Attempt to generalize fractional-order electric elements to complex-order ones

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The complex derivative $D^{\alpha\pm j\beta}$, with $\alpha, \beta \in R+$ is a generalization of the concept of integer derivative, where $\alpha = 1$, $\beta = 0$. Fractional-order electric elements and circuits are becoming more and more attractive. In this paper, the complex-order electric elements concept is proposed for the first time, and the complex-order elements are modeled and analyzed. Some interesting phenomena are found that the real part of the order affects the phase of output signal, and the imaginary part affects the amplitude for both the complex-order capacitor and complex-order memristor. More interesting is that the complex-order capacitor can do well at the time of fitting electrochemistry impedance spectra. The complex-order memristor is also analyzed. The area inside the hysteresis loops increases with the increasing of the imaginary part of the order and decreases with the increasing of the real part. Some complex case of complex-order memristors hysteresis loops are analyzed at last, whose loop has touching points beyond the origin of the coordinate system.

Keywords: complex derivative, fractional-order elements, imaginary and real part, memristor

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1. Introduction

Fractional calculus is a 300 years old topic which was put forward first by Leibniz and L'Hospital. It is a generalization of integration and differentiation to non-integer orders. Unfortunately, the work was forgotten for some decades. Recent studies have brought it to more widespread attention,[1] revealing that many physical phenomena can be modeled through fractional differential equations, and fractional-order systems have been a subject of increasing interest.[2] It had been applied to model many real-world phenomena in various fields of physics, engineering and economics, such as dielectric polarization,[3] electromagnetic waves,[4] viscoelastic system,[5] heat conduction,[6] biology,[7] finance,[8] and control theory.[9,10] The complex derivative $D^{\alpha\pm j\beta}$, with α, $\beta \in R+$ is a generalization of the concept of integer derivative, where $\alpha = 1$, $\beta = 0$. Recently, several studies focused on the application of complex-order derivatives,[11–13] which modeled and analyzed control theory,[14] chaos,[15] and elements[16] with it.

Fractional-order elements are an application of fractional calculus in electricity. Jonscher and his partner pointed out that there is no ideal integer-order capacitor in nature.[17] Westerlund in 1994 proposed a new linear capacitor model[18] which states the fractional capacitor. Westerlund in his work also described the behavior of a real inductor[19] using a fractional-order model. Finite element approximations offer a valuable tool by which the effect of fractional-order elements can be simulated using a standard circuit simulator, or studied experimentally.[20,21] This finite element approximation based on the possibility of emulating a fractional-order capacitor via semi-infinite RC trees as shown in Fig. 1(a). The circuit diagram of the RC equivalent circuit for the fractional order element of any order is presented as shown in Fig. 1(b).

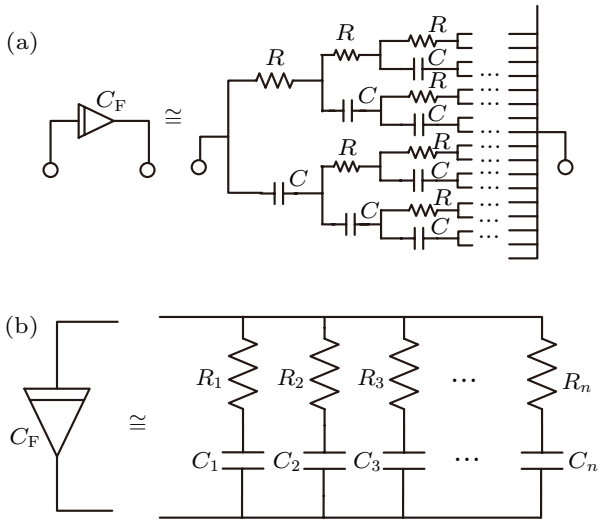


Fig. 1. (a) Equivalent *RC* tree circuit of the fractional-order element of order 0.5 and (b) equivalent *RC* tree circuit of the fractional-order element of any order.

