Exact solution of the generalized Kemmer oscillator*

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In the present work, the generalized Kemmer oscillator was introduced in one dimension. It is shown that the exact solutions of generalized Kemmer oscillator with some appropriate choices of the interactions $f(x)$ have been obtained by using the Nikiforov–Uvarov (NU) method. Moreover, several interesting cases are discussed.

Keywords: Kemmer equation, generalized oscillator, NU method

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1. Introduction

In recent years, the Dirac oscillator that has been attracted a lot of attention is introduced by replacing momentum $\boldsymbol{p}$ in the free particle's Dirac equation by combination $\boldsymbol{p} \to \boldsymbol{p} - \mathrm{i}m\omega\beta\boldsymbol{r}$, where $\boldsymbol{r}$ is the position operator, m is the particle's mass, and ω is the oscillator frequency.[1] It is given the name Dirac oscillator because it reduces to the standard harmonic oscillator with a strong spin–orbit coupling in the non-relativistic limit. This system has many applications in different fields and has been studied extensively in high energy physics,[2–8] condensed matter physics,[9–11] quantum optics,[12–17] and mathematical physics,[18–25] etc. Especially in recent, Dutta and collaborators suggested to substitute in the one-dimensional free Dirac equation the momentum operator $\boldsymbol{p}$ with $\boldsymbol{p} - \mathrm{i}f(x)$ and the corresponding model is called as generalized Dirac oscillator. The exact solutions of a couple of generalized Dirac oscillators with complex interactions have been obtained.[26] On the other hand, the Kemmer equation[27] is a Dirac-type equation of relativistic particle of spin-one which involves matrices obeying a different scheme of commutation rules[28–30] seems to be richer than the Klein–Gordon one with respect to the introduction of interactions. Moreover, the matrices in two-body Dirac equation can be reduced to a direct sum of 10×10, 5×5, and 1×1 matrices realizing irreducible representations of the Kemmer algebra. Thus the DKP equation is equivalent to the two-body Dirac equation[31] and has been investigated in several active research subjects such as the quark–antiquark bound state problem.[32–36] Motivated by the above basic idea of introducing generalized Dirac oscillator, in this work, we will introduce generalized Kemmer oscillator by replacing momentum $\boldsymbol{p}$ with $\boldsymbol{p} - \mathrm{i}f(x)$ in one-dimensional free Kemmer equation and the model can be interpreted as the interaction of the anomalous magnetic moment with a general electric field. Then we will derive a complete solution of the generalized Kemmer oscillator within appropriate choices of the interactions $f(x)$ by using the Nikiforov–Uvarov method.[37–47] This paper is organized as follows. In Section 2, we introduce the generalized one-dimensional Kemmer oscillator. In Section 3, we give the exact solutions of the Kemmer equation with appropriate choices of the interaction $f(x)$ by the Nikiforov-Uvarov method. Section 4 is devoted to the conclusion. And finally, we review the Nikiforov-Uvarov method in Appendix A.

2. Kemmer equation with generalized oscillator interaction in one dimension

The Dirac-type relativistic Kemmer equation[27–32] that describes the spin-one particles is written as (in the unit system $\hbar = c = 1$)

$$\left(\hat{\beta}^{\mu}\hat{p}_{\mu} - M\right)\psi = 0, \tag{1}$$

where $M = 2m$ is the total mass of two identical spin-1/2 particles and the matrices $\hat{\beta}^{\mu}$ $(\mu = 0,1,2,3)$ obey the following commutation relations

$$\hat{\beta}^{\mu}\hat{\beta}^{\eta}\hat{\beta}^{\nu} + \hat{\beta}^{\nu}\hat{\beta}^{\eta}\hat{\beta}^{\mu} = \hat{g}^{\mu\eta}\hat{\beta}^{\nu} + \hat{g}^{\nu\eta}\hat{\beta}^{\mu}, \tag{2}$$

with

$$\hat{\beta}^{\mu} = \hat{\gamma}^{\mu}\otimes\hat{I} + \hat{I}\otimes\hat{\gamma}^{\mu}. \tag{3}$$

