Profile function properties & optical transition formulae*

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Profile function properties with different variables are discussed, the formulae of stimulated absorption, spontaneous and stimulated emission, absorption and emission coefficients, and cross sections are deduced, and some confusing issues are clarified.

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1. Introduction

Due to defects, strain, disorder, lattice vibration, and so on in solids, an ideal optical absorption or emission transition line of rare earth ions, described by the Dirac function δ, is usually a wider band, which can be described by a profile function ϕ, such as the Voigt function,[1–4] with independent variable wavelength λ, frequency ν, or angle frequency ω, which has the following relationships:

$$\int \phi(\lambda)\,\mathrm{d}\lambda = 1, \tag{1a}$$

$$\int \phi(\nu)\,\mathrm{d}\nu = 1, \tag{1b}$$

$$\int \phi(\omega)\,\mathrm{d}\omega = 1, \tag{1c}$$

$$\phi(\nu) = \frac{\lambda^2}{c}\phi(\lambda), \tag{2a}$$

$$\phi(\omega) = \frac{\lambda^2}{2\pi c}\phi(\lambda), \tag{2b}$$

$$\phi(\omega) = \frac{1}{2\pi}\phi(\nu), \tag{2c}$$

where c is the light speed in vacuum. The frequency ν or angle frequency ω of a light source is fixed when it transmits in a medium or vacuum, so equations (2a)–(2c) are suitable for the medium or vacuum, and λ is the wavelength in vacuum. The δ function has the same relationships as in Eqs. (2a)–(2c). Based on this, a physical quantity such as spontaneous transition rate $A(\omega) = A\phi(\omega)$ can be written as

$$A(\omega) = A\phi(\omega) = \frac{\lambda^2}{2\pi c}A\phi(\lambda) = \frac{A}{2\pi}\phi(\nu). \tag{3}$$

From Eq. (3), it seems that the transition rate has very different amplitudes if a profile function has a different independent variable, e.g., $\frac{\lambda^2}{2\pi c}A$, $\frac{A}{2\pi}$, or A in Eq. (3), which easily causes much confusion in application.

On the other hand, there are many unit systems of electromagnetics, such as the rationalized MKSA system of units, the absolute electrostatic system of units (e.s.u. or CGSE), the absolute electromagnetic system of units (e.m.u. or CGSM), and the Gauss system of units (CGS). Electromagnetic quantities and formulae usually have a variety of values and expressions. For example, the dielectric constant ε_0 and permeability μ_0 in vacuum are 8.85×10^{-12} F/m and $4\pi\times10^{-7}$ H/m in MKSA, but both are 1 in CGS; electromagnetic energy density $\rho = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B})$ in MKSA, but $\rho = \frac{1}{4\pi}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{H}\cdot\boldsymbol{B})$ in the Gauss system of units. Although international system of units has been widely used recently, many very different systems of units have been used in references, which were usually not declared explicitly, resulting in confusion about published calculations. It is necessary to clarify transition formula expressions in order to perform spectral analysis or calculation.

The present work first discusses the distribution property of a physical quantity with a profile in wavelength, frequency, or angle frequency, and further clarifies the formulae of emission and absorption. The unit system of this work is SI units.

2. General properties of profile functions

Assume a physical quantity Q, such as the absorption coefficient, emission or absorption cross section, or transition probability, with the variable angle frequency ω and profile $\phi(\omega)$ is expressed as

$$Q(\omega) = Q_0\phi(\omega). \tag{4}$$

Substituting Eqs. (2b) and (2c) into Eq. (4), we obtain the following equations:

$$Q(\omega) = \frac{Q_0}{2\pi}\phi(\nu), \tag{5}$$

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$$Q(\omega) = \frac{Q_0\lambda^2}{2\pi c}\phi(\lambda). \tag{6}$$

Although equations (5) and (6) are expressed in profile $\phi(\nu)$ or $\phi(\lambda)$, either should be integrated against variable ω. In order to obtain the distribution of Q with variable ν or λ, $Q(\nu)$ and $Q(\lambda)$ should satisfy the relationships

$$\int Q(\nu)\,\mathrm{d}\nu = \int Q_0'\phi(\nu)\,\mathrm{d}\nu, \tag{7}$$

$$\int Q(\lambda)\,\mathrm{d}\lambda = \int Q_0''\phi(\lambda)\,\mathrm{d}\lambda. \tag{8}$$

