any scalar quantity that satisfies $\mathrm{d}\chi/\mathrm{d}t = 0$, the difference in χ between two infinitesimally close fluid elements is also materially conserved, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\chi_1 - \chi_2) = \frac{\mathrm{d}\delta\chi}{\mathrm{d}t} = 0. \tag{33}$$

From $\delta\chi = \nabla\chi \cdot \delta\boldsymbol{l}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\chi \cdot \delta\boldsymbol{l}) = 0, \tag{34}$$

where $\delta\boldsymbol{l}$ is the infinitesimal vector connecting two fluid elements.

On the basis of the frozen-in property of vorticity, the line element ($\delta\boldsymbol{l}$) and the vorticity ($\boldsymbol{\xi}$, or $\boldsymbol{\xi}/\rho$ in compressible fluid) obey the same equation. Thus the following equation is obtained by replacing the line element with $\boldsymbol{\xi}$ or $\boldsymbol{\xi}/\rho$ in Eq. (34):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\nabla\chi \cdot \boldsymbol{\xi}}{\rho}\right) = 0. \tag{35}$$

That is, the generalized potential vorticity, $Q = (\tilde{\boldsymbol{\xi}} \cdot \nabla\chi)$ is a material invariant. If $\chi = \theta$ (or θ_e), the conservation property of traditional potential vorticity (or equivalent potential vorticity) is proven from another way, i.e., the viewpoint of frozen-in property of vorticity, different from the traditional derivation.

4.1.2. Baroclinic case

The presence of the solenoidal term brings a little complexity to the baroclinic case. However, if it is the baroclinic vorticity $\boldsymbol{\xi}_\eta$ but not $\boldsymbol{\xi}$ (or $\boldsymbol{\xi}/\rho$) is used to substitute for the line element in Eq. (34), the case is simplified greatly. Correspondingly, equation (35) turns into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\nabla\chi \cdot \boldsymbol{\xi}_\eta}{\rho}\right) = 0. \tag{36}$$

Note that equations (29)–(32) which embody the frozen-in property of baroclinic vorticity, are utilized during above derivation towards Eq. (36) herein.

From Eq. (36), in the case that the solenoidal term $(-\nabla\eta \times \nabla S)$ in $\boldsymbol{\xi}_\eta$ vanishes, $\boldsymbol{\xi}_\eta$ falls back to $\boldsymbol{\xi}$ and equation (36) returns to Eq. (35), which recovers the barotropic fluid. Certainly, the above condition may be too strict to implement the evolution of Eq. (36) into Eq. (35). Relax the restriction of $-\nabla\eta \times \nabla S = 0$, as long as three vectors $\nabla\chi$, $\nabla\eta$, and ∇S are coplanar (which requires χ to be an arbitrary function of η and S), the mixed product $-\nabla\chi \cdot (\nabla\eta \times \nabla S) = 0$ is satisfied and that equation (36) recovers Eq. (35) is realized.

As a natural choice, χ is taken as potential temperature (PT). Because entropy S is a function of θ, then $\nabla\theta \cdot (\nabla\eta \times \nabla S) = 0$ is satisfied. Explicitly, the solenoidal term in Eq. (36) vanishes hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(\nabla\theta) \cdot (\boldsymbol{\xi} - \nabla\eta \times \nabla S)}{\rho}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\xi} \cdot \nabla\theta}{\rho}\right). \tag{37}$$

Thus the PV (and its associated PV equation) takes the same form in both barotropic and baroclinic fluids. In two different atmospheres, PV is $\boldsymbol{\xi} \cdot \nabla\theta/\rho$ and PV equation is $(\mathrm{d}/\mathrm{d}t)(\boldsymbol{\xi} \cdot \nabla\theta/\rho) = 0$, which has a single expression. However, for the GPV, two different sets of frameworks in barotropic and baroclinic are unified into one only if $\nabla\chi$, $\nabla\eta$, and ∇S are coplanar.

4.2. Example 2

This example is used to analyze the vorticity generation and growth by both ideal schematic chart and real severe weather cases.

