Noether symmetry and conserved quantity for dynamical system with non-standard Lagrangians on time scales*

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This paper focuses on studying the Noether symmetry and the conserved quantity with non-standard Lagrangians, namely exponential Lagrangians and power-law Lagrangians on time scales. Firstly, for each case, the Hamilton principle based on the action with non-standard Lagrangians on time scales is established, with which the corresponding Euler–Lagrange equation is given. Secondly, according to the invariance of the Hamilton action under the infinitesimal transformation, the Noether theorem for the dynamical system with non-standard Lagrangians on time scales is established. The proof of the theorem consists of two steps. First, it is proved under the infinitesimal transformations of a special one-parameter group without transforming time. Second, utilizing the technique of time-re-parameterization, the Noether theorem in a general form is obtained. The Noether-type conserved quantities with non-standard Lagrangians in both classical and discrete cases are given. Finally, an example in Friedmann–Robertson–Walker spacetime and an example about second order Duffing equation are given to illustrate the application of the results.

Keywords: time scale, non-standard Lagrangian, Noether symmetry, conserved quantity

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1. Introduction

The conservation law of the mechanical system is important not only for mathematics, but also for the profound physical laws. Finding the conserved quantity for a mechanical system is an aspect of the modern development of analytical mechanics. The symmetry and conserved quantity of the dynamical system have been playing an important role in both the basic theory and the specific application. In 1918, by studying the invariance of Hamiltonian action under the infinitesimal transformations of groups, Noether[1] obtained the symmetry and conserved quantity and pointed out that each symmetry had its corresponding conserved quantity, and made a connection among symmetry, invariance and conservation law of physics. It completely broke the traditional way to find conserved quantity. On the pioneer work, great progress in the study of symmetry and conserved quantity has been made in the past century.[2–7]

Non-standard Lagrangians (NSL), entitled non-natural Lagrangians, play an important role in cosmology, physics, dissipative systems, nonlinear dynamics and other fields. Non-standard Lagrangians were discussed recently in some literature and they are special classes of Lagrangians characterized by deformed kinetic energy and potential energy forms. Non-standard Lagrangians may take exponential forms, power-law forms, etc., and may or may not depend on time. In reality, the idea of non-standard Lagrangians started in 1978 but it did not catch scholars' eyes. Since the pioneer work,[8] Alekseev and Arbuvoz[9] have successfully applied the non-standard Lagrangians to the Yang–Mills quantum field theory where non-standard Lagrangians are used to describe large distance interactions in the region of applicability of classical theory, a problem which is directly related to color confinement issue. Nowadays, a series of results has been obtained for non-standard Lagrangians. Dimitrijevic and Milosevic[10] considered two classical examples with non-standard Lagrangians in modern cosmological model, namely Dirac–Born–Infeld (DBI) Lagrangians in Tachyon field and P-adic string theory Lagrangians. El-Nabulsi[11–16] applied non-standard Lagrangians to Friedmann–Robertson–Walker spacetime and discussed its application in quantum mechanics by the modified Hamilton–Jacobi equation and Schrödinger equation, and generalized dynamical systems with higher order derivatives. Musielak[17,18] studied the method of obtaining the non-standard Lagrangians for dissipative system and its existence conditions. On Musielak's work, Cieslinski and Nikiciuk[19] rederived and generalized some of their results and found more non-standard Lagrangian formulations that seemed to be new in the dissipative systems. El-Nabulsi[20–24] studied the action and differential equation of motion with non-standard Lagrangians in nonlinear dynamical systems. Zhang and Zhou,[25] and Zhou and Zhang[26] studied the Noether symmetry with non-standard Lagrangians in nonlinear dynamical systems where the conserved quanti-

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ties were given and the well-known Routh method of reduction was extended to the dynamical system with non-standard Lagrangians which reduced the complexity of this system.

So far there has been a considerable amount of interest in the theory and applications of dynamic derivatives on time scales. The time scale calculus, initiated by Hilger[27] in his PhD thesis, is a relatively new theory, which is not only significant in unifying various concepts in discrete and continuous systems but also crucial in depicting physical essences effectively. In recent years, the time scale calculus has been applied, among others, to mathematics, physics, economics and optimal control. It is nowadays a powerful tool since the excellent work of Bohner and Peterson.[28,29] In 2000, Ahlbrandt *et al.*[30] studied linear and nonlinear Hamiltonian systems on time scales, and unified symplectic flow properties in both discrete and continuous Hamiltonian systems. The second Euler–Lagrange necessary optimality condition was proved for optimal trajectories of variational problems on time scales by Torres *et al.*[31] The Noether theorem of optimal control problem was extended to time scales by Malinowska and Ammi.[32] Bartosiewicz and Torres,[33] Malinowska and Martins[34] showed that there exists a conserved quantity in Lagrangian system for each Noether symmetry on time scales by the technique of time-re-parameterization, with which Zhang[35] and Song and Zhang[36] studied the Noether theorem for Hamiltonian and Birkhoffian systems on time scales. Zu and Zhu[37] obtained the Noether theorem of nonholonomic nonconservative systems in phase space on time scales. Jin and Zhang[38] generalized the Routh and Whittaker methods of reduction to Lagrange systems, and showed that the order of differential equations of motion for the system was reduced by using the cyclic integrals or the generalized energy integrals with nabla derivatives. Atici *et al.*[39] applied the nabla derivative to economics and proposed a dynamic optimization problem, from which the time-scale model was constructed and the variational method was used to derive the solution. However, the research on the dynamical systems with non-standard Lagrangians on time scales has not been studied yet.

In this paper, we further study the Noether symmetry and conserved quantity for dynamical system with non-standard Lagrangians on time scales. The rest of this paper is organized as follows. In Sections 2 and 3, the Euler–Lagrange equations with non-standard Lagrangians on time scales are established. According to invariance of the Hamilton action with non-standard Lagrangians on time scales under the infinitesimal transformations of groups, the Noether theorem is proved by the technique of time-re-parameterization and the conserved quantity in the classical and discrete cases are given. Examples are given to illustrate the application of the results. Finally, some conclusions are drawn in Section 4.

2. Noether theorem for dynamical system with exponential Lagrangians on time scales

2.1. Hamilton action and Euler–Lagrange equations

We assume that the configuration of a mechanical system is determined by n generalized coordinates q_k $(k=1,2,\ldots,n)$. The Hamilton action with exponential Lagrangians on time scales is defined by

$$S[q_k(\cdot)]=\int_a^b \exp L\left(t,q_k^{\sigma}(t),q_k^{\Delta}(t)\right)\Delta t,\quad (q_k(t)\in\Omega;\ k=1,2,\ldots,n), \tag{1}$$

where $q_k^{\sigma}(t)=(q_k\circ\sigma)(t)$, $q_k^{\Delta}(t)$ is the delta derivative. For simplicity, we write $L=L\left(t,q_k^{\sigma},q_k^{\Delta}\right)$ $(k=1,2,\ldots,n)$ instead of $L=L\left(t,q_k^{\sigma}(t),q_k^{\Delta}(t)\right)$, $L:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ is of class C^1 and $L=L\left(t,q_k^{\sigma},q_k^{\Delta}\right)$ is the standard Lagrangian of the system on time scales. $\Omega=\left\{q_k\,|q_k:[a,b]\cap\mathbb{T}\to\mathbb{R},q_k\in C_{\mathrm{rd}}^1\right\}$. In what follows, all intervals are time scale intervals, that is, we simply write $[a,b]$ to denote the set $[a,b]\cap\mathbb{T}$.

