Multiple exp-function method for soliton solutions of nonlinear evolution equations

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We applied the multiple exp-function scheme to the (2+1)-dimensional Sawada–Kotera (SK) equation and (3+1)-dimensional nonlinear evolution equation and analytic particular solutions have been deduced. The analytic particular solutions contain one-soliton, two-soliton, and three-soliton type solutions. With the assistance of Maple, we demonstrated the efficiency and advantages of the procedure that generalizes Hirota's perturbation scheme. The obtained solutions can be used as a benchmark for numerical solutions and describe the physical phenomena behind the model.

Keywords: (2+1)-dimensional Sawada–Kotera (SK) equation, (3+1)-dimensional nonlinear evolution equation (NLEE), multiple exp-function method, multiple wave solutions

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1. Introduction

As is well known, nonlinear evolution equations (NLEEs) are encountered in many physical problems as a result of modelling related with the phenomena which occur in nature. For example, the Korteweg–de Vries (KdV) equation arises in modelling shallow water waves, and the Burgers equation appears in nonlinear acoustic waves. NLEEs emerge in a variety of contexts such as, physics, fluid mechanics, material science, optical fibers, solid state physics, chemical kinematics, and finance. Solitons which are self-reinforcing solitary wave packets are one of the most important areas of research in the field of nonlinear optics.[1] Therefore, an important problem is obtaining solutions of these equations. In the last three decades, many methods were developed for constructing the analytical solutions of NLEEs. Some of them are listed in Refs. [2]–[8].

In 2010, Ma *et al.*[9] suggested the multiple exp-function method as a new constructive way for obtaining the multi-soliton solutions of NLEEs. The useful multiple exp-function method is widely used by many such as in Refs. [10] and [11] and by the reference therein.

The main objective of this study is to build multi-soliton solutions of some multi-dimensional NLEEs by applying the multiple exp-function method. For this goal we will consider (2+1)-dimensional Sawada–Kotera (SK) equation[12]

$$u_t - u_{5x} - 5u_x u_{2x} - 5uu_{3x} - 5u^2 u_x - 5u_{2xy} - 5uu_y + 5\int u_{2y}\,\mathrm{d}x - 5u_x \int u_y\,\mathrm{d}x = 0, \tag{1}$$

and (3+1)-dimensional nonlinear evolution equation (NLEE)[14]

$$3u_{xz} - 2u_{ty} - u_{3xy} + 4u_x u_y + 2uu_{xy} + 2u_{xx}\int u_y\,\mathrm{d}x = 0. \tag{2}$$

As can be seen, the above equations comprise the integral terms; therefore, we convert Eqs. (1) and (2) to partial differential equations (PDEs) by the help of

$$u = v_x. \tag{3}$$

Then, equations (1) and (2) can be read in the following new form, respectively:

$$v_{xt} - v_{6x} - 5v_{2x}v_{3x} - 5v_x v_{4x} - 5v_x^2 v_{2x} - 5v_{3xy} - 5v_x v_{xy} + 5v_{2y} - 5v_{2x}v_y = 0, \tag{4}$$

$$3v_{2xz} - 2v_{xyt} - v_{4xy} + 4v_{xy}v_{2x} + 2v_x v_{2xy} + 2v_{3x}v_y = 0. \tag{5}$$

Up to date, the (2+1) dimensional Sawada–Kotera and (3+1)-dimensional nonlinear evolution equations have been studied by many researchers from different aspects (see, Refs. [12], [13], and [14]–[16]).

The paper is structured as follows. In Section 2, the method is oulined. In Section 3, we apply the method to the (2+1)-dimensional Sawada–Kotera equation and (3+1)-dimensional nonlinear evolution equation respectively. In the meanwhile, we plot graphics of some solutions. In the final section, concluding remarks are given.

2. Multiple exp-function method

Now, we shortly present the method step by step. The method will be given for the (1+1) dimensional NLEEs but surely it can be extended to the multi-dimensional case too. The basic steps of the multiple exp-function method are prescribed as follows.[17–19]

Step 1 We consider the following (1+1) dimensional NLEEs:

$$P(x,t,u_x,u_t,\ldots) = 0. \tag{6}$$

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We will use the following first order auxiliary equations:

$$\eta_{i,x} = k_i \eta_i, \quad \eta_{i,t} = -\omega_i \eta_i, \quad 1 \leqslant i \leqslant n, \tag{7}$$

where k_i, $1 \leqslant i \leqslant n$, are the angular wave numbers and ω_i, $1 \leqslant i \leqslant n$, are the wave frequencies. Obtaining the solutions of Eq. (7) is easy and they are given as

$$\eta_i = c_i \mathrm{e}^{\xi_i}, \quad \xi_i = k_i x - \omega_i t, \quad 1 \leqslant i \leqslant n, \tag{8}$$

where c_i, $1 \leqslant i \leqslant n$, are arbitrary constants.