To simply demonstrate how 'frozen-in' property of vorticity is used to explain vorticity generation and growth, a schematic diagram is given in Fig. 2. Several ideal cases are described respectively. As shown in Fig. 2(a), the vertical vorticity is negligible at near-surface layer and at the initial time, and two quasi-horizontal material line elements $\delta\boldsymbol{l}_1$ and $\delta\boldsymbol{l}_2$ (vectors with red arrows) are parallel to the vortex line because the vortex line (the horizontal solid line with arrow) is assumed to be frozen in the fluid. If an updraft is present between $\delta\boldsymbol{l}_1$ and $\delta\boldsymbol{l}_2$ (dash arrow), the tilting of two material line elements is accomplished by the updraft (Fig. 2(b)), which is visualized from the pointer (arrow direction) intuitively and imaginably. It can also be derived theoretically by considering a single Cartesian component of Eq. (4), e.g., in the z-direction,

$$\frac{\mathrm{d}\delta l^z}{\mathrm{d}t} = \delta l^x\frac{\partial w}{\partial x} + \delta l^y\frac{\partial w}{\partial y} + \delta l^z\frac{\partial w}{\partial z}. \tag{38}$$

The first and second terms on the right-hand side of Eq. (38) are the tilting terms. The third term may be explained in terms of stretching. It shows that the fluid element in the z direction may be generated from those in the x and y directions. At the initial time (Fig. 2(a)), δl^y and δl^z components are zero. Thus only the tilting effect, $\delta l^x\partial w/\partial x$, plays a role based on Eq. (38). Because $\delta l^x > 0$ at the initial time, the left fluid element produces upward component ($\delta l^z > 0$) due to $\partial w/\partial x > 0$, while the right line element has $\delta l^z < 0$ because of $\partial w/\partial x < 0$. Thus the updraft acts as tilting the fluid elements, also because vorticity is frozen in these material lines, the horizontal vorticity is tilted and vertical component of vorticity vector is generated from Figs. 2(a) and 2(b).

For the second ideal case (Fig. 2(c)), only vertical vorticity (represented by the solid lines with arrows) is present at a low level and at the initial time, and a vertical material line elements $\delta\boldsymbol{l}$ (vector with red arrow) is parallel to the vertical vortex line since the vortex line is glued to the fluid. If an horizontal convergence is present (dash arrow in Fig. 2(c)) and acts on $\delta\boldsymbol{l}$, the stretching of the material line elements is accomplished by compressing the both-side convergence (Fig. 2(b)), which is easy to understand in physics. According

to Eq. (38), δl^x and δl^y components are zero at the initial time (Fig. 2(c)). Thus only the stretching term ($\delta l^z \partial w/\partial z$) plays a role. Because of $\delta l^z > 0$ at $t = 0$, the fluid element continues to grow up in the vertical direction ($\mathrm{d}\delta l^z/\mathrm{d}t > 0$) due to $\partial w/\partial z > 0$. Note that the continuity equation is utilized herein by which horizontal convergence ($-(\partial u/\partial x + \partial v/\partial y) > 0$) leads to vertical stretching upwards ($\partial w/\partial z > 0$). Thus the convergence influences on the fluid elements and the stretching effect makes the short-fat material line elements turn into a new tall-slim guy. Certainly, because of the frozen-in property of vorticity, vertical vorticity rises straight from the near-ground layer. The vigorous growth of vertical vorticity from Fig. 2(c) to Fig. 2(d) is accompanied by the development of convection and cloud as shown in Fig. 2(d). In this process, vortex lines become denser and denser, and are concentrated by the convergence effect.

The above two cases are obviously oversimplified, and they generally coexist in most instances. Figure 2(e) is taken for example. If figure 2(b) is seen as an outset, although vortex line associated with fluid element $\delta \boldsymbol{l}_1$ is tilted upwards, at near-ground layer no significant vertical vorticity arises by only an updraft. Because updraft makes air rise away from the near-ground following the tilting of material line elements and associated vortex line.