The Hamilton principle on time scales is defined by

$$\delta S=0, \tag{2}$$

which satisfies the exchange relationship

$$\delta q_k^{\Delta}=(\delta q_k)^{\Delta}, \tag{3}$$

and the given boundary conditions

$$\delta q_k|_{t=a}=\delta q_k|_{t=b}=0,\quad (k=1,2,\ldots,n). \tag{4}$$

Substituting Eq. (1) into Eq. (2), we obtain

$$\delta S=\int_a^b\left(\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\delta q_k^{\sigma}+\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\delta q_k^{\Delta}\right)\Delta t=0. \tag{5}$$

By Eqs. (3) and (4), we have

$$\begin{aligned}&\int_a^b \exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\delta q_k^{\sigma}\Delta t\\ &=\int_a^b\frac{\Delta}{\Delta t}\left\{\left[\delta q_k\int_a^t\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\Delta\tau\right]-\delta q_k^{\Delta}\int_a^t\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\Delta\tau\right\}\Delta t\\ &=\left[\delta q_k\int_a^t\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\Delta\tau\right]\Big|_a^b-\int_a^b\left[(\delta q_k)^{\Delta}\int_a^t\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\Delta\tau\right]\Delta t\\ &=-\int_a^b\left[(\delta q_k)^{\Delta}\int_a^t\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\Delta\tau\right]\Delta t.\end{aligned} \tag{6}$$

Substituting Eq. (6) into Eq. (5), we obtain

$$\begin{aligned}\delta S&=\int_a^b(\delta q_k)^{\Delta}\left\{\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}-\int_a^t\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\Delta\tau\right\}\Delta t\\ &=0.\end{aligned} \tag{7}$$

By Lemma Dubois–Reymond,[28] we have

$$\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}-\int_a^t \exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\Delta\tau=C,\quad (k=1,2,\ldots,n), \tag{8}$$

with the operations of delta derivative on both sides of Eq. (8), we obtain

$$\frac{\Delta}{\Delta t}\left[\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\right]-\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}=0,\quad (k=1,2,\ldots,n). \tag{9}$$

Equation (9) is called the Euler–Lagrange equations for dynamical system with exponential Lagrangians on time scales.

If $\mathbb{T}=\mathbb{R}$, equation (9) reduces to the classical Euler–Lagrange equations for dynamical system with exponential Lagrangians[25]

$$\exp(L)\left(\frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}L}{\mathrm{d}t}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}\right)=0,\quad (k=1,2,\ldots,n). \tag{10}$$

2.2. Noether symmetry and conserved quantity

We establish the Noether theorem with exponential Lagrangians on time scales. The proof of the theorem consists of two steps. First, it is proved under the infinitesimal transformations of a special one-parameter group without transforming time. Second, utilizing the technique of time-reparameterization, the Noether theorem in a general form is obtained. Let $\mathbb{T}$ be a given time scale. Suppose that $a,b\in\mathbb{T}$ and $a<b$. We just consider one-parameter infinitesimal transformations of q_k

$$\bar{q}_k=q_k+\varepsilon\xi_k(t,q_s), \tag{11}$$

where ε is an infinitesimal parameter and $\xi_k,(k=1,2,\ldots,n)$ are called the infinitesimal generators of the transformation (11). Then, we define the Noether symmetry with exponential Lagrangians for dynamical system on time scales under the infinitesimal transformations (11), and the corresponding conserved quantity is given.

Definition 1 Let U be a set of C_{rd}^1 functions and $q_k:[a,b]\to\mathbb{R}$. A functional S is said to be invariant under transformations (11) if and only if

$$\int_{t_a}^{t_b}\exp L\left(t,q_k^{\sigma},q_k^{\Delta}\right)\Delta t=\int_{t_a}^{t_b}\exp L\left(t,\bar{q}_k^{\sigma},\bar{q}_k^{\Delta}\right)\Delta t \tag{12}$$

holds for any subinterval $[t_a,t_b]\subseteq[a,b]$ with $t_a,t_b\in\mathbb{T}$.

Theorem 1 If functional S is invariant on U under transformations (11), then

$$\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\xi_k^{\sigma}+\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\xi_k^{\Delta}=0 \tag{13}$$

for all $t\in[a,b]$ and $q_k\in U$, where $\xi_k^{\sigma}(t,q_s)=\xi_k(\sigma(t),q_s(\sigma(t)))$, $\xi_k^{\Delta}(t,q_s)=\frac{\Delta}{\Delta t}\xi_k(t,q_s)$.

Proof By Definition 1, we have

$$\int_{t_a}^{t_b}\exp L\left(t,q_k^{\sigma},q_k^{\Delta}\right)\Delta t=\int_{t_a}^{t_b}\exp L\left(t,q_k^{\sigma}+\varepsilon\xi_k^{\sigma},q_k^{\Delta}+\varepsilon\xi_k^{\Delta}\right)\Delta t. \tag{14}$$

Since $[t_a,t_b]$ is any subinterval of $[a,b]$, it follows from formula (14) that

$$\exp L\left(t,q_k^{\sigma},q_k^{\Delta}\right)=\exp L\left(t,q_k^{\sigma}+\varepsilon\xi_k^{\sigma},q_k^{\Delta}+\varepsilon\xi_k^{\Delta}\right). \tag{15}$$

Differentiate both sides of Eq. (15) with the respect to ε and let $\varepsilon=0$, then we will obtain Eq. (13).

Theorem 2 If functional S is invariant on U under transformations (11), then we have the conserved quantity

$$I_N=\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\xi_k. \tag{16}$$

Proof By the derivative formula of the product on time scales,[28] we have

$$\frac{\Delta}{\Delta t}I_N=\frac{\Delta}{\Delta t}\left(\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\right)\xi_k^{\sigma}+\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\xi_k^{\Delta}. \tag{17}$$

Substituting Eq. (9) into Eq. (17), we can derive

$$\frac{\Delta}{\Delta t}I_N=\exp(L)\frac{\partial L}{\partial q_k^{\sigma}}\xi_k^{\sigma}+\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\xi_k^{\Delta}=0. \tag{18}$$

Let us consider the following infinitesimal transformations with respect to the time:

$$\begin{aligned}\bar{t}&=T_{\varepsilon}(t,q_s)=t+\varepsilon\xi_0(t,q_s),\\ \bar{q}_k&=Q_{\varepsilon}^k(t,q_s)=q_k+\varepsilon\xi_k(t,q_s),\end{aligned} \tag{19}$$

where ε is an infinitesimal parameter, and $\xi_0:[a,b]\times\mathbb{R}^n\to\mathbb{R}$ and $\xi_k:[a,b]\times\mathbb{R}^n\to\mathbb{R}$ are delta differentiable functions.