Step 2 Now, we give the solutions of Eq. (6) as the following rational function based on Eq. (8). We consider the following rational form:

$$u(x,t) = \frac{p(\eta_1, \eta_2, \ldots, \eta_n)}{q(\eta_1, \eta_2, \ldots, \eta_n)}, \tag{9}$$

with

$$p = \sum_{r,s=1}^{n} \sum_{i,j=0}^{M} p_{rs,ij} \eta_r^i \eta_s^j,$$

$$q = \sum_{r,s=1}^{n} \sum_{i,j=0}^{N} q_{rs,ij} \eta_r^i \eta_s^j,$$

where $p_{kl,ij}$ and $q_{kl,ij}$ are all constants to be determined from the original equation (6). After taking all the derivatives of Eq. (9) with respect to t and x and then plugging these derivatives into the original equation, we obtain a rational function equation in the new variables η_i $(1 \leqslant i \leqslant n)$

$$Q(x,t,\eta_1,\eta_2,\ldots,\eta_n) = 0. \tag{10}$$

Step 3 We set the numerator of the resulting transformed $Q(x,t,\eta_1,\eta_2,\ldots,\eta_n) = 0$ to zero. Then a system of algebraic equations of all variables $k_i, \omega_i, p_{kl,ij}, q_{kl,ij}$ will be obtained. We solve this system with the assistance of Maple to determine two polynomials p and q and the wave exponents ξ_i, $1 \leqslant i \leqslant n$. Eventually, the multiple wave solution u is constructed and given by

$$u(x,t) = \frac{p(c_1 \mathrm{e}^{k_1 x - w_1 t}, \ldots, c_n \mathrm{e}^{k_n x - w_n t})}{q(c_1 \mathrm{e}^{k_1 x - w_1 t}, \ldots, c_n \mathrm{e}^{k_n x - w_n t})}. \tag{11}$$

3. Application of the multiple exp-function method

3.1. Application to the (2+1)-dimensional SK equation

In this subsection, our goal is to apply the extended multiple exp-function to Eq. (4).

3.1.1. One-wave solution

We consider the following form as one wave solution:

$$v(x,y,t) = \frac{p}{q} = \frac{A_0 + A_1 \mathrm{e}^{k_1 x + l_1 y - \omega_1 t}}{B_0 + B_1 \mathrm{e}^{k_1 x + l_1 y - \omega_1 t}}, \tag{12}$$

where A_0, A_1, B_0, and B_1 are constants. Plugging Eq. (12) into Eq. (4) and making the cumbersome calculations by Maple, we obtain the dispersion relation w_1 and A_1 as follows:

$$\omega_1 = -\frac{k_1^6 + 5k_1^3 l_1 - 5l_1^2}{k_1}, \tag{13}$$

$$A_1 = -\frac{-6B_0 B_1 k_1 - A_0 B_1}{B_0}. \tag{14}$$

3.1.2. Two-wave solution

We now construct the two-wave solution with the following ansatz:

$$v(x,y,t) = \frac{p}{q}, \tag{15}$$

where p and q are defined by

$$\begin{aligned} p &= \gamma(k_1 \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} + k_2 \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} \\ &\quad + A_{12}(k_1 + k_2) \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} \mathrm{e}^{k_2 x + l_2 y - \omega_2 t}), \end{aligned} \tag{16}$$

$$\begin{aligned} q &= 1 + \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} + \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} \\ &\quad + A_{12} \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} \mathrm{e}^{k_2 x + l_2 y - \omega_2 t}, \end{aligned} \tag{17}$$

with γ being a constant.

Applying the multiple exp-function algorithm by Maple leads to the following solution:

$$\begin{aligned} \gamma &= 6, \\ A_{12} &= \frac{\alpha}{\beta}, \\ \alpha &= k_1^6 k_2^2 - 3k_1^5 k_2^3 + 4k_1^4 k_2^4 - 3k_1^3 k_2^5 + k_1^2 k_2^6 \\ &\quad + k_1^4 k_2 l_2 + 2k_1^3 k_2^2 l_1 - 3k_1^3 k_2^2 l_2 - 3k_1^2 k_2^3 l_1 \\ &\quad + 2k_1^2 k_2^3 l_2 + k_1 k_2^4 l_1 + k_1^2 l_2^2 - 2k_1 k_2 l_1 l_2 + k_2^2 l_1^2, \\ \beta &= k_1^6 k_2^2 + 3k_1^5 k_2^3 + 4k_1^4 k_2^4 + 3k_1^3 k_2^5 + k_1^2 k_2^6 \\ &\quad + k_1^4 k_2 l_2 + 2k_1^3 k_2^2 l_1 + 3k_1^3 k_2^2 l_2 + 3k_1^2 k_2^3 l_1 \\ &\quad + 2k_1^2 k_2^3 l_2 + k_1 k_2^4 l_1 + k_1^2 l_2^2 - 2k_1 k_2 l_1 l_2 + k_2^2 l_1^2, \\ \omega_1 &= -\frac{k_1^6 + 5k_1^3 l_1 - 5l_1^2}{k_1}, \quad \omega_2 = -\frac{k_2^6 + 5k_2^3 l_2 - 5l_2^2}{k_2}. \end{aligned} \tag{18}$$