In Eq. (3), $\otimes$ is a direct product operation and $\hat{I}$, $\hat{\gamma}^{\mu}$ can be written respectively as

$$\hat{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{\gamma}^{0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$\hat{\gamma}^{j} = \begin{pmatrix} 0 & -\mathrm{i}\hat{\sigma}_j \\ \mathrm{i}\hat{\sigma}_j & 0 \end{pmatrix}, \quad j = 1,2,3, \tag{4}$$

where $\hat{\sigma}_j$ are the so-called Pauli matrices.

Now let us substitute the momentum operator $\boldsymbol{p}$ with $\boldsymbol{p} - \mathrm{i}f(x)$ in one-dimensional free Kemmer equation (1), then

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the Kemmer equation with generalized oscillator interaction can be written as

$$\left[\left(\hat{\gamma}^0\otimes\hat{I}+\hat{I}\otimes\hat{\gamma}^0\right)E-\left(\hat{\gamma}^0\otimes\hat{\sigma}_x+\hat{\sigma}_x\otimes\hat{\gamma}^0\right)\cdot\left(\hat{p}_x-\mathrm{i}\hat{B}f(x)\right)-M\hat{\gamma}^0\otimes\hat{\gamma}^0\right]\psi=0, \tag{5}$$

where the operator $\hat{B}$ is chosen as $\hat{B}=\hat{\gamma}^0\otimes\hat{\gamma}^0$ with $\hat{B}^2=\hat{I}$. Especially, equation (5) will reduce to Kemmer equation with general oscillator interaction for $f(x)=m\omega x$.[32,48,49] The stationary state of Eq. (5) can be described by a four-component spinor

$$\psi=\left(\begin{array}{cccc}\psi_1 & \psi_2 & \psi_3 & \psi_4\end{array}\right)^{\mathrm{T}} \tag{6}$$

with T the transpose operator.

Putting Eq. (6) into Eq. (5), we can easily obtain four equations

$$(2E-M)\psi_1-[\hat{p}_x+\mathrm{i}f(x)]\psi_2-[\hat{p}_x+\mathrm{i}f(x)]\psi_3=0, \tag{7}$$

$$-[\hat{p}_x-\mathrm{i}f(x)]\psi_1+M\psi_2+[\hat{p}_x-\mathrm{i}f(x)]\psi_4=0, \tag{8}$$

$$-[\hat{p}_x-\mathrm{i}f(x)]\psi_1+M\psi_3+[\hat{p}_x-\mathrm{i}f(x)]\psi_4=0, \tag{9}$$

$$[\hat{p}_x+\mathrm{i}f(x)]\psi_2+[\hat{p}_x+\mathrm{i}f(x)]\psi_3-(2E+M)\psi_4=0. \tag{10}$$

From Eqs. (7), (8), (9), and (10), it is not difficult to find the following relations obeying four-component wave functions

$$\begin{aligned}\psi_2&=\psi_3,\\ \psi_1&=\frac{2}{2E-M}[\hat{p}_x+\mathrm{i}f(x)]\psi_2,\\ \psi_4&=\frac{2}{2E+M}[\hat{p}_x+\mathrm{i}f(x)]\psi_2.\end{aligned} \tag{11}$$

Substituting Eq. (11) into Eq. (8) will lead to the following equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{\mathrm{d}f(x)}{\mathrm{d}x}-f^2(x)+\left(E^2-m^2\right)\right]\psi_2=0. \tag{12}$$

In the following section, the generalized parametric NU method is used to find the solution of Eq. (12) for appropriate choices of the interactions $f(x)$.

3. The exact solution of Kemmer equation with appropriate choices of interactions

In this section, we concentrate our efforts in the Kemmer equation with two different appropriate choices of the interactions $f(x)$. The corresponding exact solutions are obtained by using the generalized parametric NU method.

