Tunable ground-state solitons in spin–orbit coupling Bose–Einstein condensates in the presence of optical lattices*

Huafeng Zhang(张华峰)[1,2,†], Fang Chen(陈方)[1,2], Chunchao Yu(郁春潮)[1,2], Lihui Sun(孙利辉)[1,2], and Dahai Xu(徐大海)[2]

[1] *Institute of Quantum Optics and Information Photonics, Yangtze University, Jingzhou* 434023, *China*

[2] *School of Physics and Optoelectronic Engineering, Yangtze University, Jingzhou* 434023, *China*

(Received 19 February 2017; revised manuscript received 12 April 2017; published online 20 June 2017)

Properties of the ground-state solitons, which exist in the spin–orbit coupling (SOC) Bose–Einstein condensates (BEC) in the presence of optical lattices, are presented. Results show that several system parameters, such as SOC strength, lattice depth, and lattice frequency, have important influences on properties of ground state solitons in SOC BEC. By controlling these parameters, structure and spin polarization of the ground-state solitons can be effectively tuned, so manipulation of atoms may be realized.

Keywords: spin–orbit coupling, Bose–Einstein condensates, optical lattices

PACS: 03.75.Be, 03.75.Hh, 03.75.Lm **DOI:** 10.1088/1674-1056/26/8/080304

1. Introduction

Spin–orbit coupling (SOC), the intrinsic interaction between a quantum particle's spin and its momentum, plays a crucial role in various areas of condensed matter physics,[1–7] such as spin Hall effects,[8] spintronics,[9] topological insulators,[10] and so on. Generally speaking, neutral atoms do not have the spin–orbit coupling effect when they are in their original state. Nevertheless, the recent experimental results show that synthetic SOC can be realized for ultracold neutral atoms by employing atom–light interactions.[11] In a landmark experiment, a synthetic spin–orbit coupling in a neutral atomic Bose–Einstein condensate (BEC) is engineered by dressing two atomic spin states with a pair of counter-propagating Raman laser lights.[12] The idea of such an experiment is that the two Raman laser beams are used to couple with a two-component BEC, and the momentum can be transferred between lasers and atoms generating a synthetic spin–orbit coupling.[13–26] The SOC opens a new door for artificially controlling atoms, and stimulates much research interest in it. However, the experiments on SOC BEC are have only ever been realized in a one-dimensional system before. Encouragingly, not so long ago, group Pan has received two-dimensional SOC BEC in their experiments.[21]

The matter–wave soliton in BEC is one of the most interesting topics. The most fascinating and well-known feature of this localized wave packet is that it can propagate without changing its shape as a result of the balance between nonlinearity and dispersion.[27,28] So far, the matter–wave soliton in BEC in the absence of SOC has been most extensively studied and discussed through experiments and theoretical analyses, including such aspects as the soliton dynamics,[29,30] vortex dynamics,[31,32] interference patterns,[33,34] domain walls in binary BEC,[35,36] and the modulational instability.[37,38] However, for the reason of the complexity in physics, solitons in spin–orbit coupled BEC have been paid little attention, only a few researchers focused on this subject. For example, Zhang presented a detailed study on gap solitons in a spin–orbit coupled BEC in two cases relating to a spin-dependent parity symmetry;[39] Kartashov reported a diversity of stable gap solitons with different physical symmetries (with respect to parity (P), time (T), and internal degree of freedom, i.e., spin (C), inversions) in a spin–orbit coupled BEC subject to a spatially periodic Zeeman field;[40] Gautam addressed vortex-bright solitons in a quasi-two-dimensional spin–orbit coupled hyperfine spin-1 three-component BEC by using variational method and numerical solution of a mean-field model;[41] dynamical behavior of solitons in one-dimensional uniform BEC subjected to a spin–orbit coupling was studied in detail.[42]

In this paper, we present ground-state solitons and their properties in an SOC BEC in the presence of an optical lattice. Our results suggest that atoms that make up the solitons can be artificially manipulated by controlling the system parameters. Furthermore, the spin polarization of the spin–orbit coupled BEC can also be efficiently dominated. These results will be helpful for us to understand deeply the physics of the system in which the SOC is significant.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11547007 and 11304024), the Yangtze Youth Fund (Grant No. 2016cqn55), and the Yangtze Fund for Youth Teams of Science and Technology Innovation (Grant No. 2015cqt03).

†Corresponding author. E-mail: hfzhang@yangtzeu.edu.cn

© 2017 Chinese Physical Society and IOP Publishing Ltd

2. Model

We addressed one-dimensional (1D) mixture of a two-component BEC with an experimentally realized SOC which is described by a spinor wave function $\Psi = (\Psi_\uparrow, \Psi_\downarrow)^{\mathrm{T}}$, where $\Psi^\dagger\Psi = |\Psi_\uparrow|^2 + |\Psi_\downarrow|^2$ and

$$\int \Psi^\dagger \cdot \Psi \, \mathrm{d}X = \int \left(|\Psi_\uparrow|^2 + |\Psi_\downarrow|^2 \right) \mathrm{d}X = N$$

with N being the total number of atoms. The components $\Psi_\uparrow$ and $\Psi_\downarrow$ are the wave functions of two (pseudo)spin states, with $\Psi_\uparrow$ for up spin and $\Psi_\downarrow$ for down spin. The dynamics of the spin–orbit coupled BEC in an optical lattice are governed by the following Gross–Pitaevskii equation (GPE)