Let the above U be a set of C_{rd}^1 functions and $q_k:[a,b]\to\mathbb{R}$. We assume that for each $q_k\in U$ and ε, the map $t\in[a,b]\mapsto\alpha(t):=T_{\varepsilon}(t,q_s)\in\mathbb{R}$ is an increasing function of C_{rd}^1 and its image is a new time scale with the forward jump operator $\bar{\sigma}$ and the delta derivative $\bar{\Delta}$, then we have the following property:

$$\bar{\sigma}\circ\alpha=\alpha\circ\sigma. \tag{20}$$

Definition 2 A functional S is said to be invariant on U under transformations (19) if and only if

$$\int_{t_a}^{t_b}\exp L(t,q_k^{\sigma},q_k^{\Delta})\Delta t=\int_{\alpha(t_a)}^{\alpha(t_b)}\exp L(\bar{t},\bar{q}_k^{\bar{\sigma}},\bar{q}_k^{\bar{\Delta}})\bar{\Delta}\bar{t} \tag{21}$$

holds for any subinterval $[t_a,t_b]\subseteq[a,b]$.

Theorem 3 If functional S is invariant on U under transformations (19), then we have

$$\exp(L)\left[\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_k^{\sigma}}\xi_k^{\sigma}+\frac{\partial L}{\partial q_k^{\Delta}}\left(\xi_k^{\Delta}-q_k^{\Delta}\xi_0^{\Delta}\right)+\xi_0^{\Delta}\right]=0. \tag{22}$$

Equation (22) is called the Noether identity for dynamical system with exponential Lagrangians on time scales.

Proof By Definition 2, we have

$$\begin{aligned}&\int_{t_a}^{t_b}\exp L\left(t,q_k^{\sigma},q_k^{\Delta}\right)\Delta t=\int_{\alpha(t_a)}^{\alpha(t_b)}\exp L\left(\bar{t},\bar{q}_k^{\bar{\sigma}},\bar{q}_k^{\bar{\Delta}}\right)\bar{\Delta t}\\&=\int_{t_a}^{t_b}\exp L\left(\alpha(t),\bar{q}_k^{\bar{\sigma}}(\alpha(t)),\frac{(\bar{q}_k\circ\alpha)^{\Delta}(t)}{\alpha^{\Delta}(t)}\right)\alpha^{\Delta}(t)\Delta t\\&=\int_{t_a}^{t_b}\exp L\left(T_{\varepsilon},\left(Q_{\varepsilon}^{k}\right)^{\sigma},\frac{\left(Q_{\varepsilon}^{k}\right)^{\Delta}}{T_{\varepsilon}^{\Delta}}\right)T_{\varepsilon}^{\Delta}\Delta t.\end{aligned}\tag{23}$$

Since $[t_a,t_b]$ is any subinterval of $[a,b]$, we can obtain

$$\exp L\left(t,q_k^{\sigma},q_k^{\Delta}\right)=\exp L\left(T_{\varepsilon},\left(Q_{\varepsilon}^{k}\right)^{\sigma},\frac{\left(Q_{\varepsilon}^{k}\right)^{\Delta}}{T_{\varepsilon}^{\Delta}}\right)T_{\varepsilon}^{\Delta}.\tag{24}$$

Differentiating both sides of Eq. (24) with respect to ε, we have

$$\begin{aligned}0=&\exp L\left(T_{\varepsilon},\left(Q_{\varepsilon}^{k}\right)^{\sigma},\frac{\left(Q_{\varepsilon}^{k}\right)^{\Delta}}{T_{\varepsilon}^{\Delta}}\right)\\&\times\left[\frac{\partial L}{\partial T_{\varepsilon}}\xi_0+\frac{\partial L}{\partial\left(Q_{\varepsilon}^{k}\right)^{\sigma}}\xi_k^{\sigma}+\frac{\partial L}{\partial\left(\left(Q_{\varepsilon}^{k}\right)^{\Delta}/T_{\varepsilon}^{\Delta}\right)}\right.\\&\left.\times\frac{\xi_k^{\Delta}\left(1+\varepsilon\xi_0^{\Delta}\right)-\left(q_k^{\Delta}+\varepsilon\xi_k^{\Delta}\right)\xi_0^{\Delta}}{\left(1+\varepsilon\xi_0^{\Delta}\right)^2}\right]\\&\times\left(1+\varepsilon\xi_0^{\Delta}\right)+\exp L\left(T_{\varepsilon},\left(Q_{\varepsilon}^{k}\right)^{\sigma},\frac{\left(Q_{\varepsilon}^{k}\right)^{\Delta}}{T_{\varepsilon}^{\Delta}}\right)\xi_0^{\Delta}.\end{aligned}\tag{25}$$

Let $\varepsilon=0$, then we will obtain

$$0=\exp(L)\left[\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_k^{\sigma}}\xi_k^{\sigma}+\frac{\partial L}{\partial q_k^{\Delta}}\left(\xi_k^{\Delta}-q_k^{\Delta}\xi_0^{\Delta}\right)+\xi_0^{\Delta}\right].$$

Theorem 4 If functional S is invariant on U under transformations (19), then

$$\begin{aligned}I_N=&\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\xi_k+\exp(L)\left[1-\frac{\partial L}{\partial t}\mu(t)-\frac{\partial L}{\partial q_k^{\Delta}}q_k^{\Delta}\right]\xi_0\\=&\text{ const.},\end{aligned}\tag{26}$$

which is a conserved quantity for dynamical system with exponential Lagrangians on time scales.

Proof Denote

$$\exp\bar{L}(t;s,q;r,v):=\exp L\left(s-\mu(t)r,q,\frac{v}{r}\right)r,\tag{27}$$

where $t\in[a,b],q,v\in\mathbb{R}^n,s,r\in\mathbb{R},r\neq0$. Observe that when $s(t)=t,q:[a,b]\to\mathbb{R}^n$,

$$\exp L\left(t,q_k^{\sigma},q_k^{\Delta}\right)=\exp\bar{L}\left(t;s^{\sigma},q_k^{\sigma};s^{\sigma},q_k^{\Delta}\right)\tag{28}$$

and

$$\bar{S}[s(\cdot),q_k(\cdot)]:=\int_a^b\exp\bar{L}\left(t;s^{\sigma},q_k^{\sigma};s^{\sigma},q_k^{\Delta}\right)\Delta t,\tag{29}$$

then we will have

$$S[q_k(\cdot)]=\bar{S}[s(\cdot),q_k(\cdot)].\tag{30}$$

By Definition 2, for $s(t)=t$, we have

$$\begin{aligned}&\bar{S}[s(\cdot),q_k(\cdot)]\\=&S[q_k(\cdot)]\\=&\int_a^b\exp L\left(t,q_k^{\sigma},q_k^{\Delta}\right)\Delta t=\int_{\alpha(a)}^{\alpha(b)}\exp L\left(\bar{t},\bar{q}_k\bar{\sigma},\bar{q}_k^{\bar{\Delta}}\right)\bar{\Delta t}\\=&\int_a^b\exp L\left(\alpha(t),\bar{q}_k^{\bar{\sigma}}(\alpha(t)),\frac{(\bar{q}_k\circ\alpha)^{\Delta}(t)}{\alpha^{\Delta}(t)}\right)\alpha^{\Delta}(t)\Delta t\\=&\int_a^b\exp L\left(\alpha^{\sigma}(t)-\mu(t)\alpha^{\Delta}(t),(\bar{q}_k\circ\alpha)^{\sigma}(t),\right.\\&\left.\frac{(\bar{q}_k\circ\alpha)^{\Delta}(t)}{\alpha^{\Delta}(t)}\right)\alpha^{\Delta}(t)\Delta t\\=&\int_a^b\exp\bar{L}\left(t;\alpha^{\sigma}(t),(\bar{q}_k\circ\alpha)^{\sigma}(t);\alpha^{\Delta}(t),(\bar{q}_k\circ\alpha)^{\Delta}(t)\right)\Delta t\\=&\bar{S}[\alpha(\cdot),(\bar{q}_k\circ\alpha)(\cdot)].\end{aligned}\tag{31}$$