3.1. The exact solution of generalized Kemmer equation with $f(x)$ to be Coulomb potential plus linear potential

Now let us take the function $f(x)$ to be Coulomb potential plus linear potential,[50,51]

$$f(x)=ax-\frac{b}{x},\quad b\geq 0,\quad a>0. \tag{13}$$

The Coulomb potential plus the linear potential has received a great deal of attention in particle physics. More correctly it is used to describe systems of quark and anti-quark bound states in the context of meson spectroscopy and it is also used to represent a radial Stark effect in hydrogen in atomic and molecular physics.[52–54] In this case, equation (12) is read as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_2-\left[a^2x^2+\frac{b^2+b}{x^2}+\zeta^2\right]\psi_2=0, \tag{14}$$

where $\zeta^2=a-\varepsilon^2-2ab$, $\varepsilon^2=E^2-m^2$. In order to find the possible bound state solution we introduce a new variable such as $s=x^2$, then the equation (14) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\psi_2+\frac{1}{2s}\frac{\mathrm{d}}{\mathrm{d}s}\psi_2-\frac{a^2s^2+\zeta^2s+b^2+b}{4s^2}\psi_2=0. \tag{15}$$

Obviously, equation (15) has the same form with the basic equation of the NU method.[37–47] So, it is straight-forward to gain the eigenenergy and the corresponding eigenfunction by using the NU method. By comparing Eq. (15) with Eq. (A1) in Appendix A, the following parameters can be easily given

$$\begin{aligned}&c_1=\frac{1}{2},\quad c_2=c_3=0,\quad \xi_1=\frac{a^2}{4},\\ &\xi_2=-\frac{\zeta^2}{4},\quad \xi_3=\frac{b^2+b}{4}.\end{aligned} \tag{16}$$

Based on the NU method, the corresponding eigenenergy and eigenfunction satisfy the following equation

$$(c_2-c_3)n+c_3n^2-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+c_7+2c_3c_8+2\sqrt{c_8c_9}=0, \tag{17}$$

$$\psi_2(s)=Ns^{c_{12}}(1-c_3s)^{-c_{12}-(c_{13}/c_3)}\times\mathrm{P}_n^{(c_{10}-1,(c_{11}/c_3)-c_{10}-1)}(1-2c_3s), \tag{18}$$

where $\mathrm{P}_n^m(s)$ is Jacobi polynomial. And the other coefficients are determined by Eq. (A4) as

$$\begin{aligned}&c_4=\frac{1}{4},\quad c_5=0,\quad c_6=\frac{a^2}{4},\quad c_7=\frac{\zeta^2}{4},\\ &c_8=\frac{(2b+1)^2}{16},\quad c_9=\frac{a^2}{4},\quad c_{10}=b+\frac{3}{2},\\ &c_{11}=a,\quad c_{12}=\frac{b+1}{2},\quad c_{13}=-\frac{a}{2}.\end{aligned} \tag{19}$$

In Eq. (18), for the parameter $c_3=0$, the wave function will become into the form as follows:[37–47]

$$\psi_2(s)=Ns^{c_{12}}\mathrm{e}^{c_{13}s}\mathrm{L}_n^{c_{10}-1}(c_{11}s), \tag{20}$$

where $\mathrm{L}_n^m(s)$ is the Laguerre polynomial. With the help of Eqs. (16), (17), (18), (19), and some simple mathematical manipulations, the corresponding eigenenergy and eigenfunction can be expressed respectively as follows:

$$E^2 = m^2 + 4a(n+1), \tag{21}$$

$$\psi(x) = \begin{pmatrix} \dfrac{2}{2E-M}\left[\hat{p}_x + \mathrm{i}f(x)\right] \\ 1 \\ 1 \\ \dfrac{2}{2E+M}\left[\hat{p}_x + \mathrm{i}f(x)\right] \end{pmatrix} \times Nx^{b+1}\mathrm{e}^{-(a/2)x^2}\mathrm{L}_n^{b+(1/2)}\left(ax^2\right), \tag{22}$$

where N is the normalization constant. For the various values of quantum numbers n and $m = 1$, the square of energy spectrum of the Coulomb potential plus linear potential versus parameter a has been given in Fig. 1 (in natural units).