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial \tau} = \left[\frac{1}{2m}\left(p_X + \hbar\kappa\sigma_2\right)^2 + \hbar\Delta\sigma_3 - g\Psi^\dagger\cdot\Psi + \widetilde{V}_{\mathrm{latt}} \right]\Psi. \tag{1}$$

The nonlinear coefficients $g_{\uparrow\uparrow}$ (or $g_{\downarrow\downarrow}$) and $g_{\uparrow\downarrow}$ (or $g_{\downarrow\uparrow}$) characterizes inter- and intra-spin interaction strengths, respectively, which can be adjusted by employing Feshbach resonances.[43] Generally, the difference between inter- and intra-spin two-body interactions can be made as small as necessary,[12,44] so we have taken $g_{\uparrow\uparrow} = g_{\downarrow\downarrow} = g_{\uparrow\downarrow} = g_{\downarrow\uparrow} = g$ for simplicity and $g > 0$ for attractive interatomic interactions below. The SOC is induced experimentally by using two counter-propagating Raman lasers along the z direction to couple two hyperfine ground states $\Psi_\uparrow$ and $\Psi_\downarrow$, which leads to constituting a pseudo-spin-1/2 system. The SOC strength κ is determined by the relative incident angle of the Raman beams and is tunable by changing the angle.[13] The Rabi frequency Δ can be easily controlled by varying the intensities of the Raman lasers, σ_2 and σ_3 are Pauli matrices. The spin–orbit coupled BEC is loaded into a 1D optical lattice $\widetilde{V}_{\mathrm{latt}} = v\sin^2(k_{\mathrm{latt}}X)$ by adiabatically ramping up the lattice beams, k_{latt} is the wave number of the lattice beams, and v is lattice depth which can be tuned by changing the intensity of the lattice beams.

To study numerically solitons in the system described by Eq. (1), we begin with the following dimensionless GP equation,

$$\mathrm{i}\frac{\partial \Phi}{\partial t} = \left[\frac{-1}{2}\frac{\partial^2}{\partial x^2} + \mathrm{i}\alpha\sigma_2\frac{\partial}{\partial x} + \sigma_3 - \gamma\Phi^\dagger\cdot\Phi - v_0\cos(\omega_{\mathrm{L}}x) \right]\Phi \tag{2}$$

by scaling energy with $\hbar\Delta$, time with $1/\Delta$, and length with $\sqrt{\hbar/m\Delta}$. The dimensionless parameters $\alpha = -\kappa\sqrt{\hbar\Delta/m}$, $\omega_{\mathrm{L}} = 2k_{\mathrm{latt}}\sqrt{\hbar/m\Delta}$, $v_0 = v/(2\hbar\Delta)$, and $\gamma = Ng\sqrt{m/\hbar\Delta}/\hbar$. In addition, $V_{\mathrm{latt}} = -v_0\cos(\omega_{\mathrm{L}}x)$ is the optical lattices. The dimensionless wave functions $\Phi = (\Phi_\uparrow, \Phi_\downarrow)^{\mathrm{T}}$ satisfy

$$\int \left(|\Phi_\uparrow|^2 + |\Phi_\downarrow|^2 \right) \mathrm{d}x = 1.$$

It is worth pointing out that the SOC term $\mathrm{i}\alpha\sigma_2(\partial/\partial x)$ in Eq. (2) suggests that the spin σ_2 only couples the momentum in the x direction. One can define

$$\langle\sigma_3\rangle = \int \Phi^\dagger\sigma_3\Phi\,\mathrm{d}x = \int \left[|\Phi_\uparrow|^2 - |\Phi_\downarrow|^2 \right] \mathrm{d}x$$

as the spin polarization of the spin σ_3. The energy functional of our system is

$$E = \int \mathrm{d}x[\frac{1}{2}|\partial_x\Phi_\uparrow|^2 + \frac{1}{2}|\partial_x\Phi_\downarrow|^2 + |\Phi_\uparrow|^2 - |\Phi_\downarrow|^2 + \alpha\Phi_\uparrow^*\partial_x\Phi_\downarrow - \alpha\Phi_\downarrow^*\partial_x\Phi_\uparrow$$
$$-\frac{\gamma}{2}\left(|\Phi_\uparrow|^4 + |\Phi_\downarrow|^4 \right) + 2|\Phi_\uparrow|^2|\Phi_\downarrow|^2 - v_0\cos(\omega_{\mathrm{L}}x)\left(|\Phi_\uparrow|^2 + |\Phi_\downarrow|^2 \right)].$$

To investigate the properties of the SOC BEC in the system described by Eq. (2), one can numerically solve Eq. (2) by using imaginary time-evolution method.[44–46] The numerical results show that the ground states of BEC in the system (2) happen to be stationary bright solitons.