Based on this method, the authors in Refs. [22] and [23] studied the Buck–Boost converter, while the authors in Ref. [24] designed and analyzed fractional-order multi-phase oscillators. Fractional-order circuit theory has attracted the attention of many researchers recently. Many fractional-order circuits have been studied, such as Chua's fractional-order

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system,[25,26] and filter circuit.[27,28] In this paper, a complex-order derivative is introduced to circuit elements, and many interesting phenomena are found. The real part of the order affects the phase of the output signal, and the imaginary part affects the amplitude for both the complex-order capacitor and complex-order memristor. The complex-order capacitor can do well at the time of fitting electrochemistry impedance spectra. The area inside the hysteresis loops of complex-order memristors increases with the increasing of the imaginary part of the order and decreases with the increasing of the real part. Some complex cases of complex-order memristors hysteresis loops are analyzed at last.

This paper is organized as follows. Section 2 presents the preliminary concepts of the fractional derivative and the complex derivative. In Section 3, we present fractional-order and complex-order elements models. In Section 4, simulation of complex-order elements are presented. Some discussions are given in Section 5. Finally, our concluding remarks and conclusions are given in Section 6.

2. Preliminary concepts

In fractional calculus, the fundamental operator ${}_aD_t^\alpha$ denotes a generalization of integrations and differentiations to an arbitrary order differ-integral operator. It can be defined as

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \displaystyle\int_a^t (\mathrm{d}\tau)^{-\alpha}, & \alpha < 0, \end{cases} \tag{1} $$

where $\alpha \in R$ is the order of the operation, a and t are bounds of the operation. There are several different definitions of fractional derivatives. Riemann–Liouville definition is a generalization[29] of the Grunwald–Letnikov which is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,\mathrm{d}\tau, \tag{2} $$

where $\Gamma(\cdot)$ is the Gamma function, and $n \in N$, n is the smallest integer larger than α, $n-1 < \alpha < n$.

The Riemann–Liouville definition of the α-th order fractional integral operator ${}_aD_t^{-\alpha}$ can be written as

$$ {}_aD_t{}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{3} $$

Grunwald–Letnikov definition[30] of fractional derivatives is given by

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{\gamma=0}^{[t-a/h]} (-1)^\gamma \binom{\alpha}{\gamma} f(t-\gamma h). \tag{4} $$

The Grunwald–Letnikov definition evolved from the n-order integer derivative. In the same way, it can be used in complex derivatives. The Grunwald–Letnikov definition of complex integration and derivative is the following

$$ {}_aD_t^\varsigma f(t) = \lim_{h\to 0} \frac{1}{h^\varsigma} \sum_{k=0}^{[t-a/h]} (-1)^j \binom{\varsigma}{k} f(t-hk), \tag{5} $$

where

$$ \binom{\varsigma}{k} = \frac{\varsigma(\varsigma-1)\cdots(\varsigma-k-1)}{k!}. $$

This formula can be reduced to

$$ {}_0D_t^\varsigma y(t_m) \approx h^{-\varsigma} \sum_{k=0}^{m} \omega_k{}^\varsigma y_m - k, \tag{6} $$

where h is the time step and ς is the complex number. The complex derivative $D^{\alpha\pm \mathrm{j}\beta}$, with $\alpha, \beta \in R+$ is a generalization of the concept of integer derivative, where $\alpha = 1$, $\beta = 0$.

3. Fractional-order elements and complex-order elements

In classical linear circuit theory the three fundamental elements are constituted by the resistor R, inductor L, and capacitor C.

$$ v(t) = Ri(t), \tag{7} $$

$$ \varphi(t) = Li(t), \tag{8} $$

$$ q(t) = Cv(t), \tag{9} $$

where $v(t)$ and $i(t)$ represent the electrical voltage and current, respectively, and t denotes time. The fractional-order capacitor and inductor can be given as follows when fractional calculus is introduced into circuit elements

$$ I(t) = C\frac{\mathrm{d}^\alpha v(t)}{\mathrm{d}t^\alpha} \equiv C\,{}_0D_t{}^\alpha v(t), \tag{10} $$

$$ v(t) = L\frac{\mathrm{d}^\beta I(t)}{\mathrm{d}t^\beta} \equiv L\,{}_0D_t{}^\beta I(t), \tag{11} $$

where $0 < \alpha, \beta < 1, (\alpha, \beta \in R)$. The impedance of fractional-order electric elements can be obtained.