3.1.3. Three-wave solution

We next consider the three-wave function

$$v(x,y,t) = \frac{p}{q}, \tag{19}$$

with p and q being defined by

$$\begin{aligned} p &= \gamma(k_1 \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} + k_2 \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} + k_3 \mathrm{e}^{k_3 x + l_3 y - \omega_3 t} \\ &\quad + A_{12}(k_1 + k_2) \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} \\ &\quad + A_{13}(k_1 + k_3) \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} \mathrm{e}^{k_3 x + l_3 y - \omega_3 t} \\ &\quad + A_{23}(k_2 + k_3) \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} \mathrm{e}^{k_3 x + l_3 y - \omega_3 t} \\ &\quad + A_{12} A_{13} A_{23}(k_1 + k_2 + k_3) \\ &\quad \times \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} \mathrm{e}^{k_3 x + l_3 y - \omega_3 t}), \\ q &= 1 + \mathrm{e}^{k_1 x + l_1 y - \omega_1 t} + \mathrm{e}^{k_2 x + l_2 y - \omega_2 t} + \mathrm{e}^{k_3 x + l_3 y - \omega_3 t} \end{aligned} \tag{20}$$

$$
\begin{aligned}
&+A_{12}\,\mathrm{e}^{k_1x+l_1y-\omega_1t}\,\mathrm{e}^{k_2x+l_2y-\omega_2t}\\
&+A_{13}\,\mathrm{e}^{k_1x+l_1y-\omega_1t}\,\mathrm{e}^{k_3x+l_3y-\omega_3t}\\
&+A_{23}\,\mathrm{e}^{k_2x+l_2y-\omega_2t}\,\mathrm{e}^{k_3x+l_3y-\omega_3t}\\
&+A_{12}A_{13}A_{23}\,\mathrm{e}^{k_1x+l_1y-\omega_1t}\,\mathrm{e}^{k_2x+l_2y-\omega_2t}\,\mathrm{e}^{k_3x+l_3y-\omega_3t}, \quad (21)
\end{aligned}
$$

where γ is a constant.

Applying the multiple exp-function algorithm by Maple leads to the following solution:

$$
\begin{aligned}
\gamma &= 6,\\
A_{12} &= \frac{\alpha}{\beta},\\
\alpha &= k_1^6k_2^2-3k_1^5k_2^3+4k_1^4k_2^4-3k_1^3k_2^5+k_1^2k_2^6\\
&\quad +k_1^4k_2l_2+2k_1^3k_2^2l_1-3k_1^3k_2^2l_2-3k_1^2k_2^3l_1\\
&\quad +2k_1^2k_2^3l_2+k_1k_2^4l_1+k_1^2l_2^2-2k_1k_2l_1l_2+k_2^2l_1^2,\\
\beta &= k_1^6k_2^2+3k_1^5k_2^3+4k_1^4k_2^4+3k_1^3k_2^5+k_1^2k_2^6\\
&\quad +k_1^4k_2l_2+2k_1^3k_2^2l_1+3k_1^3k_2^2l_2+3k_1^2k_2^3l_1\\
&\quad +2k_1^2k_2^3l_2+k_1k_2^4l_1+k_1^2l_2^2-2k_1k_2l_1l_2+k_2^2l_1^2,\\
A_{13} &= \frac{\alpha}{\beta},\\
\alpha &= k_1^6k_3^2-3k_1^5k_3^3+4k_1^4k_3^4-3k_1^3k_3^5+k_1^2k_3^6\\
&\quad +k_1^4k_3l_3+2k_1^3k_3^2l_1-3k_1^3k_3^2l_3-3k_1^2k_3^3l_1\\
&\quad +2k_1^2k_3^3l_3+k_1k_3^4l_1+k_1^2l_3^2-2k_1k_3l_1l_3+k_3^2l_1^2,\\
\beta &= k_1^6k_3^2+3k_1^5k_3^3+4k_1^4k_3^4+3k_1^3k_3^5+k_1^2k_3^6\\
&\quad +k_1^4k_3l_3+2k_1^3k_3^2l_1+3k_1^3k_3^2l_3+3k_1^2k_3^3l_1\\
&\quad +2k_1^2k_3^3l_3+k_1k_3^4l_1+k_1^2l_3^2-2k_1k_3l_1l_3+k_3^2l_1^2,\\
A_{23} &= \frac{\alpha}{\beta},\\
\alpha &= k_2^6k_3^2-3k_2^5k_3^3+4k_2^4k_3^4-3k_2^3k_3^5+k_2^2k_3^6\\
&\quad +k_2^4k_3l_3+2k_2^3k_3^2l_2-3k_2^3k_3^2l_3-3k_2^2k_3^3l_2\\
&\quad +2k_2^2k_3^3l_3+k_2k_3^4l_2+k_2^2l_3^2-2k_2k_3l_2l_3+k_3^2l_2^2,\\
\beta &= k_2^6k_3^2+3k_2^5k_3^3+4k_2^4k_3^4+3k_2^3k_3^5+k_2^2k_3^6\\
&\quad +k_2^4k_3l_3+2k_2^3k_3^2l_2+3k_2^3k_3^2l_3+3k_2^2k_3^3l_2\\
&\quad +2k_2^2k_3^3l_3+k_2k_3^4l_2+k_2^2l_3^2-2k_2k_3l_2l_3+k_3^2l_2^2,\\
\omega_1 &= -\frac{k_1^6+5k_1^3l_1-5l_1^2}{k_1}, \quad \omega_2=-\frac{k_2^6+5k_2^3l_2-5l_2^2}{k_2},\\
\omega_3 &= -\frac{k_3^6+5k_3^3l_3-5l_3^2}{k_3}. \qquad (22)
\end{aligned}
$$