3. Results and discussions

By instinct, in the absence of the spin–orbit coupling effect, i.e., $\alpha = 0$, one of two spin components will disappear. Figure 1 presents the typical profile of bright solitons which exist in the system (2). One can find from Fig. 1, when the spin–orbit coupling strength α is equal to zero, the up component $\Phi_\uparrow$ vanishes, and only the down component $\Phi_\downarrow$ survives. Further studies reveal that optical lattices have an important impact on the properties of the down component in the case when $\alpha = 0$. As can be seen from Figs. 1(a)–1(c), the magnitude of the lattice depth v_0 significantly influences the profile of bright solitons for the down component. More specifically, with an increase of the lattice depth v_0, the peak amplitude of the down component increases, and its root-mean-square width decreases, as shown in Figs. 2(a) and 2(b).

Yet, the influence of the lattice frequency ω_{L} on the bright soliton for the down component is more complex (see Figs. 1(d)–1(f)). When the lattice frequency ω_{L} increases, the changes in root-mean-square (RMS) width and the peak amplitude of the down component is nonmonotonic. Concretely speaking, with an increase of the lattice frequency ω_{L} from zero, the peak amplitude of the down component first increases, then it decreases and finally tends to be a constant (see Fig. 2(c)); whereas for its width, the reverse phenomenon happens and it initially decreases (see Fig. 2(d)). More remarkably, by comparing Fig. 1(a) with Fig. 1(f) and further

numerical simulations, one can see that, for $\omega_{\rm L} \to \infty$, the peak amplitude of the down component and its width respectively tend to a constant, which is the right one at $\omega_{\rm L} = 0$ (without optical lattices). The reason for this phenomenon is that the averages of the lattice potential $\langle V_{\rm latt} \rangle$ is nearly equal to zero within the existence space of the down component. In addition, the spin polarization $\langle \sigma_3 \rangle$ remains to be equal to -1 in the case of $\alpha = 0$.

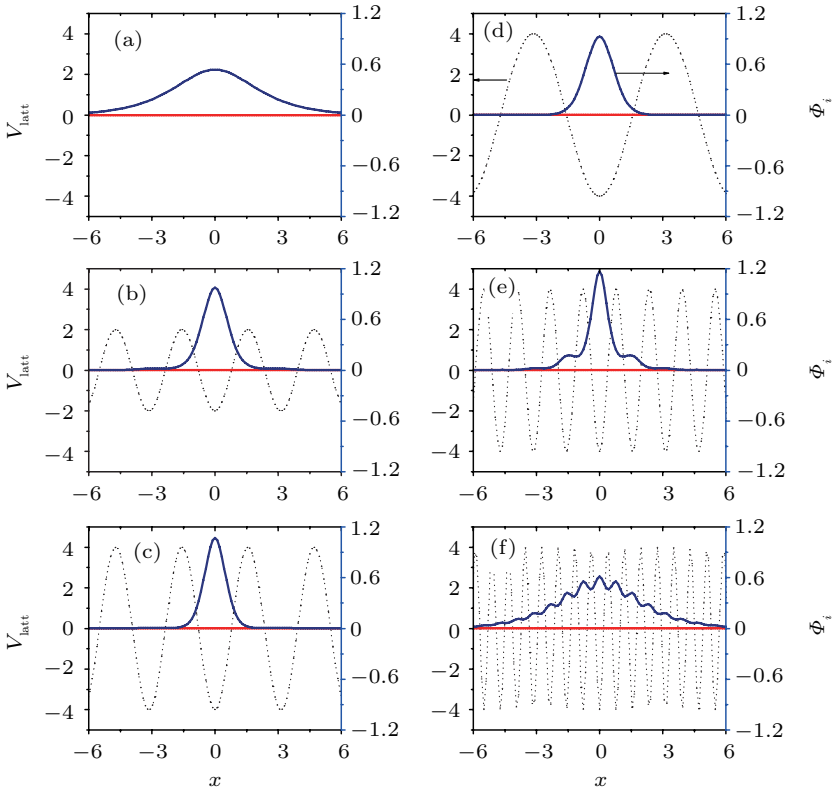


Fig. 1. (color online) Stationary bright solitons (red lines for up spin and blue lines for down spin) and optical lattices $V_{\rm latt}$ (dotted lines) in the system (2) at (a) $v_0 = 0$, (b) $v_0 = 2$, and (c) $v_0 = 4$ for $\omega_{\rm L} = 2$; (d) $\omega_{\rm L} = 1$, (e) $\omega_{\rm L} = 4$, and (f) $\omega_{\rm L} = 8$ for $v_0 = 4$. The other parameters are $\alpha = 0$ and $\gamma = 1$.