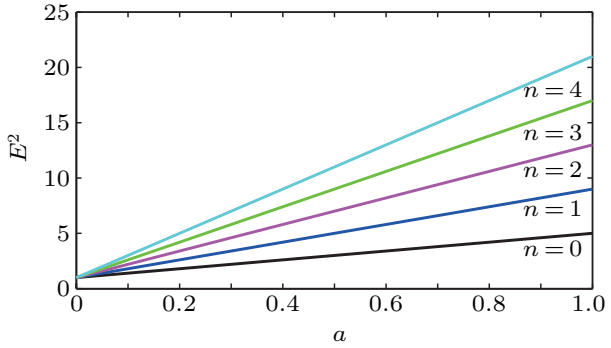


Fig. 1. (color online) The square of energy spectrum of the Coulomb potential plus linear potential versus a for different quantum numbers n and $m = 1$.

Next let us make $b = 0, a = M\omega$. In the case, the interaction $f(x)$ will become the linear potential, that is to say, the generalized Kemmer oscillator will reduce to the general Kemmer oscillator. Then, the corresponding eigenvalue and eigenfunction will change into

$$E = \pm m\sqrt{1+4r(n+1)}, \quad r = \frac{2\omega}{m}, \tag{23}$$

$$\psi(x) = \begin{pmatrix} \dfrac{2}{2E-M}\left[\hat{p}_x + \mathrm{i}f(x)\right] \\ 1 \\ 1 \\ \dfrac{2}{2E+M}\left[\hat{p}_x + \mathrm{i}f(x)\right] \end{pmatrix} \times Nx\exp\left(-\frac{1}{2}M\omega x^2\right)\mathrm{L}_n^{1/2}\left(M\omega x^2\right). \tag{24}$$

The results are in good agreement with those given by Ref. [28].

3.2. The exact solution of generalized Kemmer equation with $f(x)$ to be exponential-type potential

As we all know, the exponential-type potentials such as the Eckart potential,[55–57] the Hulth'en-type potential,[58–60] the Woods–Saxon potential,[61–64] and the Morse-type potential[65–68] are a typical diatomic molecular potential model which has been used to describe the interaction between two atoms in a diatomic molecule and has been widely applied in physics and chemical physics.[69] In the rest part of this paper, we will try to find the closed-form analytic solution of the Kemmer equation with the interactions $f(x)$ to be chosen as generalized exponential potential:

$$f(x) = \frac{A_1 + A_2\mathrm{e}^{-\alpha x}}{B_1 + B_2\mathrm{e}^{-\alpha x}}, \tag{25}$$

with A_1, A_2, B_1, B_2, and α being real parameters. If these parameters are chosen suitably, the interactions $f(x)$ will turn into the Hulth'en-type potential, the Woods–Saxon-type potential, and the Morse-type potential. Substituting Eq. (25) into Eq. (12) gives rise to the following equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_2 + \left[\frac{(A_2B_1 - A_1B_2)\,\alpha\,\mathrm{e}^{-\alpha x} - (A_1 + A_2\mathrm{e}^{-\alpha x})^2}{(B_1 + B_2\mathrm{e}^{-\alpha x})^2} + \varepsilon^2\right]\psi_2 = 0, \tag{26}$$

where $\varepsilon^2 = E^2 - m^2$. Then, making $y = \mathrm{e}^{-\alpha x}$ equation (26) can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_2 + \frac{(B_1 + B_2y)}{y(B_1 + B_2y)}\frac{\mathrm{d}}{\mathrm{d}y}\psi_2 + \frac{-D_1y^2 + F_1y - G_1}{y^2(B_1 + B_2y)^2}\psi_2 = 0, \tag{27}$$