$$ Z_C(\mathrm{j}\omega) = \frac{1}{C(\mathrm{j}\omega)^\alpha}, \tag{12} $$

$$ Z_L(\mathrm{j}\omega) = L(\mathrm{j}\omega)^\beta. \tag{13} $$

When we generalize fractional-order elements to complex-order, the complex-order capacitor and inductor are

$$ I(t) = C\frac{\mathrm{d}^{\alpha\pm\mathrm{j}\beta} v(t)}{\mathrm{d}t^{\alpha\pm\mathrm{j}\beta}} = C\,{}_0D^{\alpha\pm\mathrm{j}\beta} v(t), \tag{14} $$

$$ v(t) = L\frac{\mathrm{d}^{\alpha\pm\mathrm{j}\beta} I(t)}{\mathrm{d}t^{\alpha\pm\mathrm{j}\beta}} = L\,{}_0D^{\alpha\pm\mathrm{j}\beta} I(t). \tag{15} $$

In Ref. [15] the authors analyzed the chaos in Chua's complex-order system and its chaotic synchronization for secure communication. The authors in Ref. [11] modeled the van der Pol oscillator with complex derivative. Chua's system circuit and van der Pol oscillator circuit are shown in Fig. 2. The circuit models in the two papers can be obtained by using the complex-order elements as shown above.

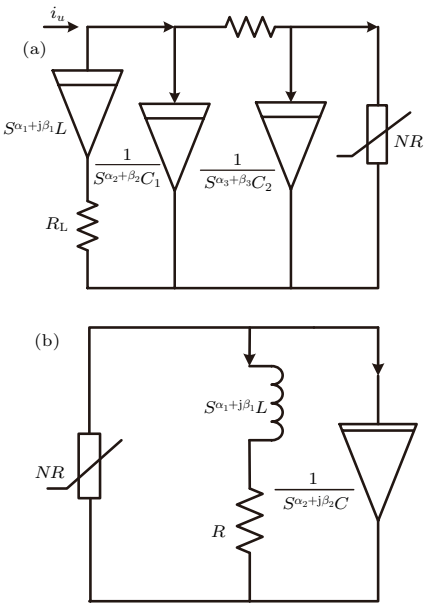


Fig. 2. (a) Chua's circuit and (b) van der Pol oscillator circuit.

4. Simulation of complex-order elements

For a sine function, at steady-state, the complex derivative of order $\alpha \pm j\beta$ is given by[11–14]

$$\begin{aligned} D^{\alpha\pm j\beta}\sin(\omega t) &= \omega^{\alpha}\cdot[\cos[\beta\ln(\omega)] \pm j\sin[\beta\ln(\omega)]] \\ &\quad \cdot\left[\sin\left(\omega t+\alpha\frac{\pi}{2}\right)cosh\left(\beta\frac{\pi}{2}\right)\right. \\ &\quad \left.\pm j\cos\left(\omega t+\alpha\frac{\pi}{2}\right)\sinh\left(\beta\frac{\pi}{2}\right)\right], \quad (16) \end{aligned}$$

where $j=\sqrt{-1}$ and ω is the angular frequency. For obtaining real-valued results, we group the conjugate orders, into the operators:

$$\begin{aligned} &CO_1^{(\alpha,\beta)}\sin(\omega t) \\ &= \frac{1}{2}\left[D^{\alpha+j\beta}+D^{\alpha-j\beta}\right]\sin(\omega t), \quad (17) \\ &CO_1^{(\alpha,\beta)}\sin(\omega t) \\ &= \omega^{\alpha}\left[\cos\left[\beta\ln(\omega)\sin\left(\omega t+\alpha\frac{\pi}{2}\right)\right]\cosh\left(\beta\frac{\pi}{2}\right)\right. \\ &\quad \left.-\sin\left[\beta\ln(\omega)\cos\left(\omega t+\alpha\frac{\pi}{2}\right)\right]\sinh\left(\beta\frac{\pi}{2}\right)\right], \quad (18) \\ &CO_2^{(\alpha,\beta)}\sin(\omega t) \\ &= \frac{1}{2j}\left[D^{\alpha+j\beta}-D^{\alpha-j\beta}\right]\sin(\omega t), \quad (19) \\ &CO_2^{(\alpha,\beta)}\sin(\omega t) \\ &= \omega^{\alpha}\left[\cos\left[\beta\ln(\omega)\cos\left(\omega t+\alpha\frac{\pi}{2}\right)\right]\sinh\left(\beta\frac{\pi}{2}\right)\right. \\ &\quad \left.+\sin\left[\beta\ln(\omega)\sin\left(\omega t+\alpha\frac{\pi}{2}\right)\right]\cosh\left(\beta\frac{\pi}{2}\right)\right]. \quad (20) \end{aligned}$$