The above solutions are plotted in Figs. 1–9.

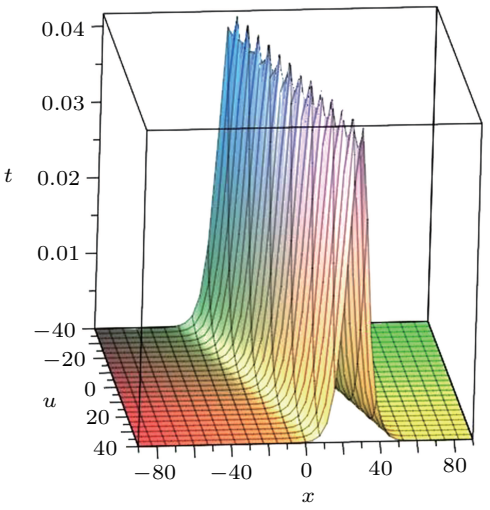


Fig. 1. (color online) Plot of Eq. (13) when $A_0 = 0.2$, $B_0 = 0.3$, $B_1 = 0.6$, $k_1 = 1/6$, $l_1 = 0.1$.

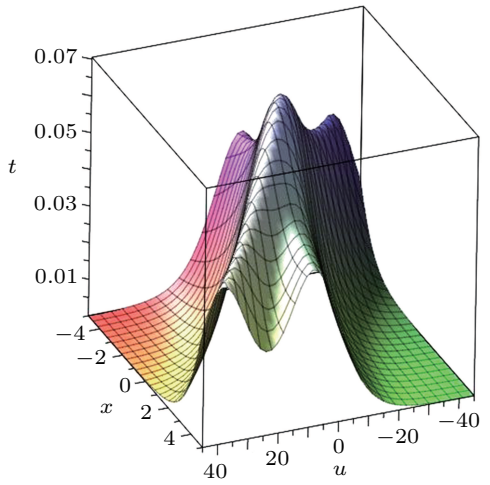


Fig. 2. (color online) Pot of Eq. (16) when $k_1 = 1/6$, $k_2 = 1/6$, $l_1 = 0.1$, $l_2 = 0.2$.

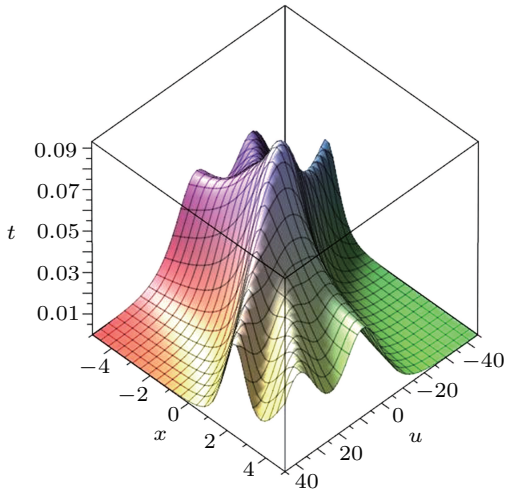


Fig. 3. (color online) Plot of Eq. (20) when $k_1 = 1/6$, $k_2 = 1/6$, $k_3 = 1/6$, $l_1 = 0.1$, $l_2 = 0.2$, $l_3 = 0.3$.

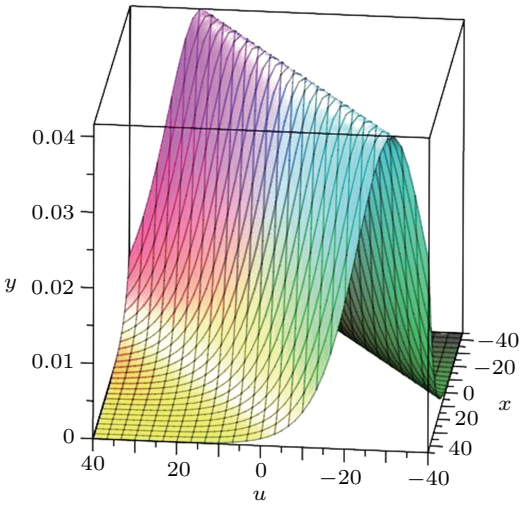


Fig. 4. (color online) Plot of Eq. (13) when $A_0 = 0.2$, $B_0 = 0.3$, $B_1 = 0.6$, $k_1 = 1/6$, $l_1 = 0.1$.