where the following parameters have been defined:

$$D_1 = \frac{A_2^2 - \varepsilon^2B_2^2}{\alpha^2},$$

$$F_1 = \frac{(A_2B_1 - A_1B_2)\,\alpha - 2A_1A_2 + 2\varepsilon^2B_1B_2}{\alpha^2},$$

$$G_1 = \frac{A_1^2 - \varepsilon^2B_1^2}{\alpha^2}.$$

It is obvious that equation (27) also has the same form with the basic equation of the NU method. By comparing Eq. (27) with Eq. (A1) in Appendix A the following parameters can be easily obtained

$$c_1 = 1, \quad c_2 = c_3 = -\frac{B_2}{B_1},$$

$$\xi_1 = \frac{D_1}{B_1^2}, \quad \xi_2 = \frac{F_1}{B_1^2}, \quad \xi_3 = \frac{G_1}{B_1^2}. \tag{28}$$

In addition, other coefficients in the NU method are deter-

mined by Eq. (A4) in Appendix A and can be written as

$$c_4=0,\quad c_5=\frac{B_2}{2B_1},\quad c_6=\frac{B_2^2}{4B_1^2}+\frac{D_1}{B_1^2},\quad c_7=-\frac{F_1}{B_1^2},$$
$$c_8=\frac{G_1}{B_1^2},\quad c_9=\frac{B_2^2G_1}{B_1^4}+\frac{B_2F_1}{B_1^3}+c_6,\quad c_{10}=1+\frac{2\sqrt{G_1}}{B_1},$$
$$c_{11}=2\sqrt{c_9}-\frac{2B_2}{B_1}\left(1+\frac{\sqrt{G_1}}{B_1}\right),$$
$$c_{12}=\frac{\sqrt{G_1}}{B_1},\quad c_{13}=\frac{B_2}{B_1}\left(\frac{1}{2}+\frac{\sqrt{G_1}}{B_1}\right)-\sqrt{c_9}. \tag{29}$$

By using the NU method again, we can obtain the eigenfunction and eigenenergy

$$\psi(x)=\begin{pmatrix}\frac{2}{2E-M}\left[\hat{p}_x+\mathrm{i}f(x)\right]\\1\\1\\\frac{2}{2E+M}\left[\hat{p}_x+\mathrm{i}f(x)\right]\end{pmatrix}N\mathrm{e}^{-\alpha\vartheta x}\left(1-c_3\mathrm{e}^{-\alpha x}\right)^{(1-\rho)/2}\mathrm{P}_n^{(2\vartheta,-\rho)}\left(1-2c_3\mathrm{e}^{-\alpha x}\right), \tag{30}$$

$$E^2=m^2+\frac{A_1^2}{B_1^2}-\left\{\frac{\alpha B_1\left[\left(2n^2+\lambda_2\right)\alpha B_1B_2-\lambda_2\sqrt{\lambda_1}+2\kappa\right]-4A_1\kappa}{2B_1\left[\sqrt{\lambda_1}-\alpha\lambda_2B_1B_2\right]}\right\}^2, \tag{31}$$

where $\lambda_1=4\kappa^2+\alpha B_1B_2\left(4\kappa+\alpha B_1B_2\right)$, $\lambda_2=(2n+1)$, $\kappa=A_2B_1-A_1B_2$, $\rho=\sqrt{\lambda_1}/\alpha B_1B_2$, $\vartheta=\sqrt{G_1}/B_1$, and N is the new normalization constant.

Now let us consider a few special interest cases.