4.1. Simulations of complex-order capacitor

Applying sinusoidal excitation signals $v(t)=\sin(t)$ to the capacitor (see Fig. 3), for different models, and the current waveforms can be obtained. Figure 4 show the current waveforms with $C=10^{-5}$ F. As shown in Fig. 4, the real part of the order affects the phase of current. We can clearly see that the imaginary part of the order affects the amplitude of current, and there is a larger amplitude when the imaginary part of the order is bigger. A more important point is that there is no interaction between the imaginary part and the real part. Figure 5 shows the current waveforms with different frequencies. Table 1 shows the amplitudes for the imaginary part of CO_1 and CO_2.

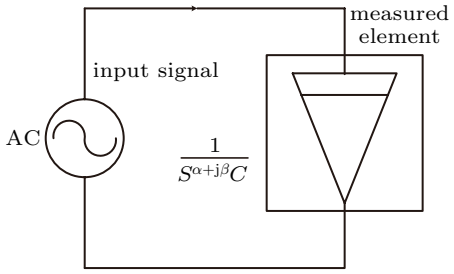


Fig. 3. Test circuit.

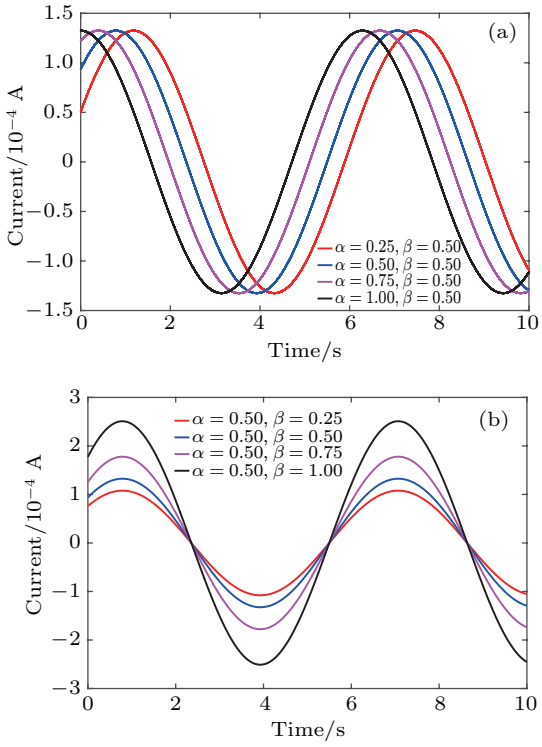


Fig. 4. (color online) Simulation of complex-order (CO_1) capacitor current versus time when $\omega=1$.

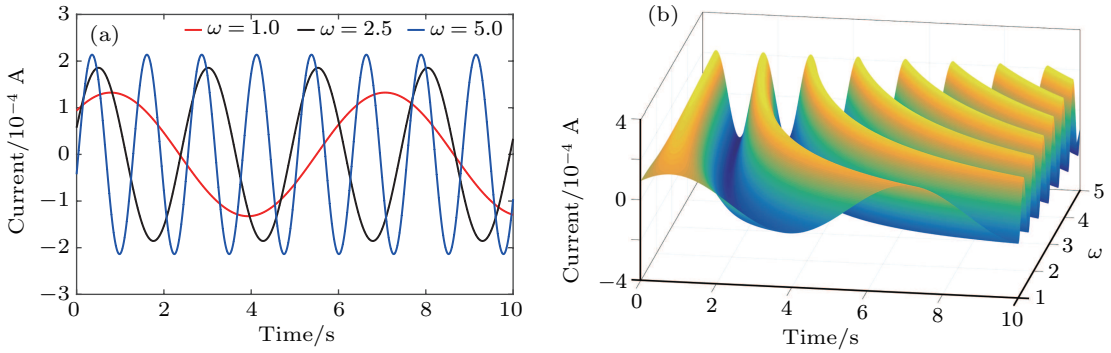


Fig. 5. (color online) Simulation of current versus time for different frequencies when $\alpha = 0.50$, $\beta = 0.50$.

Table 1. Amplitudes of current (in unit mA) for different values of imaginary part of CO_1 and CO_2.