Then, integrating Eq. (5) versus ω, the left is changed as

$$\int Q(\omega)\mathrm{d}\omega = \int Q(\omega)2\pi\mathrm{d}\nu,$$

and the right is changed as

$$\int \frac{Q_0}{2\pi}\phi(\nu)\mathrm{d}\omega = \int Q_0\phi(\nu)\mathrm{d}\nu.$$

The left integral should be equal to the right integral, so

$$\int Q(\omega)2\pi\mathrm{d}\nu = \int Q_0\phi(\nu)\,\mathrm{d}\nu. \tag{9}$$

Comparing Eq. (9) with Eq. (7), we obtain the following equation:

$$Q(\nu) = 2\pi Q(\omega), \quad Q_0' = Q_0,$$

and then

$$Q(\nu) = 2\pi Q(\omega) = Q_0\phi(\nu), \tag{10}$$

which should be integrated against ν on both sides directly to obtain the total quantity Q_0.

Integrate Eq. (10) versus ω, using $\mathrm{d}\omega = \frac{2\pi c}{\lambda^2}\mathrm{d}\lambda$, through the same procedure, we obtain $Q(\lambda) = \frac{c}{\lambda^2}Q(\omega)$ and $Q_0'' = Q_0$, and then

$$Q(\lambda) = \frac{c}{\lambda^2}Q(\omega) = Q_0\phi(\lambda), \tag{11}$$

which should be integrated against λ directly on both sides to obtain the total quantity Q_0.

In summary, if a physical quantity Q can be expressed as the product of amplitude and a profile function with variable wavelength, frequency, or angle frequency, it has the same distribution expression and amplitude Q_0 in different variable spaces, and the profile function value is normalized in the respective variable spaces. This reflects the fact that a physical quantity should be the same no matter how it is described by a profile function in variables of wavelength, frequency, or angle frequency.

3. Transition formulae in absorption and emission spectra

3.1. Optical absorption and emission

Let us consider a plane electromagnetic wave, its electric and magnetic vector can be written as

$$\boldsymbol{E} = \boldsymbol{E}_0\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}), \tag{12}$$

$$\boldsymbol{B} = \mu\boldsymbol{H} = \mu\sqrt{\frac{\varepsilon}{\mu}}\frac{\boldsymbol{k}}{k}\times\boldsymbol{E} = \frac{n}{c}\frac{\boldsymbol{k}}{k}\times\boldsymbol{E}, \tag{13}$$

where μ is the permeability, ε is the dielectric constant, n is the refractive index, c is the light speed in vacuum, and $\boldsymbol{k}$ is the wave vector. In the atomic size region, $\boldsymbol{k}\cdot r \approx \boldsymbol{r}/\lambda \ll 1$, $\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) \approx \cos(\omega t)$, the electric field can be seen as a uniform field whose energy density for a linear medium[5] is

$$\begin{aligned}\rho &= \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right)\\ &= \varepsilon E^2 = \varepsilon E_0^2\cos^2(\omega t).\end{aligned} \tag{14}$$

During a period $T = 2\pi/\omega$, the average energy density

$$\rho = \varepsilon E_0^2\frac{1}{T}\int_0^{2\pi/\omega}\cos^2(\omega t)\mathrm{d}t = \frac{1}{2}\varepsilon E_0^2, \tag{15}$$

and then the average energy density per unit angle frequency ρ_ω can be written as

$$\begin{aligned}\rho_\omega &= \rho\delta(\omega) = \frac{1}{2}\varepsilon E_0^2\delta(\omega)\\ &= \frac{1}{2}n^2\varepsilon_0 E_0^2\delta(\omega),\end{aligned} \tag{16a}$$

and $\delta(\omega)$ should have dimension Hz^{-1}, which satisfies the relationship $\rho = \int\rho_\omega\mathrm{d}\omega$. If the energy is distributed in a finite region near ω, then $\delta(\omega)$ should be replaced by $\phi(\omega)$:

$$\begin{aligned}\rho_\omega &= \rho\phi(\omega) = \frac{1}{2}\varepsilon E_0^2\phi(\omega)\\ &= \frac{1}{2}n^2\varepsilon_0 E_0^2\phi(\omega).\end{aligned} \tag{16b}$$