Case 1 $(A_2=0)$

As the first special case of $A_2=0$, the interaction of the generalized exponential potential will turn into the Hulth'en-type potential.[58–60] So, from Eqs. (30) and (31) the corresponding eigenfunction and eigenvalue change into

$$\psi(x)=\begin{pmatrix}\frac{2}{2E-M}\left[\hat{p}_x+\mathrm{i}f(x)\right]\\1\\1\\\frac{2}{2E+M}\left[\hat{p}_x+\mathrm{i}f(x)\right]\end{pmatrix}N\mathrm{e}^{-\alpha\vartheta x}\left(1-c_3\mathrm{e}^{-\alpha x}\right)^{(1-\tau)/2}\mathrm{P}_n^{(2\vartheta,-\tau)}\left(1-2c_3\mathrm{e}^{-\alpha x}\right), \tag{32}$$

$$E^2=m^2+\frac{A_1^2}{B_1^2}-\left\{\frac{B_1\alpha\left[\left(2n^2+\lambda_2\right)B_1\alpha-\lambda_2\sqrt{\chi_1}\right]-2A_1\left(2A_1-B_1\alpha\right)}{2B_1\left(\sqrt{\chi_1}-\lambda_2B_1\alpha\right)}\right\}^2, \tag{33}$$

where the parameter ρ and λ_1 will turn into the parameter $\tau=\sqrt{\chi_1}/\alpha B_1$ and $\chi_1=4A_1\left(A_1-\alpha B_1\right)+\alpha^2B_1^2$, respectively.

Case 2 $(A_1=0)$

If we make $A_1=0$, the interaction of the generalized exponential potential will transform into the Woods–Saxon-type potential.[61–64] The eigenenergy and the corresponding eigenfunction can be written as follows:

$$E^2=m^2-\left[\frac{\left(2n^2+\lambda_2\right)B_2\alpha^2-\alpha\lambda_2\sqrt{\chi_2}+2\alpha A_2}{2\left(\sqrt{\chi_2}-\alpha\lambda_2B_2\right)}\right]^2, \tag{34}$$

$$\psi(x)=\begin{pmatrix}\frac{2}{2E-M}\left[p_x+\mathrm{i}f(x)\right]\\1\\1\\\frac{2}{2E+M}\left[p_x+\mathrm{i}f(x)\right]\end{pmatrix}N\mathrm{e}^{-\mathrm{i}\varepsilon x}\left(1-c_3\mathrm{e}^{-\alpha x}\right)^{(1-\upsilon)/2}\mathrm{P}_n^{(2i\varepsilon/\alpha,-\upsilon)}\left(1-2c_3\mathrm{e}^{-\alpha x}\right), \tag{35}$$

where the parameter ρ and λ_1 will turn into the parameter $\upsilon=\sqrt{\chi_2}/\alpha B_2$ and $\chi_2=4A_2\left(A_2+\alpha B_2\right)+B_2^2\alpha^2$, respectively.

Case 3 $(B_2=0)$

The last case of $B_2=0$, the generalized exponential potential will reduce to the Morse-type potential.[65–68] From the Eq. (31), the corresponding eigenenergy can be obtained as follows:

$$E^2=m^2+\frac{A_1^2}{B_1^2}-\left[\frac{n\alpha B_1+A_1}{B_1}\right]^2. \tag{36}$$

The parameter c_3 will be zero due to $B_2=0$, so the corre-

sponding eigenfunction can be given by

$$\psi(x) = \begin{pmatrix} \frac{2}{2E-M}\left[\hat{p}_x + \mathrm{i}f(x)\right] \\ 1 \\ 1 \\ \frac{2}{2E+M}\left[\hat{p}_x + \mathrm{i}f(x)\right] \end{pmatrix} N\mathrm{e}^{-\alpha\vartheta x}\mathrm{e}^{-(A_2/\alpha B_1)\mathrm{e}^{-\alpha x}} \times \mathrm{L}_n^{2\vartheta}\left(2A_2\mathrm{e}^{-\alpha x}/\alpha B_1\right), \quad (37)$$

where N is the normalization constant. For the various values of quantum numbers n and $m = A_1 = A_2 = B_1 = B_2 = 1$, the square of energy spectrum of the generalized exponential-type potential versus parameter α has been given in Fig. 2 (in natural units).