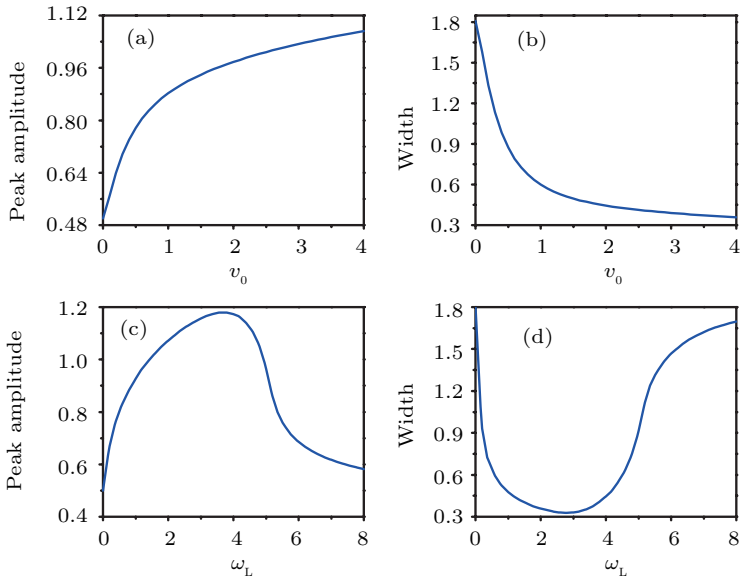


Fig. 2. (color online) The dependence of [(a) and (c)] peak amplitude and [(b) and (d)] root-mean-square width of the down spin component on the lattice depth v_0 for $\omega_{\rm L} = 2$; dependence of (a) peak amplitude and (b) root-mean-square width of the down spin component on the lattice frequency $\omega_{\rm L}$ for $v_0 = 4$. The other parameters are $\alpha = 0$ and $\gamma = 1$.

In the presence of spin–orbit coupling effect, i.e., $\alpha \neq 0$, the properties of ground-state solitons of BEC in the system described by Eq. (2) is markedly different from the case for $\alpha = 0$. As seen in Fig. 3, with an increase of the SOC strength α from zero, the up component grows out of nothing and goes from strength to strength (comparing with Figs. 3(a)–3(f)). Apparently, when $\alpha \neq 0$, the spin polarization $\langle \sigma_3 \rangle$ no longer keeps equaling to -1, instead it changes with the alteration

of the SOC strength. The physical mechanism of such a phenomenon is easy to understand. When the SOC strength α is not equal to zero, the atoms, which are originally at a lower level (ground state, labeled as "spin down") are coupled to a higher level (excited state, labeled as "spin up"). If the SOC strength α slowly increases from zero, the atoms at the higher level get more and more, and those at the lower level become less. There is a critical SOC strength $\alpha_{\rm cr}$, when α is above $\alpha_{\rm cr}$, the number of the atoms at the higher level is equal to that at the lower level, and the total spin is zero. As seen from Fig. 5(a), there exists a quantum phase transition from a spin-polarized to a spin-balanced ground-state soliton in a lattice-trapped SOC Bose–Einstein condensate with an increase of the SOC strength from zero. It is worth noting that there is a unique symmetry for the system described by Eq. (2), the up component $\Phi_\uparrow$ is an odd function of x while the down component $\Phi_\downarrow$ is even when the SOC strength changes. Besides, there are several "nodes" in these ground-state bright solitons, and the more the SOC strength α, the more the node number (see Fig. 3). The occurrence of such type of soliton in Fig. 3 is the result of synergetic action of three factors, such as SOC effect, optical lattices and interatomic interaction. The spin–orbit coupling effect may make the BEC be in a spin mixed or "stripe" phase,[43] that is why the "nodes" appear in Fig. 3.