$CO_i \downarrow \beta \rightarrow$	0.1	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00
CO_1	0.1012	0.1050	0.1113	0.1204	0.1325	0.1478	0.1668	0.1899	0.2177	0.2509
CO_2	0.0158	0.0319	0.0489	0.0671	0.0859	0.1088	0.1334	0.1614	0.1934	0.2305

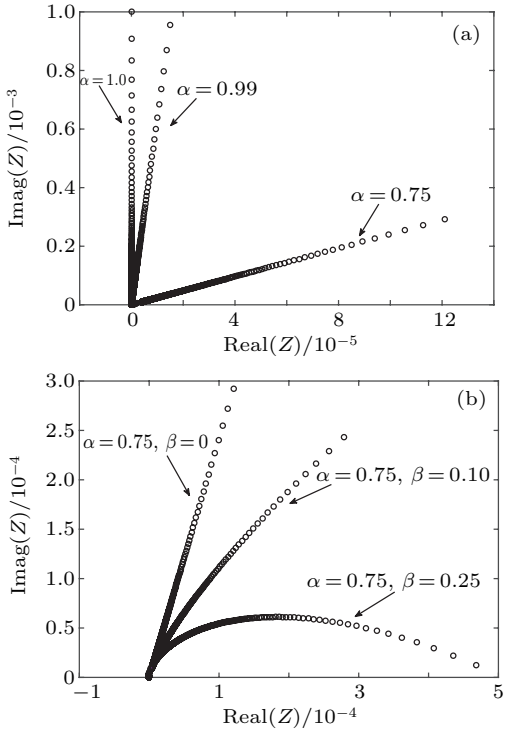


Fig. 6. Nyquist diagram of the impedance (a) $Z(\mathrm{j}\omega, \alpha)$ of fractional-order capacitor (b) $Z(\mathrm{j}\omega, \alpha, \beta)$ of complex-order (CO_1) capacitor.

It is well known that, in electrochemical systems with diffusion, the impedance is modelled using the so-called Warburg element. The Warburg element arises from one-dimensional diffusion of an ionic species to the electrode. If the impedance is under an infinite diffusion layer, the Warburg impedance is given by Eq. (21), which can be regard as fractional capacitance.

$$Z = \frac{1}{T(\mathrm{j}\omega)^{\alpha}}, \quad 0.5 \leq \alpha \leq 1. \tag{21}$$

Figure 6(a) shows the Nyquist plot of the ideal capacitor, real capacitor,[18] and fractional-order capacitor impedance, respectively, when $\alpha = 1$, 0.99, and 0.75. Figure 6(b) shows the Nyquist plot of the complex-order (CO_1) capacitor impedance. References [31] and [32] are mainly about the theory, experiment, and applications of impedance spectroscopy, from which we can know, the complex-order capacitor can fit well the impedance spectra of the oxide films obtained at different concentrations of $Ce(SO_4)_2$ when $\alpha = 0.75$, $\beta = 0.25$, and it also can be used to fit the impedance spectra of TiO_2 and $CuInS_2$ at different potentials and temperatures.

4.2. Complex-order memristors

Chua proposed that there should be a fourth basic element M, which he called the 'memristor',[33] for memory resistor, completing the set of relations with $\varphi = f(q)$. The first passive realization was introduced by HP-lab a few years ago.[34] In Ref. [35] the authors proposed the first neural network chip based on a memristor in 2015,[36,37] and researched the application of memory elements in the circuits system. Meristors and memory elements will bring a technological revolution in the future. Considering the memristor model as $\varphi(q) = \frac{1}{3}q^3$, with the input signal $i(t)$, and the integrated $i(t)$ can lead to charge $q(t)$. Figure 7 shows the complex-order (CO_1) memristor waveforms of voltage and R_{M} versus time.

As shown in Fig. 7, the real part of the order affects the phase of voltage and R_{M}. We also can see that the imaginary part of the order affects the amplitude, the larger the imaginary part of the order, the larger the amplitude. Figure 8 shows the pinched hysteretic loop of the complex-order (CO_1) memristor. There are great changes when fractional-order elements are generalized to complex-order. The area inside the hysteresis loops increases with the increasing of the imaginary part of the order and decreases with the increasing of the real part. The trends of the I–V curves that collapse to a line when $\alpha = 1.0$, $\beta = 1.0$ can be clearly observed as shown in Fig. 8(b).