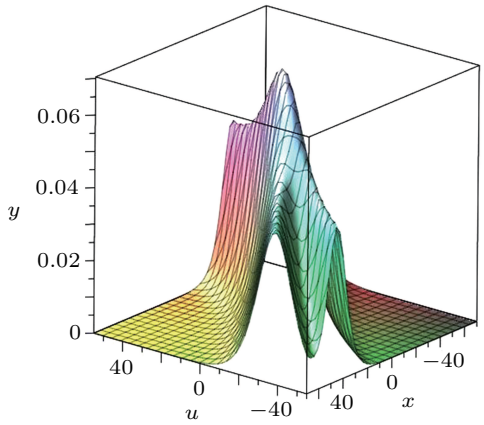


Fig. 5. (color online) Plot of Eq. (16) when $k_1 = 1/6$, $k_2 = 1/6$, $l_1 = 0.1$, $l_2 = 0.2$.

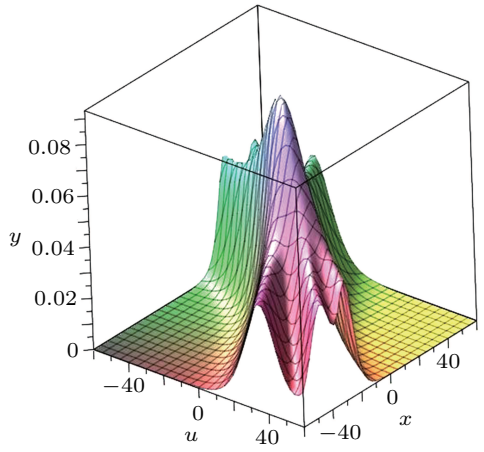


Fig. 6. (color online) Plot of Eq. (20) when $k_1 = 1/6$, $k_2 = 1/6$, $k_3 = 1/6$, $l_1 = 0.1$, $l_2 = 0.2$, $l_3 = 0.3$.

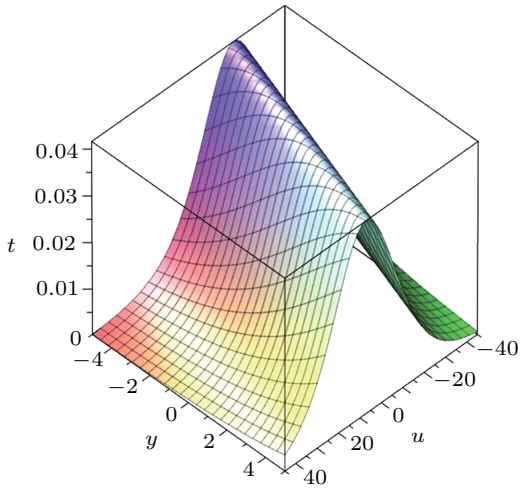


Fig. 7. (color online) Plot of Eq. (13) when $A_0 = 0.2$, $B_0 = 0.3$, $B_1 = 0.6$, $k_1 = 1/6$, $l_1 = 0.1$.

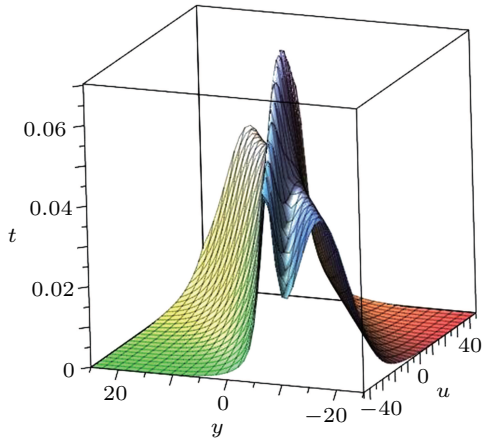


Fig. 8. (color online) Plot of Eq. (16) when $k_1 = 1/6$, $k_2 = 1/6$, $l_1 = 0.1$, $l_2 = 0.2$.

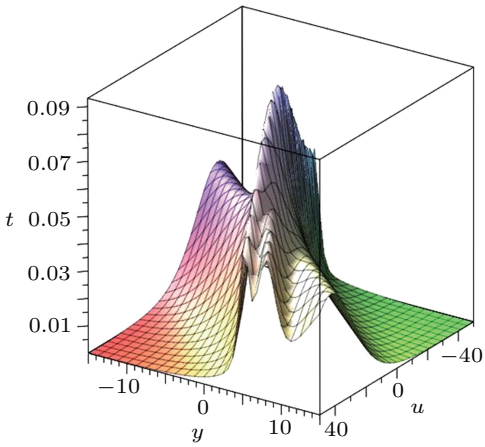


Fig. 9. (color online) Plot of Eq. (20) when $k_1 = 1/6$, $k_2 = 1/6$, $k_3 = 1/6$, $l_1 = 0.1$, $l_2 = 0.2$, $l_3 = 0.3$.