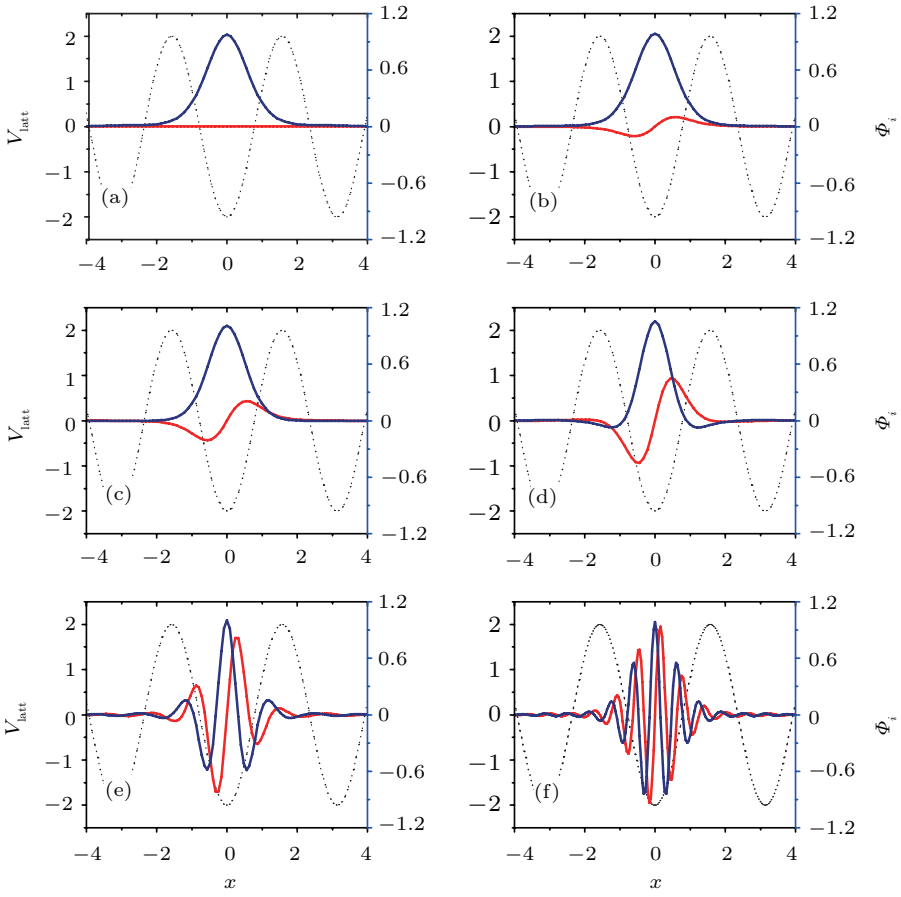


Fig. 3. (color online) Stationary bright solitons (red lines for up spin and blue lines for down spin) and optical lattices $V_{\rm latt}$ (dotted lines) in the system (2) at (a) $\alpha = 0$, (b) $\alpha = 0.5$, (c) $\alpha = 1$, (d) $\alpha = 2$, (e) $\alpha = 5$, and (f) $\alpha = 10$. The other parameters are $v_0 = 2$, $\omega_{\rm L} = 2$, and $\gamma = 1$.

Lattice depth v_0 has a significant influence on the properties of ground-state bright soliton of spin–orbit coupling BEC. When the lattice depth v_0 increases from zero, these bright solitons tend to become more localized in the space domain, as shown in Fig. 4. When the SOC strength α is small, the shape of either spin-component of the ground-state bright soliton has not changed much with a change of lattice depth v_0, in this case, the number of nodes in either spin-component keeps unchanged, as illustrated by Figs. 4(a)–4(c). However, for the larger SOC strength α, one can see from Figs. 4(d)–4(f) that its shape has changed significantly when the lattice depth v_0 increases; the larger the lattice depth, the simpler the structure of the ground-state soliton and the fewer its node number. Figure 5(a) presents the dependence of the spin polarization $\langle\sigma_3\rangle$ on the SOC strength α for different lattice depths. Moreover, one can see from the figure that the spin polarization $\langle\sigma_3\rangle$ may also be effectively regulated by adjusting the lattice depth v_0. It is interesting that the spin polarization $\langle\sigma_3\rangle$ can be observably changed by altering the lattice depth v_0 only when the SOC strength is medium, as shown in Fig. 5(b). On the contrary, for smaller or larger SOC strength, no matter how one tries to change the lattice depth, the spin polarization $\langle\sigma_3\rangle$ experiences an ignorable difference.

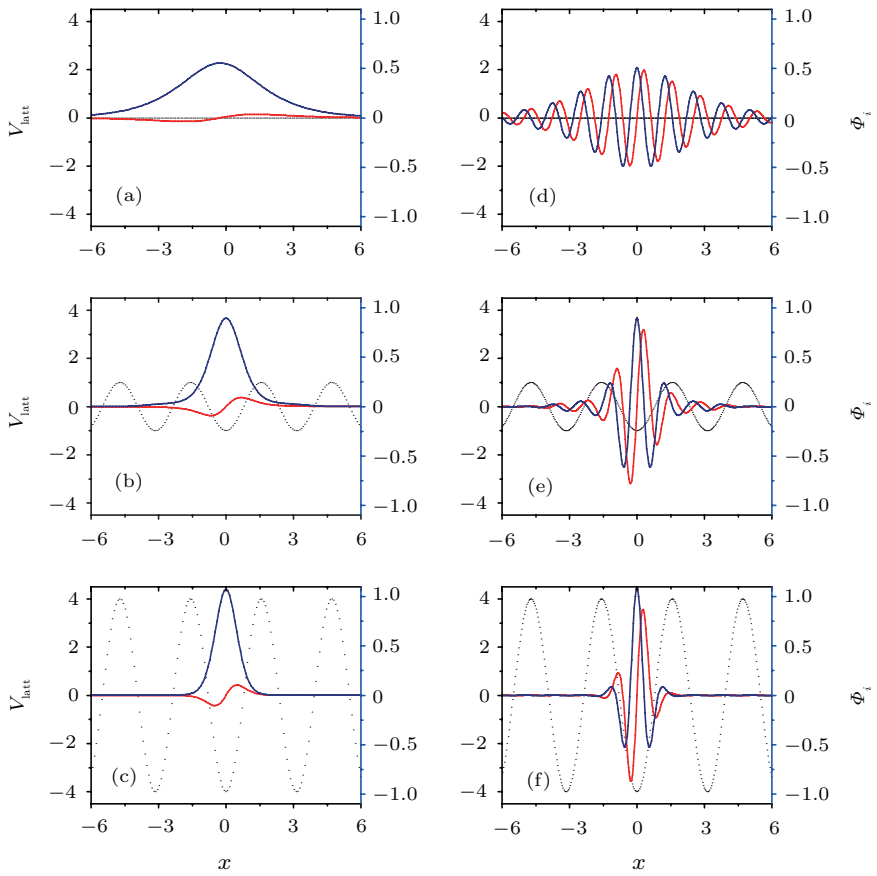


Fig. 4. (color online) Stationary bright solitons (red lines for up spin and blue lines for down spin) and optical lattices $V_{\rm latt}$ (dotted lines) in the system (2) at (a) $v_0 = 0$, (b) $v_0 = 1$, and (c) $v_0 = 4$ for $\alpha = 0.5$; (d) $v_0 = 0$, (e) $v_0 = 1$, and (f) $v_0 = 4$ for $\alpha = 5$. The other parameters are $\omega_{\rm L} = 2$ and $\gamma = 1$.