Also, for $s(t)=t$, we have

$$\begin{aligned}(\alpha(t),(\bar{q}_k\circ\alpha)(t))=&(T_{\varepsilon}(t,q_k),Q_{\varepsilon}(t,q_k))\\=&(T_{\varepsilon}(s(t),q_k),Q_{\varepsilon}(s(t),q_k)).\end{aligned}\tag{32}$$

By Definition 1, we observe that $\bar{S}$ is invariant on $\bar{U}=\{(s,q_k)|s(t)=t,q_k\in U\}$ under transformations (19). Applying Theorem 2 for $s(t)=t$, we have

$$\begin{aligned}I_N=&\frac{\partial}{\partial q_k^{\Delta}}\exp\bar{L}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\xi_k\\&+\frac{\partial}{\partial s^{\Delta}}\exp\bar{L}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\xi_0.\end{aligned}\tag{33}$$

Since

$$\begin{aligned}&\frac{\partial}{\partial q_k^{\Delta}}\exp\bar{L}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\\=&\exp(L)\frac{\partial L}{\partial q_k^{\Delta}}\frac{1}{s^{\Delta}}s^{\Delta}=\exp(L)\frac{\partial L}{\partial q_k^{\Delta}},\end{aligned}\tag{34}$$

$$\begin{aligned}&\frac{\partial}{\partial s^{\Delta}}\exp\bar{L}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\\=&-\exp(L)\frac{\partial L}{\partial\left(s^{\sigma}-\mu(t)s^{\Delta}\right)}\mu(t)s^{\Delta}\\&-\exp(L)\frac{\partial L}{\partial\frac{q_k^{\Delta}}{s^{\Delta}}}\frac{q_k^{\Delta}}{\left(s^{\Delta}\right)^2}s^{\Delta}+\exp(L)\\=&\exp(L)-\exp(L)\frac{\partial L}{\partial\left(s^{\sigma}-\mu(t)s^{\Delta}\right)}\mu(t)s^{\Delta}\\&-\exp(L)\frac{\partial L}{\partial\frac{q_k^{\Delta}}{s^{\Delta}}}\frac{q_k^{\Delta}}{s^{\Delta}}.\end{aligned}\tag{35}$$

For $s(t)=t$, we have

$$\begin{aligned}&\frac{\partial}{\partial q_k^{\Delta}}\exp\bar{L}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)=\exp(L)\frac{\partial L}{\partial q_k^{\Delta}},\\&\frac{\partial}{\partial s^{\Delta}}\exp\bar{L}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\end{aligned}\tag{36}$$

$$= \exp(L) - \exp(L)\frac{\partial L}{\partial t}\mu(t) - \exp(L)\frac{\partial L}{\partial q_k^{\Delta}}q_k^{\Delta}. \quad (37)$$

Substituting Eqs. (36) and (37) into Eq. (33), we obtain the intended conclusion.

By the above theorems, we obtain the following results.

Corollary 1 If we take $\mathbb{T} = \mathbb{R}$, equation (22) reduces to the classical Noether identity with exponential Lagrangians[25]

$$\exp(L)\left[\frac{\partial L}{\partial t}\xi_0 + \frac{\partial L}{\partial q_k}\xi_k + \frac{\partial L}{\partial \dot{q}_k}\left(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0\right) + \dot{\xi}_0\right] = 0. \quad (38)$$

And the conserved quantity (26) reduces to the classical one[25]

$$I_N = \exp(L)\,\xi_0 + \exp(L)\frac{\partial L}{\partial \dot{q}_k}(\xi_k - \dot{q}_k\xi_0) = \text{const}. \quad (39)$$

Corollary 2 If we take $\mathbb{T} = \mathbb{Z}$, from Eq. (22), we have

$$\begin{aligned}&\exp(L)\left\{\frac{\partial L(t)}{\partial t}\xi_0 + \frac{\partial L(t)}{\partial q_k(t+1)}\xi_k(t+1)\right.\\&+\frac{\partial L(t)}{\partial\left(q_k(t+1)-q_k(t)\right)}\left[\xi_k(t+1)-\xi_k(t)\right.\\&-\left(q_k(t+1)-q_k(t)\right)\left(\xi_0(t+1)-\xi_0(t)\right)\left.\right]\\&\left.+\left(\xi_0(t+1)-\xi_0(t)\right)\right\} = 0. \end{aligned} \quad (40)$$

From Eq. (26), we have

$$\begin{aligned}I_N =\ & \exp(L)\frac{\partial L(t)}{\partial\left(q_k(t+1)-q_k(t)\right)}\xi_k + \exp(L)\,\xi_0\\&\times\left\{1 - \frac{\partial L(t)}{\partial t} - \frac{\partial L(t)}{\partial\left(q_k(t+1)-q_k(t)\right)}\right.\\&\left.\times\left[q_k(t+1)-q_k(t)\right]\right\} = \text{const}. \end{aligned} \quad (41)$$

Equations (40) and (41) are called the discrete Noether identity and conserved quantity with exponential Lagrangians, respectively.

2.3. Example

Consider the dynamical system with a certain exponential Lagrangian in Friedmann–Robertson–Walker spacetime. The Hamilton action is[12]

$$\begin{aligned}S =\ & \int_a^b \exp(L)\,\Delta t\\ =\ & \int_a^b \exp\left(\frac{6q^{\sigma}\left(q^{\Delta}\right)^2}{N} + 16\pi GN\rho\left(q^{\sigma}\right)^3\right)\Delta t, \end{aligned} \quad (42)$$

where L is the Arnowitt–Deser–Misner (ADM) Lagrangian, G is the gravitational coupling constant, ρ is the energy density of the barotropic fluid, $q(t)$ is the scale factor, and N is the lapse function. When $N = 1$, from the Euler–Lagrange equations for dynamical system with exponential Lagrangians on time scales (Eq. (9)), we have

$$\begin{aligned}&\frac{\Delta}{\Delta t}\left[2q^{\sigma}q^{\Delta}\exp\left(6q^{\sigma}\left(q^{\Delta}\right)^2 + 16\pi G\rho\left(q^{\sigma}\right)^3\right)\right]\\&-\left(\left(q^{\Delta}\right)^2 + 8\pi G\rho\left(q^{\sigma}\right)^2\right)\\&\times\exp\left(6q^{\sigma}\left(q^{\Delta}\right)^2 + 16\pi G\rho\left(q^{\sigma}\right)^3\right) = 0. \end{aligned} \quad (43)$$