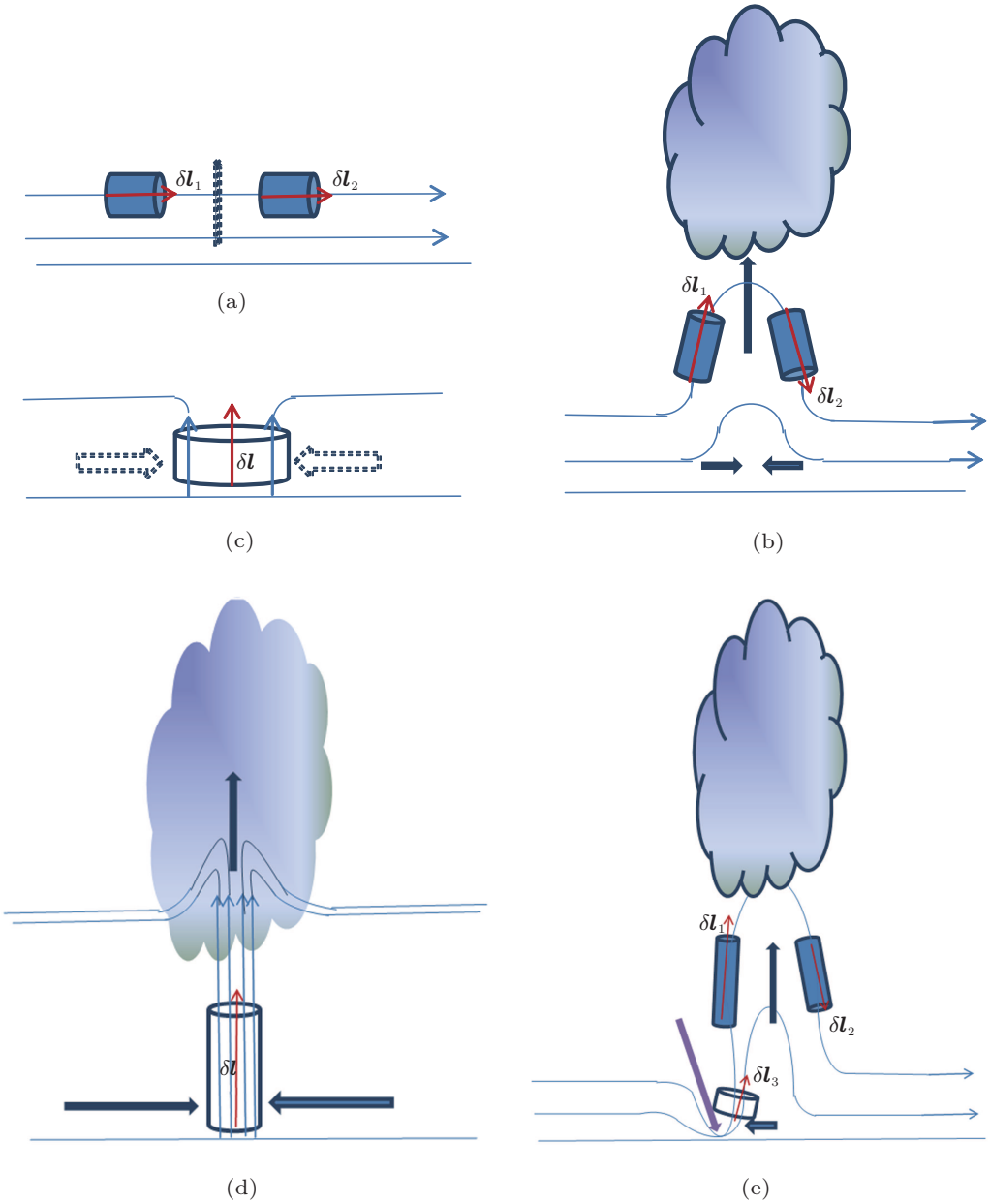


Fig. 2. (color online) (a) and (b) Ideal case where the fluid elements ($\delta \boldsymbol{l}_1$ and $\delta \boldsymbol{l}_2$) tilt from horizontal to vertical directions by only updraft, and related vorticity vector tied in material element evolves. Large and thick arrows denote updraft in panel (a) and convergence of airflows in panel(b). Thin solid lines with blue arrows refer to vortex lines, red arrows denote material line element vectors. (c) and (d) Another ideal case where the vertical component of fluid element δl grows associated vorticity intensifies and vortex lines are concentrated because of its frozen-in property. Large opposite arrows denote convergence of airflows, thin solid lines with blue arrows refer to vortex lines and vorticity vectors, red arrow represents material line vector. (e) The same as panel (b), except for low-level material line element $\delta \boldsymbol{l}_3$ developing upwards and the associated near-ground vorticity generation after the downdraft (purple arrow) joins the team.

If a downdraft is involved as shown in Fig. 2(e) (purple arrow), for δl_1, its components in the x and z directions remain positive ($\delta l^x > 0$, $\delta l^z > 0$), while the downdraft's joining the team leads to $\partial w/\partial x > 0$ and $\partial w/\partial z > 0$, which makes δl^z continue to intensify ($\mathrm{d}\delta l^z/\mathrm{d}t > 0$) from both tilting and stretching. Thus air is sinking toward the ground as shown in Fig. 2(e) producing a positive contribution to the vertical vorticity tendency. From Fig. 2(e), both δl_1 and the frozen-in vorticity in the z direction are enhanced. Meanwhile, the downward intrusion brings horizontal convergence near surface, the material line element δl_3 will be stretched and the associated near-ground vorticity tied in δl_3 is generated and intensified. At the subsequent time, the vortex line tends to become dense.