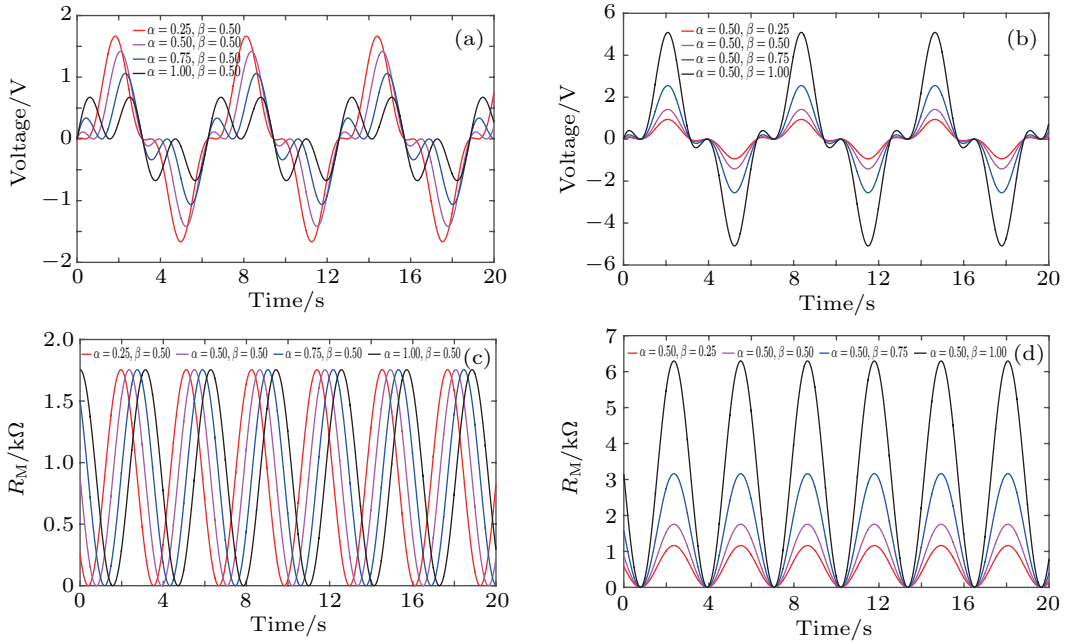


Fig. 7. (color online) Simulation of memristor when $\omega = 1$. (a) and (b) Waveforms of voltage versus time. (c) and (d) Waveforms of R_M versus time.

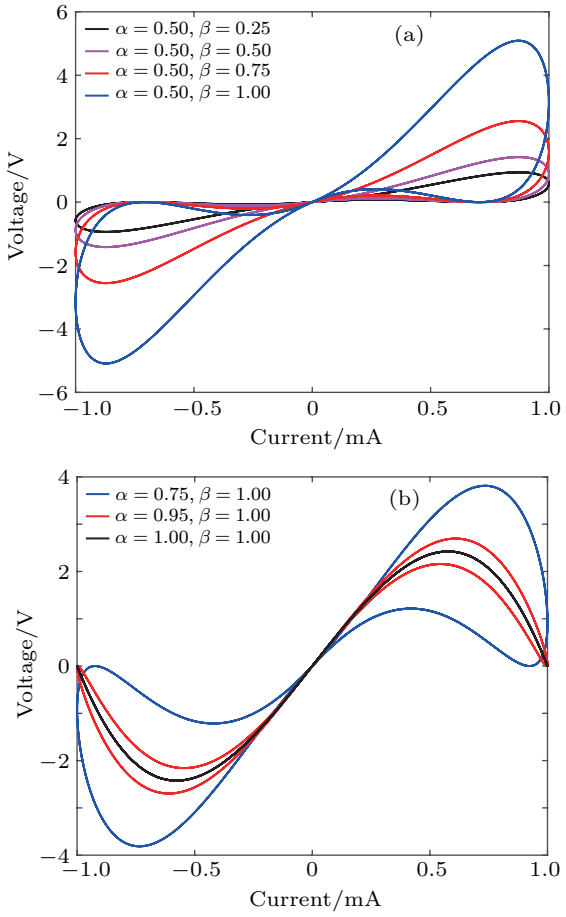


Fig. 8. (color online) Simulations of the pinched hysteretic loop when the frequency $\omega = 1$.