The additional energies $\mathcal{H}'$ induced by electric and magnetic dipole $\boldsymbol{\mu}_\mathrm{e}$ and $\boldsymbol{\mu}_\mathrm{m}$ in the time-dependent electromagnetic field are

$$\mathcal{H}_\mathrm{e}' = \boldsymbol{\mu}_\mathrm{e}\cdot\boldsymbol{E} = E_0\cos\theta\mu_\mathrm{e}, \tag{17}$$

$$\mathcal{H}_\mathrm{m}' = \boldsymbol{\mu}_\mathrm{m}\cdot\boldsymbol{B}. \tag{18}$$

With Eq. (13), equation (18) is written as

$$\mathcal{H}_\mathrm{m}' = \sqrt{\mu\varepsilon}\mu_\mathrm{m}E\cos\theta = E\cos\theta\frac{n}{c}\mu_\mathrm{m}. \tag{19}$$

Consider a transition from the initial state $|Ii\rangle$ $(i = 1,\ldots,g_I)$ with lower energy E_I to $|Ff\rangle$ $(f = 1,\ldots,g_F)$ with higher energy E_F. Where i and f are used to disignate degenerated states, the transition probability caused by the electric or magnetic dipole should be written with respect to angle frequency ω according to Eq. (2.4.7) in Ref. [6]:

$$W_{if}(\omega) = \frac{\pi}{2\hbar^2}|\langle Ff|H'|Ii\rangle|^2\phi(\omega - \omega_{IF}), \tag{20}$$

where the δ function in the reference has been replaced with a profile function ϕ, $\omega_{IF} = (E_F - E_I)/\hbar$, $\hbar = h/2\pi$, h is Planck's

constant, ε_0 is the dielectric constant in vacuum. For randomly oriented dipoles, mean value $\langle\cos\theta\rangle$ over all dipole orientations is 1/3, so substituting Eq. (16), (17), or (19) into Eq. (20) gives

$$W_{if}(\omega)=\frac{\pi}{3n^2\varepsilon_0\hbar^2}\left|\langle Ff|\hat{\boldsymbol{O}}|Ii\rangle\right|^2\rho\phi(\omega-\omega_{IF}), \quad (21a)$$

with $\hat{\boldsymbol{O}}=\boldsymbol{\mu}_{\mathrm{e}}$ for the electric dipole and $\hat{\boldsymbol{O}}=\frac{n}{c}\boldsymbol{\mu}_{\mathrm{m}}$ for the magnetic dipole. Equation (21a) implies the following equations:

$$\begin{aligned}&W_{if}(\omega)=W_0\phi(\omega-\omega_{IF}),\\&W_0=\frac{\pi}{3n^2\varepsilon_0\hbar^2}\left|\langle Ff|\hat{\boldsymbol{O}}|Ii\rangle\right|^2\rho,\\&\int W_{if}(\omega)\,\mathrm{d}\omega=W_0.\end{aligned}$$

For convenient comparison, we can substitute Eq. (2c) into Eq. (21a), yielding

$$W_{if}=\frac{2\pi^2}{3n^2\varepsilon_0h^2}\left|\langle Ff|\hat{\boldsymbol{O}}|Ii\rangle\right|^2\rho\phi(\nu-\nu_{IF}), \quad (21b)$$

which corresponds to Eq. (2.4.30) in Ref. [6]

$$W=\frac{2\pi^2}{3n^2\varepsilon_0h^2}|\mu|^2\rho g_{\mathrm{t}}(\nu-\nu_0). \quad (21c)$$

It can be seen that the right side of either Eq. (21a) or (21b) is equal to that of Eq. (21c), but all of them should be integrated versus ω in order to obtain the total transition probability

$$\begin{aligned}W_0&=\int W\,\mathrm{d}\omega=\int 2\pi W\,\mathrm{d}\nu\\&=\frac{4\pi^3}{3n^2\varepsilon_0h^2}|\mu|^2\rho\int g_{\mathrm{t}}(\nu-\nu_0)\,\mathrm{d}\nu\\&=\frac{4\pi^3}{3n^2\varepsilon_0h^2}|\mu|^2\rho.\end{aligned}$$