Figure 2(e) can be regarded as a blend of two ideal cases (Figs. 2(a) and 2(b); Figs. 2(c) and 2(d)). The situation has become complicated because both initial conditions join the battle, and even more complex in real case. However, by decomposing it step by step, the above-mentioned theory could be used to simply explain the vorticity generation and growth partly in tornado, squall line, or other strong convective weather accompanied with vortexes.

For example, figure 3(a) shows the vorticity and vertical velocity in south China at 06UTC 15 in June, 2010. From 14 June to 16 June, a warm-sector heavy rainfall occurred in south China (the National Meteorological Center, private communication). During the period no obvious large weather systems pass through the territory, the strong heavy precipitation is localized very much. As a matter of fact, it should be classified in more detail as a squall line process with strong line convection at the front side, bow echoes, bookend vortexes, etc. In Fig. 3(a), several positive-negative coupled vortex pairs (shaded) are clear. Columns of strong vertical velocities (blue solid lines with a spacing of 3 m/s in between) are distributed between the couple of vortexes. Theses vortex couples connect with updrafts between them closely. For instance, the strongest vortex couple is located on both sides of the strongest vertical velocity at 113.1° E, stretching upwards up to > 12 km. The stable westerly dominates east of 114° E, where there is no significant updraft. Correspondingly, accompanying vortex-couple pattern is absent. It is worth mentioning that the sandwiched construct of updraft surrounded by vortex-couple at 113.1° E is very similar to what is shown in Fig. 2(e). Above 4 km, positive(/negative) vorticity is intensified at left(/right) branch. Below 4km level, the intrusion of downdraft (purple arrow in Fig. 3(a)) is favorable to the generation of near-ground vorticity just as shown in Figs. 2(e) and 3(a). The theory embodied in the first ideal case in Figs. 2(a) and 2(b) could be very aptly used to explain the growth of middle-level vortex-couple and generation of near-ground positive vorticity.

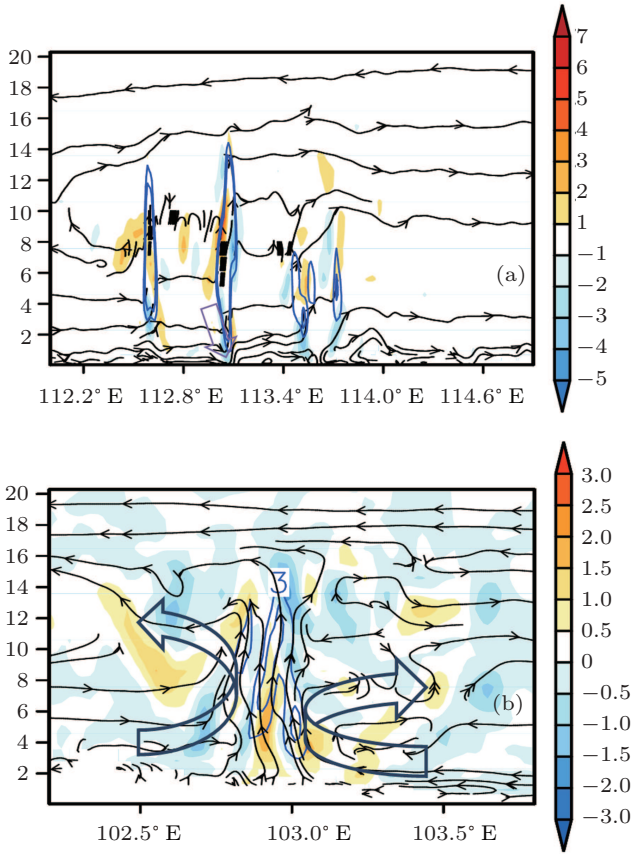


Fig. 3. (color online) (a) An example demonstrating the configuration among several positive-negative coupled vortex pairs (shaded, 10^{-3} m·s^{-1}), columns of strong vertical velocities (blue solid lines with a spacing of 3 m·s^{-1} in between) distributed between the vortex couple and streamline (thin solid line with vector arrow) along 23.3° N at 06 UTC on 15 June, 2010. The purple arrow refers to the intrusion of downdraft, which is favorable to the generation of near-ground vorticity; (b) Another example illustrating how the low-level convergence makes vorticity develop and vortex line concentrated. In detail, it shows the zonal-vertical cross sections of vertical vorticity (shaded, 10^{-3} m·s^{-1}), the vertical velocity (isolines with a spacing of 3 m·s^{-1} in between) and streamlines (thin solid line with vector arrow) along 29.82° N at 16 UTC on 19 August, 2010. The MPS circulation west of 102.9° E and vertical updraft east of 102.9° E induced by SWV are marked as large hollow curve arrow.