From the Noether identity for dynamical system with exponential Lagrangians on time scales (22), we have

$$\begin{aligned}&\exp\left(6q^{\sigma}\left(q^{\Delta}\right)^2 + 16\pi G\rho\left(q^{\sigma}\right)^3\right)\\&\times\left[\left(6\left(q^{\Delta}\right)^2 + 48\pi G\rho\left(q^{\sigma}\right)^2\right)\xi^{\sigma}\right.\\&\left.+12q^{\sigma}q^{\Delta}\left(\xi^{\Delta} - q^{\Delta}\xi_0^{\Delta}\right) + \xi_0^{\Delta}\right] = 0. \end{aligned} \quad (44)$$

Taking calculation, we obtain that

$$\xi_0 = -1, \quad \xi = 0 \quad (45)$$

satisfies formula (44). Generator (45) corresponds to the Noether symmetry with exponential Lagrangians on time scales.

Finally, using Theorem 4, we obtain that the system has the following conserved quantity

$$\begin{aligned}I_N =\ & (12q^{\sigma}(q^{\Delta})^2 - 1)\exp(6q^{\sigma}(q^{\Delta})^2 + 16\pi G\rho(q^{\sigma})^3)\\ =\ & \text{const}. \end{aligned} \quad (46)$$

If we take $\mathbb{T} = \mathbb{R}$, by Theorem 4, we obtain the classical conserved quantity

$$\begin{aligned}I_N =\ & \left(12q\dot{q}^2 - 1\right)\exp\left(6q\dot{q}^2 + 16\pi G\rho q^3\right)\\ =\ & \text{const}. \end{aligned} \quad (47)$$

If we take $\mathbb{T} = \mathbb{Z}$, by Theorem 4, we obtain the discrete conserved quantity

$$\begin{aligned}I_N =\ & \left[12q(t+1)\left(q(t+1)-q(t)\right)^2 - 1\right]\\&\times\exp\left[6q(t+1)\left(q(t+1)-q(t)\right)^2 + 16\pi G\rho\left(q(t+1)\right)^3\right]\\ =\ & \text{const}. \end{aligned} \quad (48)$$

3. Noether theorem for dynamical system with power-law Lagrangians on time scales

3.1. Hamilton action and Euler–Lagrange equations

The Hamilton action with power-law Lagrangians on time scales is defined by

$$\begin{aligned}&A[q_k(\cdot)] = \int_a^b L^{1+\alpha}\left(t, q_k^{\sigma}, q_k^{\Delta}\right)\Delta t,\\&(q_k \in \Omega;\ \ k = 1, 2, \ldots, n). \end{aligned} \quad (49)$$

The Hamilton principle on time scales is defined by

$$\delta A = 0, \quad (50)$$

which satisfies the exchange relationship (Eq.(3)) and the given boundary conditions (Eq. (4)).

Substituting Eq. (49) into Eq. (50), we obtain

$$\delta A = \int_a^b \left[(1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\delta q_k^\sigma + (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta}\delta q_k^\Delta\right]\Delta t = 0. \tag{51}$$

By Eqs. (3) and (4), we have

$$\begin{aligned}
&\int_a^b (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\delta q_k^\sigma \Delta t \\
&= \int_a^b \left\{\frac{\Delta}{\Delta t}\left[\delta q_k \int_a^t (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\Delta\tau\right]\right. \\
&\quad \left. - \delta q_k^\Delta \int_a^t (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\Delta\tau\right\}\Delta t \\
&= \left[\delta q_k \int_a^t (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\Delta\tau\right]\Big|_a^b \\
&\quad - \int_a^b \left[(\delta q_k)^\Delta \int_a^t (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\Delta\tau\right]\Delta t \\
&= -\int_a^b \left[(\delta q_k)^\Delta \int_a^t (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\Delta\tau\right]\Delta t.
\end{aligned} \tag{52}$$

Substituting Eq. (52) into Eq. (51), we obtain

$$\begin{aligned}
\delta A = &\int_a^b (\delta q_k)^\Delta \left\{(1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta}\right. \\
&\left. - \int_a^t (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\Delta\tau\right\}\Delta t = 0.
\end{aligned} \tag{53}$$

By Lemma Dubois–Reymond,[28] we have

$$(1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta} - \int_a^t (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\Delta\tau = C, \quad (k=1,2,\ldots,n). \tag{54}$$

With the operations of delta derivative on both sides of Eq. (54), we obtain

$$\frac{\Delta}{\Delta t}\left[(1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta}\right] - (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma} = 0, \quad (k=1,2,\ldots,n). \tag{55}$$

Equation (55) is called the Euler–Lagrange equations for dynamical system with power-law Lagrangians on time scales. When $\alpha = 0$, equation (55) reduces to the classical Euler–Lagrange equations on time scales.[33]

If $\mathbb{T} = \mathbb{R}$, equation (55) reduces to the classical Euler–Lagrange equations for dynamical system with power-law Lagrangians[25]

$$(1+\alpha)L^\alpha \left(\frac{\alpha}{L}\frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}L}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}\right) = 0, \quad (k=1,2,\ldots,n). \tag{56}$$

3.2. Noether symmetry and conserved quantity

We establish the Noether theorem with power-law Lagrangians on time scales. We define the Noether symmetry with power-law Lagrangians for dynamical system on time scales under one-parameter infinitesimal transformations (Eq. (11)), and the corresponding conserved quantity is given.

Definition 3 Let U be a set of C_{rd}^1 functions and $q_k : [a,b] \to \mathbb{R}$. A functional A is said to be invariant under transformations (11) if and only if

$$\int_{t_a}^{t_b} L^{1+\alpha}\left(t, q_k^\sigma, q_k^\Delta\right)\Delta t = \int_{t_a}^{t_b} L^{1+\alpha}\left(t, \bar{q}_k^\sigma, \bar{q}_k^\Delta\right)\Delta t \tag{57}$$

holds for any subinterval $[t_a,t_b] \subseteq [a,b]$ with $t_a, t_b \in \mathbb{T}$.

Theorem 5 If functional A is invariant on U under transformations (11), then

$$(1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\xi_k^\sigma + (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta}\xi_k^\Delta = 0 \tag{58}$$

for all $t \in [a,b]$ and $q_k \in U$, where

$$\xi_k^\sigma(t,q_s) = \xi_k(\sigma(t), q_s(\sigma(t))), \quad \xi_k^\Delta(t,q_s) = \frac{\Delta}{\Delta t}\xi_k(t,q_s).$$

Proof By Definition 3, we have

$$\begin{aligned}
&\int_{t_a}^{t_b} L^{1+\alpha}\left(t, q_k^\sigma, q_k^\Delta\right)\Delta t \\
&= \int_{t_a}^{t_b} L^{1+\alpha}\left(t, q_k^\sigma + \varepsilon\xi_k^\sigma, q_k^\Delta + \varepsilon\xi_k^\Delta\right)\Delta t.
\end{aligned} \tag{59}$$

Since $[t_a,t_b]$ is any subinterval of $[a,b]$, it follows from formula (14) that

$$L^{1+\alpha}\left(t, q_k^\sigma, q_k^\Delta\right) = L^{1+\alpha}\left(t, q_k^\sigma + \varepsilon\xi_k^\sigma, q_k^\Delta + \varepsilon\xi_k^\Delta\right). \tag{60}$$

Differentiate both sides of Eq. (60) with the respect to ε and let $\varepsilon = 0$, then we will obtain Eq. (58).