If we want to know the total value W_0 of W by integrating (21c) versus ν directly on both sides, W should be written as

$$W(\nu)=2\pi W=\frac{4\pi^3}{3n^2\varepsilon_0h^2}|\mu|^2\rho g_{\mathrm{t}}(\nu-\nu_0). \quad (22)$$

In fact, this has been proved in Section 2; the factor 2π is from the condition that equation (21c) should be integrated against ω, even though it contains the profile function $g_{\mathrm{t}}(\nu-\nu_0)$. Furthermore, there exists a difference of a factor of 2π in the absorption probability formulae (24a)–(24d), (28), and emission probability formulae (30a)–(30c) in the following contexts.

Consider that the local electric field correction E_0 for the electric dipole transition should be multiplied by a correct factor χ' in Eqs. (21a)–(21c). According to Eqs. (17) and (19) in Ref. [7], for an electron in a cubic insulator lattice site with relative permittivity ε_{r}, χ' should be

$$\chi'=\begin{cases}\dfrac{\varepsilon_{\mathrm{r}}+2}{3}=\dfrac{n^2+2}{3}, & \text{for electric dipole,}\\ 1, & \text{for magnetic dipole.}\end{cases} \quad (23)$$

Then the average transition rate W_{IF} of $|Ii\rangle\rightarrow|Ff\rangle|$ is

$$\begin{aligned}W_{IF}(\omega)&=\frac{1}{g_I}\sum_{i,f}\chi'^2W_{if}=\frac{\pi}{3n^2\varepsilon_0\hbar^2}\frac{\chi'^2}{g_I}\\&\times\sum_{if}\left|\langle Ff|\hat{\boldsymbol{O}}|Ii\rangle\right|^2\rho\phi(\omega-\omega_{IF}). \quad (24a)\end{aligned}$$

Furthermore, n is separated from $\hat{\boldsymbol{O}}$ to form a medium correction coefficient for the magnetic dipole transition, and equation (24a) is written as

$$W_{IF}(\omega)=\frac{\pi}{3n^2\varepsilon_0\hbar^2}\frac{\chi}{g_I}\sum_{if}\left|\langle Ff|\hat{\boldsymbol{\mu}}|Ii\rangle\right|^2\rho\phi(\omega-\omega_{IF}), \quad (24b)$$

$$\chi=\begin{cases}\dfrac{(n^2+2)^2}{9}, & \text{for electric dipole,}\\ n^2, & \text{for magnetic dipole,}\end{cases} \quad (25)$$

$$\hat{\boldsymbol{\mu}}=\begin{cases}\boldsymbol{\mu}_{\mathrm{e}}, & \text{for electric dipole,}\\ \dfrac{\boldsymbol{\mu}_{\mathrm{m}}}{c}=\dfrac{\mu_B}{c}\dfrac{1}{\hbar}(\boldsymbol{L}+2\boldsymbol{S}), & \text{for magnetic dipole,}\end{cases} \quad (26)$$

where $\mu_{\mathrm{B}}=\frac{e\hbar}{2m}$ is the Bohr magneton, e and m are the charge and mass of an electron, $\hbar$ is Planck's constant divided by 2π, and $\boldsymbol{L}$ and $\boldsymbol{S}$ are the orbital and spin momentum operators. According to the conclusion in Section 2, W_{IF} can be written in terms of frequency or wavelength immediately as

$$\begin{aligned}W_{IF}(\nu)&=2\pi W_{FI}(\omega)=\frac{\pi}{3n^2\varepsilon_0\hbar^2}\frac{\chi}{g_I}\\&\times\sum_{if}\left|\langle Ff|\hat{\boldsymbol{\mu}}|Ii\rangle\right|^2\rho\phi(\nu-\nu_{IF}), \quad (24c)\\W_{IF}(\lambda)&=\frac{2\pi c}{\lambda^2}W_{FI}(\omega)=\frac{\pi}{3n^2\varepsilon_0\hbar^2}\frac{\chi}{g_I}\\&\times\sum_{if}\left|\langle Ff|\hat{\boldsymbol{\mu}}|Ii\rangle\right|^2\rho\phi(\lambda-\lambda_{IF}), \quad (24d)\end{aligned}$$