3.2. Application to the (3+1)-dimensional nonlinear evolution equation

In this subsection, our aim is to apply the extended multiple exp-function method to Eq. (5).

3.2.1. One-wave solution

We consider again the following ansatz for obtaing the one-wave solution of Eq. (5):

$$p = A_0 + A_1 \mathrm{e}^{k_1 x + l_1 y + m_1 z - \omega_1 t}, \tag{23}$$

$$q = B_0 + B_1 \mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t}. \tag{24}$$

Then the corresponding one-wave solution is given by

$$v(x,y,z,t) = \frac{p}{q}, \tag{25}$$

with

$$A_1 = \frac{B_1(-3B_0k_1 + A_0)}{B_0}, \tag{26}$$

$$\omega_1 = \frac{k_1\left(k_1^2l_1 - 3m_1\right)}{2l_1}. \tag{27}$$

3.2.2. Two-wave solution

For the two-wave solution of Eq. (5), we obtain

$$\begin{aligned} p =& \gamma(k_1\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t} + k_2\mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t} \\ &+ A_{12}(k_1+k_2)\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t}\mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t}), \end{aligned} \tag{28}$$

$$\begin{aligned} q =& 1 + \mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t} + \mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t} \\ &+ A_{12}\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t}\mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t}, \end{aligned} \tag{29}$$

where γ is a constant and the resulting two-wave solution is

$$v(x,y,z,t) = \frac{p}{q}, \tag{30}$$

with

$$\begin{aligned} \gamma =& -3, \\ A_{12} =& (k_1^2k_2l_1^2l_2 - k_1^2k_2l_1l_2^2 - k_1k_2^2l_1^2l_2 + k_1k_2^2l_1l_2^2 \\ &- k_1l_1l_2m_2 + k_1l_2^2m_1 + k_2l_1^2m_2 - k_2l_1l_2m_1) \\ &\times(k_1^2k_2l_1^2l_2 + k_1^2k_2l_1l_2^2 + k_1k_2^2l_1^2l_2 + k_1k_2^2l_1l_2^2 \\ &- k_1l_1l_2m_2 + k_1l_2^2m_1 + k_2l_1^2m_2 - k_2l_1l_2m_1)^{-1}, \\ \omega_1 =& \frac{k_1\left(k_1^2l_1 - 3m_1\right)}{2l_1}, \quad \omega_2 = \frac{k_2\left(k_2^2l_2 - 3m_2\right)}{2l_2}. \end{aligned} \tag{31}$$

3.2.3. Three-wave solution

For the three-wave solution, one can take

$$\begin{aligned} p =& \gamma(k_1\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t} + k_2\mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t} \\ &+ k_3\mathrm{e}^{k_3x+l_3y+m_3z-\omega_3t} \\ &+ A_{12}(k_1+k_2)\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t}\mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t} \\ &+ A_{13}(k_1+k_3)\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t}\mathrm{e}^{k_3x+l_3y+m_3z-\omega_3t} \\ &+ A_{23}(k_2+k_3)\mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t}\mathrm{e}^{k_3x+l_3y+m_3z-\omega_3t} \\ &+ A_{12}A_{13}A_{23}(k_1+k_2+k_3)\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t} \\ &\times \mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t}\mathrm{e}^{k_3x+l_3y+m_3z-\omega_3t}), \end{aligned} \tag{32}$$

$$\begin{aligned} q =& 1 + \mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t} + \mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t} \\ &+ \mathrm{e}^{k_3x+l_3y+m_3z-\omega_3t} \\ &+ A_{12}\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t}\mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t} \\ &+ A_{13}\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t}\mathrm{e}^{k_3x+l_3y+m_3z-\omega_3t} \\ &+ A_{23}\mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t}\mathrm{e}^{k_3x+l_3y+m_3z-\omega_3t} \\ &+ A_{12}A_{13}A_{23}\mathrm{e}^{k_1x+l_1y+m_1z-\omega_1t} \\ &\times \mathrm{e}^{k_2x+l_2y+m_2z-\omega_2t}\mathrm{e}^{k_3x+l_3y+m_3z-\omega_3t}, \end{aligned} \tag{33}$$

where γ is a constant and the resulting three-wave solution is

$$v(x,y,z,t) = \frac{p}{q}, \tag{34}$$

with the following cases:

Case 1 ($l_i = k_i$)

$$\begin{aligned} \gamma =& -3, \\ A_{12} =& \frac{k_1^2 - 2k_1k_2 + k_2^2}{k_1^2 + 2k_1k_2 + k_2^2}, \quad A_{13} = \frac{k_1^2 - 2k_1k_3 + k_3^2}{k_1^2 + 2k_1k_3 + k_3^2}, \\ A_{23} =& \frac{k_2^2 - 2k_2k_3 + k_3^2}{k_2^2 + 2k_2k_3 + k_3^2}, \\ \omega_1 =& \frac{k_1^3}{2} - \frac{3m_1}{2}, \quad \omega_2 = \frac{k_2^3}{2} - \frac{3m_2}{2}, \\ \omega_3 =& \frac{k_3^3}{2} - \frac{3m_3}{2}; \end{aligned} \tag{35}$$