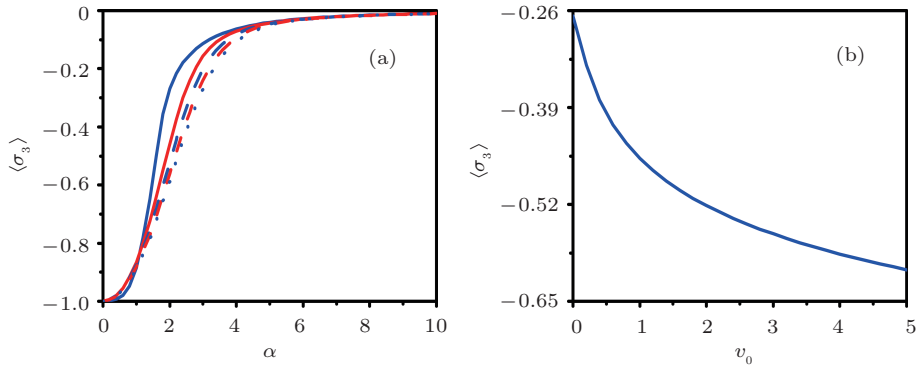


Fig. 5. (color online) The dependence of spin polarization $\langle\sigma_3\rangle$ on (a) the SOC strength α for different lattice depths $v_0 = 0$ (blue solid line), $v_0 = 1$ (red solid line), $v_0 = 2$ (blue dashed line), $v_0 = 3$ (red dashed line), and $v_0 = 4$ (blue dot line); the dependence of spin polarization $\langle\sigma_3\rangle$ on (b) the lattice depth v_0 at $\alpha = 2$. Other parameters are $\omega_{\rm L} = 2$ and $\gamma = 1$.

To investigate the influence of lattice frequency $\omega_{\rm L}$ on properties of the ground-state soliton in Bose–Einstein condensate, we have scaled a length with $\sqrt{\hbar/m\Delta}$ instead of $1/k_{\rm latt}$ in obtaining Eq. (2). Specifically speaking, when the lattice frequency $\omega_{\rm L}$ is equal to zero, this means that optical lattices do not exist. With an increase of the lattice frequency $\omega_{\rm L}$ from zero, the ground states of the spin–orbit coupling Bose–Einstein condensate firstly become more localized in space and then expanded, as illustrated in Figs. 6(a)–6(d). Remarkably, when the lattice frequency $\omega_{\rm L}$ is big enough, the ground-state soliton is very close to that in the case of $\omega_{\rm L} = 0$ (see Figs. 6(a) and 6(d)). Furthermore, if the spin–orbit coupling strength α is larger, the node number in the ground state first decreases and then increases when the lattice frequency $\omega_{\rm L}$ increases from zero. Figure 7(a) presents the dependence of the spin polarization $\langle\sigma_3\rangle$ on the SOC strength α for different lattice frequencies. When the value of the SOC strength is adjusted, a quantum phase transition, which occurs in the lattice-trapped BEC, takes on dissimilar characteristics in this case to the case for different lattice depth. For a small lattice frequency, the value of the spin polarization $\langle\sigma_3\rangle$ increases monotonically from -1 to 0 with an increase of the

SOC strength from zero. However, when the lattice frequency is larger, the increase of $\langle\sigma_3\rangle$ becomes nonmonotonic as the SOC strength increases. From Figs.7(a) and 7(b), one can see that the spin polarization $\langle\sigma_3\rangle$ can be changed dramatically by altering the lattice frequency ω_L when the SOC strength α is at an intermediate value. Also, for a medium-value SOC strength α, with an increase of the lattice frequency, the spin polarization $\langle\sigma_3\rangle$ keeps negative, its value firstly decreases and then increases, finally tending to a certain value which approaches the value at $\omega_L = 0$, as shown in Fig.7(b).