where $\lambda_{IF}=\frac{hc}{E_F-E_I}$. Consider the relationship

$$W_{IF}=B_{IF}\rho, \quad (27)$$

$$B_{IF}(\omega)=\frac{\pi}{3n^2\varepsilon_0\hbar^2}\frac{\chi}{g_I}\sum_{if}\left|\langle Ff|\hat{\boldsymbol{O}}|Ii\rangle\right|^2\phi(\omega-\omega_{IF}), \quad (28)$$

and the Einstein relation

$$\frac{B_{IF}}{B_{FI}}=\frac{g_F}{g_I},\quad \frac{A_{FI}}{B_{FI}}=\frac{\hbar\omega^3n^3}{\pi^2c^3}, \quad (29)$$

where B_{IF} is Einstein's stimulated absorption coefficient, A_{FI} is Einstein's spontaneous emission coefficient, B_{FI} is Einstein's stimulated emission coefficient. Then the spontaneous emission transition probability A_{IF} can be written as

$$\begin{aligned}A_{FI}(\omega)&=\frac{\hbar\omega^3n^3}{\pi^2c^3}\frac{g_I}{g_F}\frac{1}{g_I}\sum_{i,f}W_{if}=\frac{2\omega^3}{3\varepsilon_0hc^3}\frac{n\chi}{g_F}\\&\times\sum_{if}\left|\langle Ff|\hat{\boldsymbol{\mu}}|Ii\rangle\right|^2\phi(\omega-\omega_{IF}), \quad (30a)\\A_{FI}(\nu)&=2\pi A_{FI}(\omega)=\frac{16\pi^3\nu^3}{3\varepsilon_0hc^3}\frac{n\chi}{g_F}\\&\times\sum_{if}\left|\langle Ff|\hat{\boldsymbol{\mu}}|Ii\rangle\right|^2\phi(\nu-\nu_{IF}), \quad (30b)\end{aligned}$$

$$A_{FI}(\lambda) = \frac{2\pi c}{\lambda^2}A_{FI}(\omega) = \frac{16\pi^3}{3\varepsilon_0 h\lambda^3}\frac{n\chi}{g_F} \times \sum_{if}|\langle Ff|\hat{\mu}|Ii\rangle|^2\phi(\lambda-\lambda_{IF}). \quad (30c)$$

It is usually to call $\mathcal{S}$ line strength with dimension m^2, whose value is

$$\mathcal{S}_{IF} = \mathcal{S}_{FI} = \frac{\sum_{if}|\langle Ff|\hat{\mu}|Ii\rangle|^2}{e^2}. \quad (31)$$

3.2. Absorption coefficient and emission cross section

3.2.1. Absorption coefficient

The ion number variation rate in E_I caused by the stimulated absorption, stimulated and spontaneous emission between $|Ii\rangle$ and $|Ff\rangle$ is given by

$$-\frac{\Delta N}{\Delta t} = \frac{\Delta N_I}{\Delta t} = N_F\rho_\omega B_{FI}(\omega) - N_I\rho_\omega B_{IF}(\omega) + A_{FI}(\omega)N_F, \quad (32)$$

where N_I is the absorption ion center number in the volume passed by light beam per unit time. For a general absorption measurement, $N_F \ll N_I$,

$$\frac{\Delta N_I}{\Delta t} \approx -N_I\rho(\omega)B_{IF}(\omega). \quad (33)$$

Assume a light beam with cross section Λ and intensity $I(\omega)$, which is the energy passing through Λ per unit time, passes through a medium by $\Delta l = c/n$ and decreases by $-\Delta I(\omega)$ per unit time, then

$$\Delta N_I = -N_I\rho_\omega B_{IF}(\omega), \quad (34a)$$
$$I(\omega) = \frac{c}{n}\Lambda\rho_\omega, \quad (34b)$$
$$-\Delta I(\omega) = -\Delta N_I\hbar\omega. \quad (34c)$$