Case 2 ($m_i = k_i$)

$$\begin{aligned} \gamma =& -3, \\ A_{12} =& (k_1^2k_2l_1^2l_2 - k_1^2k_2l_1l_2^2 - k_1k_2^2l_1^2l_2 \\ &+ k_1k_2^2l_1l_2^2 + k_1^2l_2^2 - 2k_1k_2l_1l_2 + k_2^2l_1^2) \\ &\times(k_1^2k_2l_1^2l_2 + k_1^2k_2l_1l_2^2 + k_1k_2^2l_1^2l_2 \\ &+ k_1k_2^2l_1l_2^2 + k_1^2l_2^2 - 2k_1k_2l_1l_2 + k_2^2l_1^2)^{-1}, \\ A_{13} =& (k_1^2k_3l_1^2l_3 - k_1^2k_3l_1l_3^2 - k_1k_3^2l_1^2l_3 \\ &+ k_1k_3^2l_1l_3^2 + k_1^2l_3^2 - 2k_1k_3l_1l_3 + k_3^2l_1^2) \\ &\times(k_1^2k_3l_1^2l_3 + k_1^2k_3l_1l_3^2 + k_1k_3^2l_1^2l_3 \\ &+ k_1k_3^2l_1l_3^2 + k_1^2l_3^2 - 2k_1k_3l_1l_3 + k_3^2l_1^2)^{-1}, \\ A_{23} =& (k_2^2k_3l_2^2l_3 - k_2^2k_3l_2l_3^2 - k_2k_3^2l_2^2l_3 \\ &+ k_2k_3^2l_2l_3^2 + k_2^2l_3^2 - 2k_2k_3l_2l_3 + k_3^2l_2^2) \\ &\times(k_2^2k_3l_2^2l_3 + k_2^2k_3l_2l_3^2 + k_2k_3^2l_2^2l_3 \\ &+ k_2k_3^2l_2l_3^2 + k_2^2l_3^2 - 2k_2k_3l_2l_3 + k_3^2l_2^2)^{-1}, \\ \omega_1 =& \frac{k_1^2(k_1l_1 - 3)}{2l_1}, \quad \omega_2 = \frac{k_2^2(k_2l_2 - 3)}{2l_2}, \\ \omega_3 =& \frac{k_3^2(k_3l_3 - 3)}{2l_3}; \end{aligned} \tag{36}$$

Case 3 ($m_i = l_i$)

$$\begin{aligned} \gamma =& -3, \\ A_{12} =& \frac{k_1l_1 - k_1l_2 - k_2l_1 + k_2l_2}{k_1l_1 + k_1l_2 + k_2l_1 + k_2l_2}, \\ A_{13} =& \frac{k_1l_1 - k_1l_3 - k_3l_1 + k_3l_3}{k_1l_1 + k_1l_3 + k_3l_1 + k_3l_3}, \\ A_{23} =& \frac{k_2l_2 - k_2l_3 - k_3l_2 + k_3l_3}{k_2l_2 + k_2l_3 + k_3l_2 + k_3l_3}, \\ \omega_1 =& \frac{k_1^3}{2} - \frac{3k_1}{2}, \quad \omega_2 = \frac{k_2^3}{2} - \frac{3k_2}{2}, \\ \omega_3 =& \frac{k_3^3}{2} - \frac{3k_3}{2}. \end{aligned} \tag{37}$$

The above solutions are plotted in Figs. 10–15.

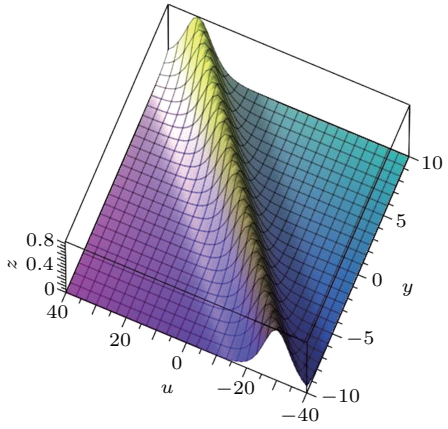


Fig. 10. (color online) Plot of Eq. (26) when $A_0 = 1$, $B_0 = 1$, $B_1 = 1$, $k_1 = -1/3$, $l_1 = 2$, $m_1 = 1$.

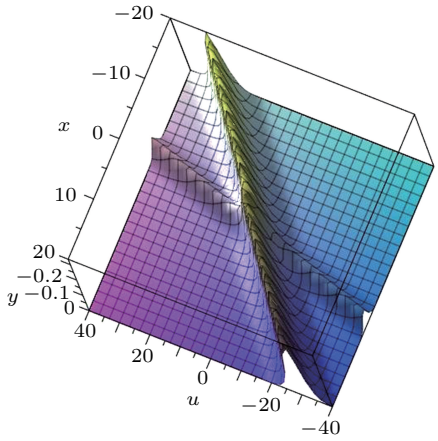


Fig. 11. (color online) Plot of Eq. (31) when $k_1 = -1/3$, $k_2 = -2/3$, $l_1 = 2$, $l_2 = 1$, $m_1 = 1$, $m_2 = 1$.