Theorem 6 If functional A is invariant on U under transformations (11), then we have the conserved quantity

$$I_N = (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta}\xi_k. \tag{61}$$

Proof By the derivative formula of the product on time scales,[28] we have

$$\frac{\Delta}{\Delta t}I_N = \frac{\Delta}{\Delta t}\left((1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta}\right)\xi_k^\sigma + (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta}\xi_k^\Delta. \tag{62}$$

Substituting Eq. (55) into Eq. (62), we can derive

$$\frac{\Delta}{\Delta t}I_N = (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\sigma}\xi_k^\sigma + (1+\alpha)L^\alpha \frac{\partial L}{\partial q_k^\Delta}\xi_k^\Delta = 0. \tag{63}$$

Now, we come to consider the infinitesimal transformations (Eq. (19)).

Definition 4 A functional A is said to be invariant on U under the transformations (19) if and only if

$$\int_{t_a}^{t_b} L^{1+\alpha}\left(t, q_k^\sigma, q_k^\Delta\right)\Delta t = \int_{\alpha(t_a)}^{\alpha(t_b)} L^{1+\alpha}\left(\bar{t}, \bar{q}_k^{\bar{\sigma}}, \bar{q}_k^{\bar{\Delta}}\right)\bar{\Delta}\bar{t} \tag{64}$$

holds for any subinterval $[t_a,t_b]\subseteq[a,b]$.

Theorem 7 If functional A is invariant on U under transformations (19), then we have

$$(1+\alpha)L^{\alpha}\left[\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_k^{\sigma}}\xi_k^{\sigma}+\frac{\partial L}{\partial q_k^{\Delta}}\left(\xi_k^{\Delta}-q_k^{\Delta}\xi_0^{\Delta}\right)\right]+L^{1+\alpha}\xi_0^{\Delta}=0. \tag{65}$$

Equation (65) is called the Noether identity for dynamical systems with power-law Lagrangians on time scales.

Proof By Definition 4, we have

$$\begin{aligned}&\int_{t_a}^{t_b}L^{1+\alpha}\left(t,q_k^{\sigma},q_k^{\Delta}\right)\Delta t=\int_{\alpha(t_a)}^{\alpha(t_b)}L^{1+\alpha}\left(\bar{t},\bar{q}_k^{\bar{\sigma}},\bar{q}_k^{\bar{\Delta}}\right)\bar{\Delta}\bar{t}\\&=\int_{t_a}^{t_b}L^{1+\alpha}\left(\alpha(t),\bar{q}_k^{\bar{\sigma}}(\alpha(t)),\frac{(\bar{q}_k\circ\alpha)^{\Delta}(t)}{\alpha^{\Delta}(t)}\right)\alpha^{\Delta}(t)\Delta t\\&=\int_{t_a}^{t_b}L^{1+\alpha}\left(T_{\varepsilon},\left(Q_{\varepsilon}^k\right)^{\sigma},\frac{\left(Q_{\varepsilon}^k\right)^{\Delta}}{T_{\varepsilon}^{\Delta}}\right)T_{\varepsilon}^{\Delta}\Delta t.\end{aligned} \tag{66}$$

Since $[t_a,t_b]$ is any subinterval of $[a,b]$, we can obtain

$$L^{1+\alpha}\left(t,q_k^{\sigma},q_k^{\Delta}\right)=L^{1+\alpha}\left(T_{\varepsilon},\left(Q_{\varepsilon}^k\right)^{\sigma},\frac{\left(Q_{\varepsilon}^k\right)^{\Delta}}{T_{\varepsilon}^{\Delta}}\right)T_{\varepsilon}^{\Delta}. \tag{67}$$

Differentiating both sides of Eq. (67) with respect to ε, we have

$$\begin{aligned}0=&(1+\alpha)L^{\alpha}\left(T_{\varepsilon},\left(Q_{\varepsilon}^k\right)^{\sigma},\left(Q_{\varepsilon}^k\right)^{\Delta}/T_{\varepsilon}^{\Delta}\right)\\&\times\left[\frac{\partial L}{\partial T_{\varepsilon}}\xi_0+\frac{\partial L}{\partial\left(Q_{\varepsilon}^k\right)^{\sigma}}\xi_k^{\sigma}+\frac{\partial L}{\partial\left(\left(Q_{\varepsilon}^k\right)^{\Delta}/T_{\varepsilon}^{\Delta}\right)}\right.\\&\left.\times\frac{\xi_k^{\Delta}\left(1+\varepsilon\xi_0^{\Delta}\right)-\left(q_k^{\Delta}+\varepsilon\xi_k^{\Delta}\right)\xi_0^{\Delta}}{\left(1+\varepsilon\xi_0^{\Delta}\right)^2}\right]\\&\times\left(1+\varepsilon\xi_0^{\Delta}\right)+L^{1+\alpha}\left(T_{\varepsilon},\left(Q_{\varepsilon}^k\right)^{\sigma},\left(Q_{\varepsilon}^k\right)^{\Delta}/T_{\varepsilon}^{\Delta}\right)\xi_0^{\Delta}.\end{aligned} \tag{68}$$

Let $\varepsilon=0$, then we will obtain

$$\begin{aligned}0=&(1+\alpha)L^{\alpha}\left[\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_k^{\sigma}}\xi_k^{\sigma}+\frac{\partial L}{\partial q_k^{\Delta}}\left(\xi_k^{\Delta}-q_k^{\Delta}\xi_0^{\Delta}\right)\right]\\&+L^{1+\alpha}\xi_0^{\Delta}.\end{aligned}$$

Theorem 8 If functional A is invariant on U under transformations (19), then

$$\begin{aligned}I_N=&L^{1+\alpha}\xi_0+(1+\alpha)L^{\alpha}\left[\frac{\partial L}{\partial q_k^{\Delta}}\left(\xi_k-q_k^{\Delta}\xi_0\right)-\frac{\partial L}{\partial t}\mu(t)\xi_0\right]\\=&\text{ const.},\end{aligned} \tag{69}$$

which is a conserved quantity for dynamical system with power-law Lagrangians on time scales.