The absorption coefficient is defined as

$$\alpha_{IF}(\omega) = \frac{-\Delta I(\omega)}{I(\omega)\Delta l}. \quad (35)$$

Substituting Eqs. (34a)–(34c) into Eq. (35) gives

$$\alpha_{IF}(\omega) = \frac{N_I\rho_\omega B_{IF}(\omega)\hbar\omega}{\frac{c}{n}\Lambda\rho(\omega)\Delta l} = \frac{nN_iB_{IF}(\omega)\hbar\omega}{c}, \quad (36)$$

where N_i is the absorption ion concentration,

$$N_i = \frac{N_I}{\Lambda\Delta l}.$$

Substituting Eq. (28) into Eq. (36) leads to

$$\alpha_{IF}(\omega) = \frac{\pi\omega N_i}{3nc\varepsilon_0\hbar}\frac{\chi}{g_I}\sum_{if}|\langle Ff|\hat{O}|Ii\rangle|^2\phi(\omega-\omega_{IF}). \quad (37a)$$

According to the conclusion in Section 2, the following equation can be written immediately:

$$\alpha_{IF}(\nu) = 2\pi\alpha_{IF}(\omega) = \frac{4\pi^3\nu N_i}{3nc\varepsilon_0 h}\frac{\chi}{g_I} \times \sum_{if}|\langle Ff|\hat{O}|Ii\rangle|^2\phi(\nu-\nu_{IF}), \quad (37b)$$

$$\alpha_{IF}(\lambda) = \frac{2\pi c}{\lambda^2}\alpha_{IF}(\omega) = \frac{4\pi^3N_i}{3n\varepsilon_0 h\lambda}\frac{\chi}{g_I} \times \sum_{if}|\langle Ff|\hat{O}|Ii\rangle|^2\phi(\lambda-\lambda_{IF}). \quad (37c)$$

$\alpha_{IF}(\omega)$ can be measured experimentally and is labeled as $\alpha_{IF}^{\rm obs}(\omega)$ here; then integrate (37c) against λ directly, take $1/\lambda$ as a constant and $1/\bar{\lambda}$ for a sharp absorption band; then

$$\int\alpha_{IF}^{\rm obs}(\omega)\,{\rm d}\lambda = \frac{2\pi^2\bar{\lambda}N_i}{3n\varepsilon_0 ch}\frac{\chi}{g_I}\sum_{if}|\langle Ff|\hat{O}|Ii\rangle|^2 = \frac{2\pi^2e^2\bar{\lambda}N_i}{3n\varepsilon_0 ch\lambda}\frac{\chi}{g_I}\mathcal{S}_{IF}. \quad (38a)$$

Based on Einstein's relationship and Eq. (38a), A_{FI} can be obtained from the absorption spectrum integral as

$$A_{FI} = \frac{8\pi cn^2}{N_i\bar{\lambda}^4}\frac{g_I}{g_F}\int\alpha_{IF}^{\rm exp}(\omega)\,{\rm d}\lambda. \quad (38b)$$

3.2.2. Emission cross section

For laser operation, spontaneous radiation does not need to be considered, so equation (32) is changed to

$$-\frac{\Delta N}{\Delta t} = -\frac{\Delta N_F}{\Delta t} = N_F\rho_\omega B_{FI}(\omega) - N_I\rho_\omega B_{IF}(\omega) = \left(N_F - \frac{g_F}{g_I}N_I\right)\rho_\omega B_{FI}(\omega), \quad (39)$$

Like the absorption coefficient, the emission coefficient or gain coefficient β is defined as

$$\beta_{IF}(\omega) = \frac{\Delta I(\omega)}{I(\omega)\Delta l}, \quad (40a)$$
$$-\Delta N_F = \left(N_F - \frac{g_F}{g_I}N_I\right)\rho_\omega B_{FI}(\omega), \quad (40b)$$
$$\Delta I(\omega) = -\Delta N_F\hbar\omega. \quad (40c)$$