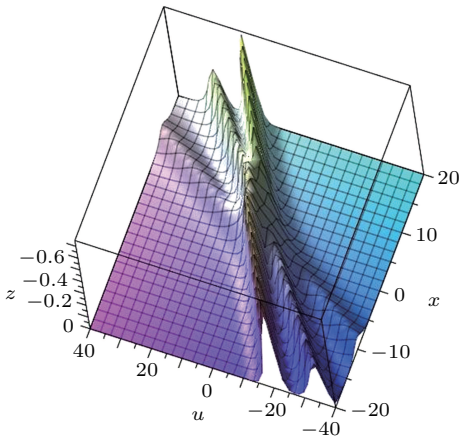


Fig. 12. (color online) Plot of Eq. (35) with Eq. (36) when $k_1 = -1/3$, $k_2 = -2/3$, $k_3 = -1$, $m_1 = 1$, $m_2 = 1$, $m_3 = 1$.

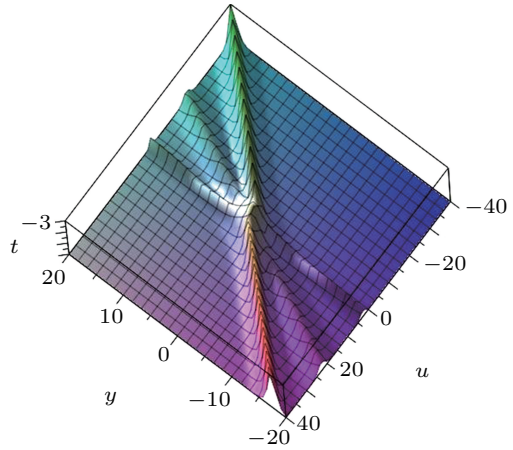


Fig. 13. (color online) Plot of Eq. (35) with Eq. (37) when $k_1 = 1$, $k_2 = 2$, $k_3 = 1$, $l_1 = 2$, $l_2 = 1$, $l_3 = 1$.

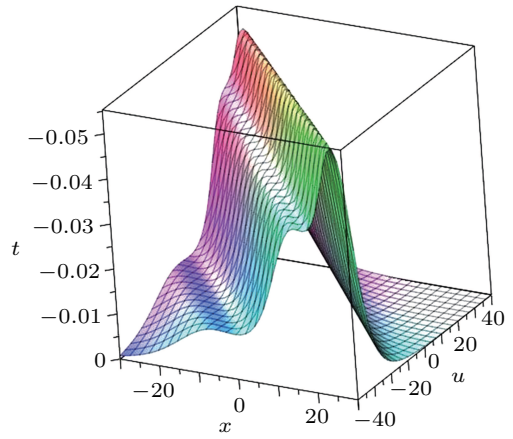


Fig. 14. (color online) Plot of Eq. (35) with Eq. (37) when $k_1 = -1/9$, $k_2 = -2/9$, $k_3 = -1/3$, $l_1 = -4/9$, $l_2 = -5/9$, $l_3 = -2/3$.

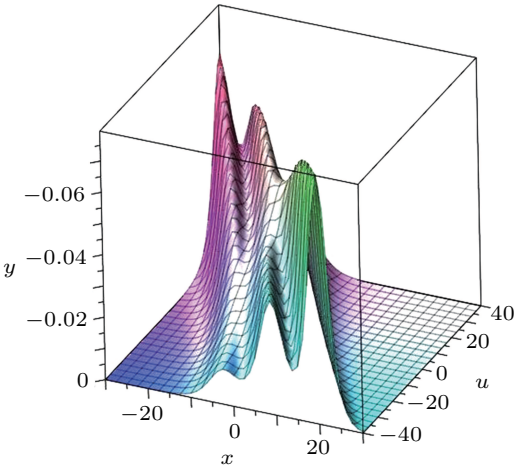


Fig. 15. (color online) Plot of Eq. (35) with Eq. (37) when $k_1 = -1/9$, $k_2 = -2/9$, $k_3 = -1/3$, $l_1 = -4/9$, $l_2 = -5/9$, $l_3 = -2/3$.

4. Concluding remarks

We considered the (2+1)-dimensional Sawada–Kotera equation and (3+1)-dimensional nonlinear evolution equation. We have used the multiple exp-function method to derive the analytic wave solutions with physical structures. The obtained solutions are one-, two-, and three-soliton solutions. The results for Eqs. (1) and (2) are in agreement with the results reported by others in the literature [20] and [21], respectively. The difference between our work and pioneering paper[9] is based upon the parameter of γ which appears in Eqs. (17), (21), (29), and (33). This parameter is taken in Ref. [9] and in other works[18,19] as 2. However, in this study we considered any parameter. As can be seen in both systems (Eqs. (1)

and (2)), we yield different values of γ. This property makes the method much more applicable to other nonlinear partial differential equations.

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