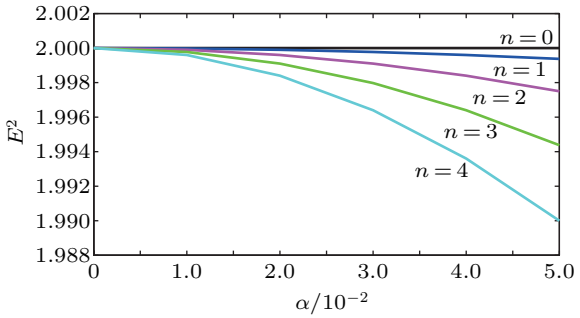


Fig. 2. (color online) The square of energy spectrum of generalized exponential-type potential versus α for $m = A_1 = A_2 = B_1 = B_2 = 1$ and different quantum numbers n.

4. Conclusion

In this paper, motivated by the above basic idea of introducing Dirac oscillator, the generalized Kemmer oscillator was introduced. The generalized Kemmer oscillator has been studied when the interactions have been chosen as Coulomb potential plus linear potential and the exponential-type potentials. By using the Nikiforov–Uvarov method, the corresponding exact solutions have been found. Several special interesting cases are discussed and the results obtained agree with those obtained in previous literature. It is worth noting that apart from the cases considered in this paper, the interactions $f(x)$ can also be taken as Mie-type potential and Kratzer–Fues potential,[70] etc.

Appendix A: The generalized parametric NU method

We present basic concepts of the NU method; for more details, we refer readers to Refs. [37]–[47]. The NU method has been proposed to solve the second-order differential equation

$$\frac{\mathrm{d}^2R}{\mathrm{d}s^2} + \frac{(c_1 - c_2 s)}{s(1-c_3 s)}\frac{\mathrm{d}R}{\mathrm{d}s} + \frac{\left(-\xi_1 s^2 + \xi_2 s - \xi_3\right)}{s^2(1-2c_3 s)^2}R = 0. \quad (A1)$$

The eigenenergy and eigenfunction can be obtained, respectively as

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0, \quad (A2)$$

$$R(s) = Ns^{c_{12}}(1-c_3 s)^{-c_{12}-(c_{13}/c_3)} \times \mathrm{P}_n^{(c_{10}-1,(c_{13}/c_3)-c_{10}-1)}(1-2c_3 s). \quad (A3)$$

Apparently, the eigenfunction can be expressed by Jacobi polynomials. These parameters can be written as

$$\begin{aligned} &c_4 = \frac{1}{2}(1-c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3), \quad c_6 = c_5^2 + \xi_1, \\ &c_7 = 2c_4c_5 - \xi_2, \quad c_8 = c_4^2 + \xi_3, \\ &c_9 = c_3c_7 + c_3^2c_8 + c_6, \quad c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \\ &c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right), \\ &c_{12} = c_4 + \sqrt{c_8}, \quad c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right). \end{aligned} \quad (A4)$$

When $c_3 = 0$, we can easily obtain

$$\lim_{c_3\to 0}(1-c_3 s)^{-c_{12}-(c_{13}/c_3)} = \mathrm{e}^{c_{13}s}, \quad (A5)$$

and

$$\lim_{c_3\to 0}\mathrm{P}_n^{(c_{10}-1,(c_{11}/c_3)-c_{10}-1)}(1-2c_3 s) = \mathrm{L}_n^{c_{10}-1}(c_{11}s). \quad (A6)$$

The solution of the eigenfunction can be written as the Laguerre polynomials when $c_3 = 0$

$$R(s) = Ns^{c_{12}}\mathrm{e}^{c_{13}s}\mathrm{L}_n^{c_{10}-1}(c_{11}s). \quad (A7)$$

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