Substitute Eqs. (34b), (40b), (40c) into Eq. (40a),

$$\beta_{IF}(\omega) = \frac{\left(N_F - \frac{g_F}{g_I}N_I\right)\rho(\omega)B_{FI}(\omega)\hbar\omega}{\frac{c}{n}\Lambda\rho(\omega)\Delta l} = \frac{nN_fB_{FI}(\omega)\hbar\omega}{c}, \quad (40d)$$

where

$$N_f = \left(N_F - \frac{g_F}{g_I}N_I\right)\Big/(\Lambda\Delta l),$$

which is the inversion activator number per unit volume. The emission cross section is defined as

$$\sigma_{FI}(\omega) = \frac{\beta_{IF}(\omega)}{N_f} = \frac{nB_{FI}(\omega)\hbar\omega}{c}. \quad (41a)$$

Substituting Eq. (28) into Eq. (41) gives

$$\sigma_{FI}(\omega) = \frac{\pi\omega}{3nc\varepsilon_0\hbar}\frac{\chi}{g_F}\sum_{if}|\langle Ff|\hat{\mu}|Ii\rangle|^2\phi(\omega-\omega_{IF}). \quad (41b)$$

Using Einstein's relationship Eq. (29), $\sigma_{FI}(\omega)$ can be written as

$$\begin{aligned}\sigma_{FI}(\omega) &= \frac{\pi^2c^2}{\omega^2n^2}A_{FI}(\omega)=\frac{c^2}{4\nu^2n^2}A_{FI}(\omega)\\ &= \frac{\lambda^2}{4n^2}A_{FI}(\omega)=\frac{\lambda^2}{4n^2}A_{FI}\phi(\omega).\end{aligned}\tag{41c}$$

Substituting Eqs. (2b) and (2c) into Eq. (41c), we obtain

$$\sigma_{FI}(\omega)=\frac{\lambda^2}{8\pi n^2}A_{FI}\phi(\nu)=\frac{\lambda^4}{8\pi cn^2}A_{FI}\phi(\lambda),\tag{41d}$$

which is the F–L formula. Note that λ in Eq. (41a)–(41d) is the wavelength in vacuum, and if we want to know the emission cross section integral value Σ_{IF}, all of them should be integrated against ω, giving

$$\begin{aligned}\Sigma_{IF} &= \int\sigma_{FI}(\omega)\,\mathrm{d}\omega=\frac{\pi\omega}{3nc\varepsilon_0\hbar}\frac{\chi}{g_F}\sum_{if}|\langle Ff|\hat{\mu}|Ii\rangle|^2\\ &= \frac{\lambda^2}{4n^2}A_{FI},\end{aligned}\tag{41e}$$

which is the amplitude of the emission cross section and can be multiplied by a profile function to give its distribution in frequency, wavelength, or angle frequency space. Comparing (41e) with Eq. (2) in Ref. [8], we find that the integrated emission cross section $\sum(\alpha,\beta)$ is unreasonable:

$$\sum(\alpha,\beta)\equiv\int\sigma(\nu)\,\mathrm{d}\nu=\frac{\bar{\lambda}^2}{8\pi n^2c}A(\alpha\beta).$$

3.3. Related aspects of full profile fitting of the emission spectrum

If the emission spectrum $\mathcal{I}(\lambda)$ of a rare earth ion is fitted by the full profile method, $\mathcal{I}(\lambda)$ is described by[9]

$$\mathcal{I}(\lambda)\equiv\sum_{I,F}\sigma_I A_{IF}(\boldsymbol{A}_{tp}^k)\phi_{IF}(\lambda,\lambda_{IF},w)\nu_{IF}.\tag{42}$$

According to the conclusion in Section 2, $\nu_{IF}\sigma_I A_{IF}(\boldsymbol{A}_{tp}^k)$ should have the same value and is independent of the variable type of ϕ_{IF}, so the intensity $\mathcal{I}$ should have the same expression and the same $\boldsymbol{A}_{tp}^k$ should be obtained regardless of whether the emission spectrum is described with wavelength, frequency or angle frequency.

4. Conclusion

A physical quantity Q such as absorption and emission that can be expressed as a product of amplitude and a profile function with variable wavelength, frequency, or angle frequency has the same distribution expression and amplitude Q_0 in any of these variable spaces, and the profile function value is normalized in the respective space. Optical transition formulae about emission and absorption were deduced, and the related confusions were clarified.

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