Proof Denoting

$$\tilde{L}^{1+\alpha}(t;s,q;r,v):=L^{1+\alpha}\left(s-\mu(t)r,q,\frac{v}{r}\right)r, \tag{70}$$

where $t\in[a,b],q,v\in\mathbb{R}^n,s,r\in\mathbb{R},$ *and* $r\neq0$. Observe that when $s(t)=t,q:[a,b]\to\mathbb{R}^n$,

$$L^{1+\alpha}\left(t,q_k^{\sigma},q_k^{\Delta}\right)=\tilde{L}^{1+\alpha}\left(t;s^{\sigma},q_k^{\sigma};s^{\sigma},q_k^{\Delta}\right) \tag{71}$$

and

$$\tilde{A}[s(\cdot),q_k(\cdot)]:=\int_a^b\tilde{L}^{1+\alpha}\left(t;s^{\sigma},q_k^{\sigma};s^{\sigma},q_k^{\Delta}\right)\Delta t, \tag{72}$$

then we will have

$$A[q_k(\cdot)]=\tilde{A}[s(\cdot),q_k(\cdot)]. \tag{73}$$

By Definition 4, for $s(t)=t$, we have

$$\begin{aligned}&\tilde{A}[s(\cdot),q_k(\cdot)]=A[q_k(\cdot)]=\int_a^bL^{1+\alpha}\left(t,q_k^{\sigma},q_k^{\Delta}\right)\Delta t\\&=\int_{\alpha(a)}^{\alpha(b)}L^{1+\alpha}\left(\bar{t},\bar{q}_k^{\bar{\sigma}},\bar{q}_k^{\bar{\Delta}}\right)\bar{\Delta}\bar{t}\\&=\int_a^bL^{1+\alpha}\left(\alpha(t),\bar{q}_k^{\bar{\sigma}}(\alpha(t)),\frac{(\bar{q}_k\circ\alpha)^{\Delta}(t)}{\alpha^{\Delta}(t)}\right)\alpha^{\Delta}(t)\Delta t\\&=\int_a^bL^{1+\alpha}\left(\alpha^{\sigma}(t)-\mu(t)\alpha^{\Delta}(t),(\bar{q}_k\circ\alpha)^{\sigma}(t),\right.\\&\left.\frac{(\bar{q}_k\circ\alpha)^{\Delta}(t)}{\alpha^{\Delta}(t)}\right)\alpha^{\Delta}(t)\Delta t\\&=\int_a^b\tilde{L}^{1+\alpha}\left(t;\alpha^{\sigma}(t),(\bar{q}_k\circ\alpha)^{\sigma}(t);\alpha^{\Delta}(t),(\bar{q}_k\circ\alpha)^{\Delta}(t)\right)\Delta t\\&=\tilde{A}[\alpha(\cdot),(\bar{q}_k\circ\alpha)(\cdot)].\end{aligned} \tag{74}$$

Also, for $s(t)=t$, we have Eq. (32). By Definition 3, we observe that $\tilde{A}$ is invariant on $\bar{U}=\{(s,q_k)|\,s(t)=t,q_k\in U\}$ under transformations (19). Using Theorem 6 for $s(t)=t$, we have

$$\begin{aligned}I_N=&\frac{\partial}{\partial q_k^{\Delta}}\tilde{L}^{1+\alpha}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\xi_k\\&+\frac{\partial}{\partial s^{\Delta}}\tilde{L}^{1+\alpha}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\xi_0,\end{aligned} \tag{75}$$

since

$$\begin{aligned}&\frac{\partial}{\partial q_k^{\Delta}}\tilde{L}^{1+\alpha}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\\&=(1+\alpha)L^{\alpha}\frac{\partial L}{\partial q_k^{\Delta}}\frac{1}{s^{\Delta}}s^{\Delta}=(1+\alpha)L^{\alpha}\frac{\partial L}{\partial q_k^{\Delta}},\end{aligned} \tag{76}$$

$$\begin{aligned}&\frac{\partial}{\partial s^{\Delta}}\tilde{L}^{1+\alpha}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)\\&=-(1+\alpha)L^{\alpha}\frac{\partial L}{\partial\left(s^{\sigma}-\mu(t)s^{\Delta}\right)}\mu(t)s^{\Delta}\\&\quad-(1+\alpha)L^{\alpha}\frac{\partial L}{\partial\frac{q_k^{\Delta}}{s^{\Delta}}}\frac{q_k^{\Delta}}{\left(s^{\Delta}\right)^2}s^{\Delta}+L^{1+\alpha}\\&=L^{1+\alpha}-(1+\alpha)L^{\alpha}\frac{\partial L}{\partial\left(s^{\sigma}-\mu(t)s^{\Delta}\right)}\mu(t)s^{\Delta}\\&\quad-(1+\alpha)L^{\alpha}\frac{\partial L}{\partial\frac{q_k^{\Delta}}{s^{\Delta}}}\frac{q_k^{\Delta}}{s^{\Delta}}.\end{aligned} \tag{77}$$

For $s(t)=t$, we have

$$\frac{\partial}{\partial q_k^{\Delta}}\tilde{L}^{1+\alpha}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)=(1+\alpha)L^{\alpha}\frac{\partial L}{\partial q_k^{\Delta}}, \quad (78)$$

$$\frac{\partial}{\partial s^{\Delta}}\tilde{L}^{1+\alpha}\left(t;s^{\sigma},q_k^{\sigma};s^{\Delta},q_k^{\Delta}\right)$$
$$=L^{1+\alpha}-(1+\alpha)L^{\alpha}\frac{\partial L}{\partial t}\mu(t)-(1+\alpha)L^{\alpha}\frac{\partial L}{\partial q_k^{\Delta}}q_k^{\Delta}. \quad (79)$$

Substituting Eqs. (78) and (79) into Eq. (75), we obtain the intended conclusion.

By the theorems above, we obtain the following results.

Corollary 3 If we take $\mathbb{T}=\mathbb{R}$, equation (65) reduces to the classical Noether identity with power-law Lagrangians[25]

$$(1+\alpha)L^{\alpha}\left[\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_k}\xi_k+\frac{\partial L}{\partial \dot{q}_k}\left(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0\right)\right]+L^{1+\alpha}\dot{\xi}_0=0. \quad (80)$$

The conserved quantity (69) reduces to the classical one[25]

$$I_N=L^{1+\alpha}\xi_0+(1+\alpha)L^{\alpha}\frac{\partial L}{\partial \dot{q}_k}(\xi_k-\dot{q}_k\xi_0)=\text{const}. \quad (81)$$

Corollary 4 If we take $\mathbb{T}=\mathbb{Z}$, from Eq. (67), we have

$$\begin{aligned}&(1+\alpha)L^{\alpha}\left\{\frac{\partial L(t)}{\partial t}\xi_0+\frac{\partial L(t)}{\partial q_k(t+1)}\xi_k(t+1)\right.\\&+\frac{\partial L(t)}{\partial\left(q_k(t+1)-q_k(t)\right)}\\&\times\left[\xi_k(t+1)-\xi_k(t)-\left(q_k(t+1)-q_k(t)\right)\right.\\&\left.\left.\times\left(\xi_0(t+1)-\xi_0(t)\right)\right]\right\}\\&+L^{1+\alpha}\left[\xi_0(t+1)-\xi_0(t)\right]=0. \quad (82)\end{aligned}$$

From Eq. (71), we have

$$\begin{aligned}I_N&=L^{1+\alpha}\xi_0+(1+\alpha)L^{\alpha}\left\{\frac{\partial L(t)}{\partial\left(q_k(t+1)-q_k(t)\right)}\right.\\&\left.\times\left[\xi_k-\left(q_k(t+1)-q_k(t)\right)\xi_0\right]-\frac{\partial L}{\partial t}\xi_0\right\}\\&=\text{const}. \quad (83)\end{aligned}$$

Equations (82) and (83) are called the discrete Noether identity and conserved quantity with power-law Lagrangians respectively.