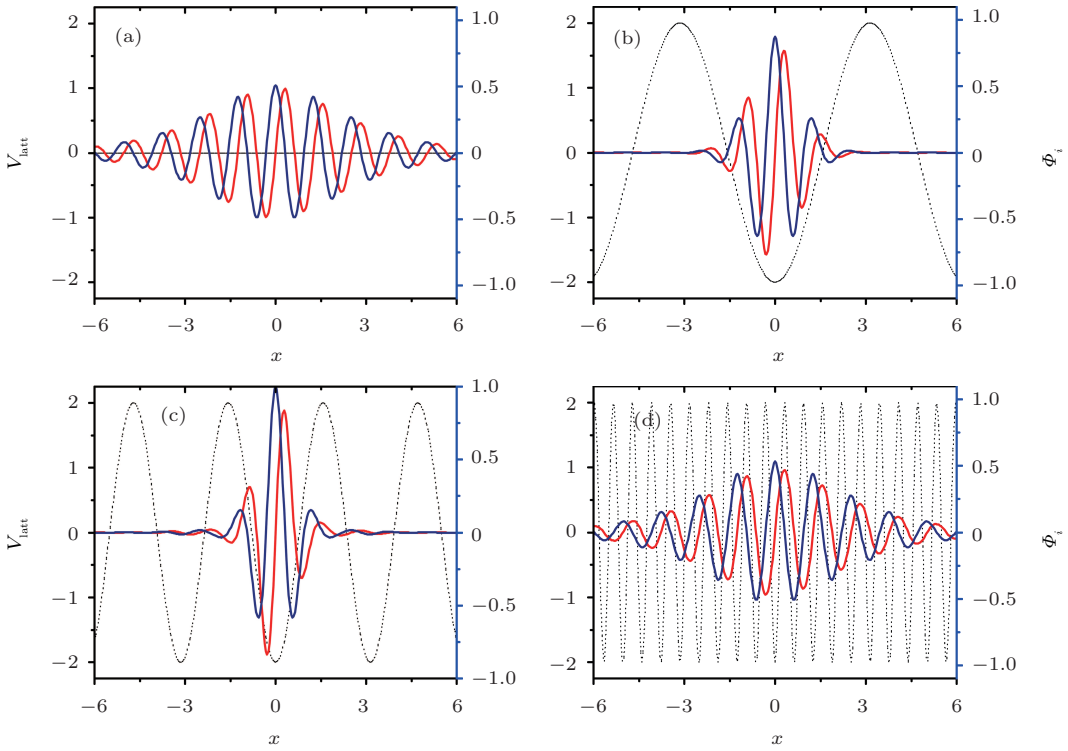


Fig. 6. (color online) Stationary bright solitons (red lines for up spin and blue lines for down spin) and optical lattices V_{latt} (dotted lines) in the system (2) at (a) $\omega_L = 0$, (b) $\omega_L = 1$, (c) $\omega_L = 2$, and (d) $\omega_L = 10$. The other parameters are $v_0 = 2$, $\alpha = 5$, and $\gamma = 1$.

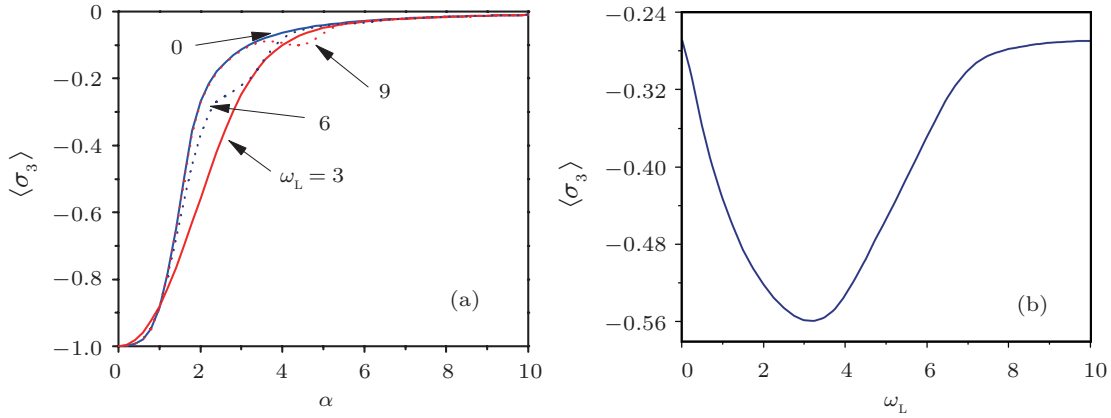


Fig. 7. (color online) The dependence of spin polarization $\langle\sigma_3\rangle$ on (a) the SOC strength α for different lattice frequencies $\omega_L = 0$ (blue solid line), $\omega_L = 3$ (red solid line), $\omega_L = 6$ (blue dashed line), and $\omega_L = 9$ (red dashed line); the dependence of spin polarization $\langle\sigma_3\rangle$ on (b) the lattice frequency ω_L at $\alpha = 2$. Other parameters are $v_0 = 2$ and $\gamma = 1$.

4. Conclusion

In summary, the properties of the ground-state solitons, which exist in the spin–orbit coupling Bose–Einstein condensates in the presence of an optical lattice, are investigated in detail. The synergetic action of three factors, including SOC effect, optical lattices, and interatomic interaction, leads to the occurrence of tunable ground-state solitons in spin–orbit coupling Bose–Einstein condensates in the presence of optical lattices. Several system parameters, such as SOC strength, lattice depth, and lattice frequency, have different influences on properties of the ground state solitons in the SOC BEC. The SOC effect remarkably changes the spin-polarization of the condensates, which can lead to a quantum phase transition. Moreover, it may make the BEC in a spin mixed or "stripe" phase, and

make several "nodes" appear in the ground-state soliton. The larger lattice depth can make the ground-state solitons become more localized. However, it does so only for the medium-value lattice frequencies. By controlling these parameters, the structure and spin polarization of the ground-state solitons can be effectively tuned.