Figure 3(b) is another example to illustrate how the low-level convergence makes vorticity develop and vortex line concentrated. It belongs to the genre shown in Figs. 2(c) and 2(d). From 18 August to 20 August, 2010, a heavy mountainous precipitation occurred in western Sichuan near the transition zone between Sichuan Basin and the Tibetan Plateau (TP) large terrain. The mountain-plain solenoid (MPS) circulation west of 102.9° E and vertically-upward motion associated with southwest vortex (SWV) east of 102.9° E coact on and maintain strong updraft near 102.9° E. The low-level convergence induced by these two circulations is helpful in fluid elements upwards stretching.Therefore the associated vorticity grows upwards, and the vortex lines are concentrated based on the theory that vortex line is a material line.

It could be seen that in the real severe convective weathers with vortexes, it is not difficult to find good examples to use the frozen-in property of vorticity to explain the generation and intensifying of vorticity. There is no necessity to derive

vorticity equation, and this method is very intuitive to explore vorticity development qualitatively. In reverse, by investigating the development of vorticity vector, it is visual to locate the associated line element vector and its evolution according to the frozen-in property of vorticity. Because of its simplicity and visualization, it should have broad application prospects to explain the vorticity growth of tornado, squall line, or other severe convective weather with vortex.

5. Summary

Considering some simple topological properties of vorticity vector, the frozen-in property of vorticity herein is reviewed and re-expressed in barotropic fluid in more detail. To relax the limits of incompressible and barotropic atmosphere, it is further derived and proved in the baroclinic case. Then two applications are given as examples from both viewpoints of theoretical and diagnostic analyses. One is the derivation of GPV conservation from the frozen-in property in both barotropic and baroclinic atmospheres, and the other is used to illuminate the vorticity generation and growth in ideal cases and real severe weather processes, e.g., in squall line, tornado, and other severe convection weather with vortex.

By theoretically analyzing the ideal and real cases, it is not difficult to find good examples to illustrate the usage of the frozen-in property of vorticity. There is no necessity to derive vorticity equation, and this method is very intuitive to explore vorticity development qualitatively. Certainly, by investigating the evolution of vortex line, it is possible to locate the associated line element vector and its development on the basis of the frozen-in property of vorticity. Because of its simplicity and visualization, it should have broad application prospects.

References

[1] Tao S Y 1980 *Rainstorms in China* (Beijing: Science Press) p. 225
[2] Fu S M, Li W L, Sun J H, Zhang J P and Zhang Y C 2014 *Atmos. Sci. Lett.* doi:10.1002/asl2. 533 1
[3] Weisman M L 1993 *J. Atmos. Sci.* **50** 645
[4] Weisman M L and Davis C A 1998 *J. Atmos. Sci.* **55** 2603
[5] Atkins N T, Arnott J M, Przybylinski R W, Wolf R A and Ketcham B D 2004 *Mon. Wea. Rev.* **132** 2224
[6] Meng Z Y, Zhang F Q, Markowski P, Wu D C and Zhao K 2012 *J. Atmos. Sci.* **69** 1182
[7] Villis GK 2006 *Atmospheric and oceanic fluid dynamics: fundamentals and large-scale circulation* (Cambridge: Cambridge University Press) p. 168
[8] Yang S and Gao S T 2014 *Chin. Phys. B* **23** 119201
[9] Xu X, Xue M and Wang Y 2015 *J. Atmos. Sci.* **72** 1963
[10] Xu X, Xue M and Wang Y 2015 *Mon. Wea. Rev.* **143** 2266
[11] Ertel H 1942 *Meteorol Z.* **59** 277
[12] Wu G X and Liu H Z 1998 *Dynam. Atmos. Oceans.* **27** 715
[13] Liu Y M, Liu H, Liu P and Wu G X 1999 *Acta Meteorol. Sin.* **57** 385 (in Chinese)
[14] Hoskins B J, McIntyre M E and Robertson A W 1985 *Quart. J. Roy. Meteorol. Soc.* **111** 877
[15] McIntyre M E and Norton W A 1991 *J. Atmos. Sci.* **57** 1214
[16] Montgomery M T and Farrell B F 1993 *J. Atmos. Sci.* **50** 285
[17] Ertel H and Rossby C C 1949 *Pure Appl. Geophys.* **14** 189
[18] Yang S, Gao S T and Chen B 2013 *Sci. China-Earth. Sci.* **43** 1788 (in Chinese)