3.3. Example

Consider the Duffing equation on time scales. The Hamilton action is

$$A=\int_a^b\frac{1}{q^{\Delta}+(q^{\sigma})^2}\Delta t. \quad (84)$$

From the Euler–Lagrange equations for dynamical system with power-law Lagrangians on time scales (55), we have

$$-\frac{\Delta}{\Delta t}\left[\frac{1}{(q^{\Delta}+(q^{\sigma})^2)^2}\right]+\frac{2q^{\sigma}}{(q^{\Delta}+(q^{\sigma})^2)^2}=0. \quad (85)$$

From the Noether identity for dynamical system with power-law Lagrangians on time scales (65), we have

$$-\frac{2q^{\sigma}\xi^{\sigma}+\xi^{\Delta}-q^{\Delta}\xi_0^{\Delta}}{\left(q^{\Delta}+(q^{\sigma})^2\right)^2}+\frac{\xi_0^{\Delta}}{q^{\Delta}+(q^{\sigma})^2}=0. \quad (86)$$

Performing calculation, we obtain that

$$\xi_0=1, \quad \xi=0 \quad (87)$$

satisfies formula (86). Generator (87) corresponds to the Noether symmetry with power-law Lagrangians on time scales.

Finally, by using Theorem 8 we obtain that the system has the following conserved quantity:

$$I_N=\frac{2q^{\Delta}+(q^{\sigma})^2}{\left(q^{\Delta}+(q^{\sigma})^2\right)^2}=\text{const}. \quad (88)$$

If we take $\mathbb{T}=\mathbb{R}$, by Theorem 8, we obtain the classical conserved quantity

$$I_N=\frac{2\dot{q}+q^2}{(\dot{q}+q^2)^2}=\text{const}. \quad (89)$$

If we take $\mathbb{T}=\mathbb{Z}$, by Theorem 8, we obtain the discrete conserved quantity

$$I_N=\frac{2\left(q(t+1)-q(t)\right)+\left(q(t+1)\right)^2}{\left(q(t+1)-q(t)+\left(q(t+1)\right)^2\right)^2}=\text{const}. \quad (90)$$

4. Conclusions

Non-standard Lagrangians on time scales unify the continuous non-standard Lagrangians and the discrete ones. Non-standard Lagrangians have some properties that standard Lagrangians do not have. When $\alpha=0$, the power-law Lagrangians reduce to the standard Lagrangians. Therefore, non-standard Lagrangians have much more universal significance. The results in this paper are more widely used because of the arbitrariness of time scales. In this paper, the Euler–Lagrange equations for two kinds of non-standard Lagrangians on time scales are derived. Eight theorems with non-standard Lagrangians on time scales are established. The arguments here improved symmetry theory on time scales and with non-standard Lagrangians as well.

References

[1] Noether A E 1918 *Nachr kgl Ges Wiss Göttingen. Math. Phys.* **KI II** 235
[2] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[3] Luo S K, Guo Y X and Mei F X 2004 *Acta Phys. Sin.* **53** 1271 (in Chinese)
[4] Shi S Y and Huang X H 2008 *Chin. Phys. B* **17** 1554
[5] Zhang Y 2009 *Chin. Phys. B* **18** 4636

[6] Jin S X and Zhang Y 2014 *Chin. Phys. B* **23** 054501
[7] Zhou Y and Zhang Y 2015 *J. Dyn. Control* **13** 410
[8] Arnold V I 1978 *Mathematical methods of classical mechanics* (New York: Springer)
[9] Alekseev A I and Arbuzov B A 1984 *Theor. Math. Phys.* **59** 372
[10] Dimitrijevic D D and Milosevic M 2012 *AIP Conf. Proc.* **1472** 41
[11] El-Nabulsi R A 2013 *Can. J. Phys.* **56** 216
[12] El-Nabulsi R A 2013 *J. Theor. Math. Phys.* **7** 1
[13] El-Nabulsi R A 2014 *J. At. Mol. Sci.* **5** 268
[14] El-Nabulsi R A 2015 *Math.* **3** 727
[15] El-Nabulsi R A 2015 *Appl. Math. Lett.* **43** 120
[16] El-Nabulsi R A 2014 *Nonlinear Dyn.* **79** 2055
[17] Musielak Z 2008 *J. Phys. A: Math. Theor.* **41** 295
[18] Musielak Z 2009 *Chaos, Solitons and Fractals* **42** 2645
[19] Cieslinski J and Nikiciuk T 2009 *J. Phys. A: Math. Theor.* **43** 1489
[20] El-Nabulsi R A 2013 *Qual. Theory Dyn. Syst.* **12** 273
[21] El-Nabulsi R A 2013 *Nonlinear Dyn.* **74** 381
[22] El-Nabulsi R A, Soulati T and Rezazadeh H 2013 *J. Adv. Res. Dyn. Control Theory* **5** 50
[23] El-Nabulsi R A 2014 *Proc. Natl. Acad. Sci. India Sect. A: Phys. Sci.* **84** 563
[24] El-Nabulsi R A 2014 *Comp. Appl. Math.* **33** 163
[25] Zhang Y and Zhou X S 2016 *Nonlinear Dyn.* **84** 1867
[26] Zhou X S and Zhang Y 2016 *Chin. Quart. Mech.* **37** 15
[27] Hilger S 1988 *Ein makettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph. D. thesis, (Universität Würzburg)
[28] Bohner M and Peterson A 2001 *Dynamic equations on time scales: An introduction with applications* (Boston: Birkhäuser)
[29] Bohner M and Peterson A 2003 *Advances in dynamic equations on time scales* (Boston: Birkhäuser)
[30] Ahlbrandt C, Bohner M and Ridenhour J 2000 *J. Math. Anal. Appl.* **250** 561
[31] Bartosiewicz Z, Martins N and Torres D F M 2011 *Eur. J. Control* **17** 9
[32] Malinowska A B and Ammi M R S 2014 *Int. J. Differ. Equ.* **ISSN** 0973
[33] Bartosiewicz Z and Torres D F M 2008 *J. Math. Anal. Appl.* **342** 1220
[34] Malinowska A B and Martins N 2013 *Abst. Appl. Anal.* **2013** 1728
[35] Zhang Y 2016 *Chin. Q. Mech.* **37** 214
[36] Song C J and Zhang Y 2015 *J. Math. Phys.* **56** 102701
[37] Zu Q H and Zhu J Q 2016 *J. Math. Phys.* **57** 082701
[38] Jin S X and Zhang Y 2017 *Chin. Phys. B* **26** 014501
[39] Atici F M, Biles D C and Lebedinsky A 2006 *Math. Comp. Mod.* **43** 718