References

[1] Qi X L and Zhang S C 2011 *Rev. Mod. Phys.* **83** 1057
[2] Li Z and Cao H 2014 *Acta Phys. Sin.* **63** 110306 (in Chinese)
[3] Gong M, Chen G, Jia S T and Zhang C W 2012 *Phys. Rev. Lett.* **109** 105302
[4] Zhang Y C, Yu Z Q, Ng T K, Zhang S Z, Pitaevskii L and Stringari S 2016 *Phys. Rev. A* **94** 033635
[5] Song S, Sun R, Zhao H, Wang X and Han B 2016 *Chin. Phys. B* **25** 040305
[6] Xu Z H, Cole W S and Zhang S Z 2014 *Phys. Rev. A* **89** 051604
[7] Xie W F, He Y and Bao C 2015 *Chin. Phys. B* **24** 060305
[8] Sinova J, Valenzuela S O, Wunderlich J, Back C H and Jungwirth T 2015 *Rev. Mod. Phys.* **87** 1213
[9] Koralek J D, Weber C P, Orenstein J, Bernevig B A, Zhang S C, Mack S and Awschalom D D 2009 *Nature* **458** 610
[10] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
[11] Zai H 2015 *Rep. Prog. Phys.* **78** 026001
[12] Lin Y J, Jiménez-García K and Spielman I B 2011 *Nature* **471** 83
[13] Zhang Y P, Chen G and Zhang C W 2013 *Sci. Rep.* **3** 1937
[14] Li Y, Qu C L, Zhang Y P and Zhang C W 2015 *Phys. Rev. A* **92** 013635
[15] Kasamatsu K 2015 *Phys. Rev. A* **92** 063608
[16] Kartashov Y V, Konotop V V and Zezyulin D A 2014 *Phys. Rev. A* **90** 063621
[17] Zhang Y P and Zhang C W 2013 *Phys. Rev. A* **87** 023611
[18] Pan J S, Zhang W, Yi W and Guo G C 2016 *Phys. Rev. A* **94** 043619
[19] Olson A J, Wang S, Niffenegger R J, Li C, Greene C H and Chen Y P 2014 *Phys. Rev. A* **90** 013616
[20] Zhang S Z, Cole W S, Paramekanti A and Trivedi N 2015 *Annual Review of Cold Atoms and Molecules* **3** 135-179
[21] Wu Z, Zhang L, Sun W, Xu X, Wang B,Ji S, Deng Y, Chen S, Liu X and Pan J W 2016 *Science* **354** 83
[22] Xu Y, Zhang Y P and Wu B 2012 *Phys. Rev. A* **87** 013614
[23] Zhang J Y, Ji S C, Chen Z, Zhang L, Du Z D, Yan B, Pan G S, Zhao B, Deng Y J, Zhai H, Chen S and Pan J W 2012 *Phys. Rev. Lett.* **109** 115301
[24] Kartashov Y V, Konotop V V, Zezyulin D A and Torner L 2016 *Phys. Rev. Lett.* **117** 215301
[25] Khamehchi M A, Hossain K, Mossman M E, Zhang Y P, Busch T, Forbes M M and Engels P 2017 *Phys. Rev. Lett.* **118** 155301
[26] Stringari S 2017 *Phys. Rev. Lett.* **118** 145302
[27] Khaykovich L, Schreck F, Ferrari G, Bourdel T, Cubizolles J, Carr L D, Castin Y and Salomon C 2002 *Science* **296** 1290
[28] Pethick C J and Smith H 2008 *Bose–Einstein condensate in dilute gases*, 2nd edn. (Cambridge: Cambridge University Press)
[29] Muryshev A, Shlyapnikov G V, Ertmer W, Sengstock K and Lewenstein M 2002 *Phys. Rev. Lett.* **89** 110401
[30] Morgan T and Busch T 2013 *Phys. Rev. A* **88** 063610
[31] Rosenbusch P, Bretin V and Dalibard J 2002 *Phys. Rev. Lett.* **89** 200403
[32] Serafini S, Barbiero M, Debortoli M, Donadello S, Larcher F, Dalfovo.F, Lamporesi G and Ferrari G 2015 *Phys. Rev. Lett.* **115** 170402
[33] Liu W M, Wu B and Niu Q 2000 *Phys. Rev. Lett.* **84** 2294
[34] Castellanos E and Rivas J I 2015 *Phys. Rev. D* **91** 084019
[35] Malomed B A, Nistazakis H E, Frantzeskakis D J and Kevrekidis P G 2004 *Phys. Rev. A* **70** 043616.
[36] Coen S and Haelterman M 2001 *Phys. Rev. Lett.* **87** 140401
[37] Wu L, Zhang J F and Li L 2007 *New J. Phys.* **9** 69
[38] Bhat I A, Mithun T, Malomed B A and Porsezian K 2015 *Phys. Rev. A* **92** 063606
[39] Zhang Y P, Xu Y and Busch T 2015 *Phys. Rev. A* **91** 043629
[40] Kartashov Y V, Konotop V V and Abdullaev F K 2013 *Phys. Rev. Lett.* **111** 060402
[41] Gautam S and Adhikari S K 2017 *Phys. Rev. A* **95** 013608
[42] Wen L, Sun Q, Chen Y, Wang D, Hu J, Chen H, Liu W M, Juzeliūnas G, Malomed B A and Ji A 2016 *Phys. Rev. A* **94** 061602
[43] Chin C, Grimm R, Julienne P and Tiesinga E 2010 *Rev. Mod. Phys.* **82** 1225
[44] Li Y, Pitaevskii L P and Stringari S 2012 *Phys. Rev. Lett.* **108** 225301
[45] Chiofalo M L, Succi S and Tosi M P 2000 *Phys. Rev. E* **62** 7438
[46] Bao W, Chern I L and Lim F Y 2006 *J. Comput. Phys.* **219** 836
[47] Yang J K and Lakoba T 2008 *Stud. Appl. Math.* **120** 265