Figure 9(b) shows a hysteresis loop, which is not the conventional double hysteresis loop. The loop has touching points beyond the origin of the coordinate system. Figure 10 shows a more complex case where the memristor generates a richer spectrum of voltage. It corresponds to a higher number of touching points on the hysteresis loop. Since the pinched hysteresis loop must be odd-symmetric, it is sufficient to focus on the first quadrant, describing processes during the first half period of signals.

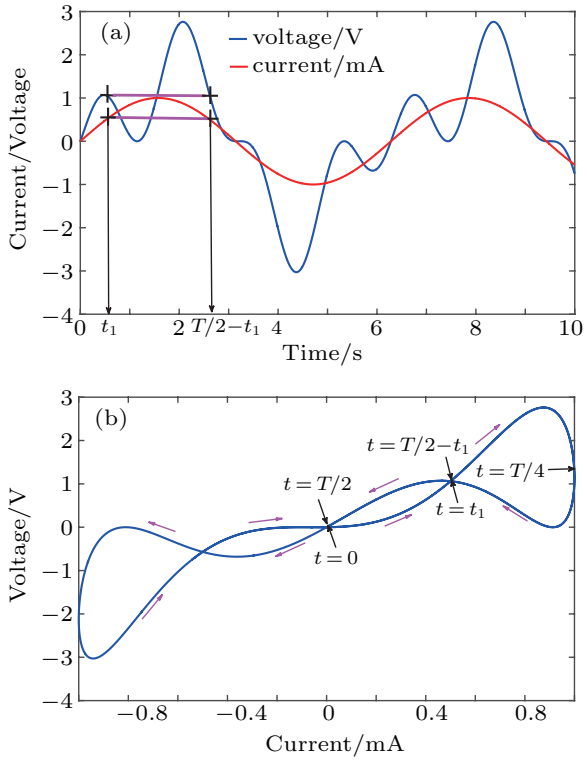


Fig. 9. (color online) Simulation of complex-order memrostor (CO_1) when $\alpha = 0.75$, $\beta = 1.0$, $\omega = 1.5$. (a) Waveforms of voltage and current versus time and (b) the pinched hysteretic loop.

If the arms of the loop are to touch at a certain point, it means that there must be two time instants t_1 and $T/2 - t_1$, $t_1 \in \langle 0, T/4 \rangle$ at which signals i and v take the same values. Graphically, this condition may be interpreted such that there

must be a rectangle whose one pair of vertices lies on the current waveform and the second pair lies on the voltage waveform (see in Fig. 9). If it is possible to find other time instants t_i, $i = 1, 2, \ldots$, this means that the hysteresis loop has more touching points (see Fig. 10).

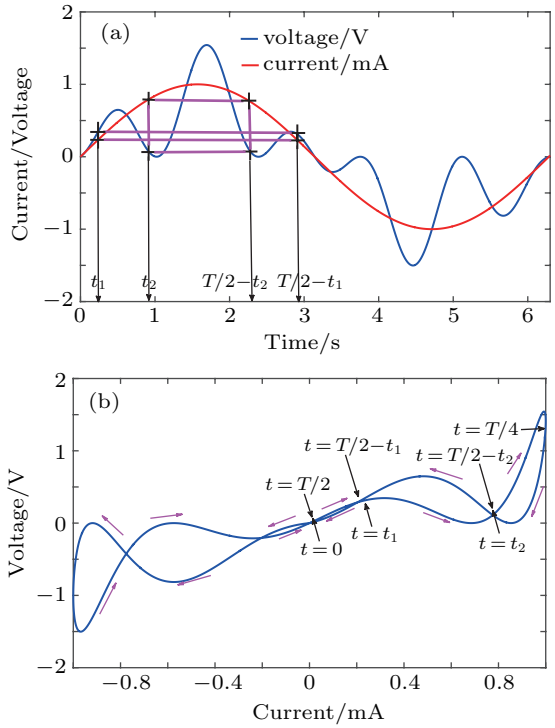


Fig. 10. (color online) Simulation of complex-order memrostor (CO_1) when $\alpha = 0.75$, $\beta = 1.0$, $\omega = 2.3$. (a) Waveforms of voltage and current versus time and (b) the pinched hysteretic loop.

5. Discussion

In this paper, we proposed the complex-order electric elements concept and modeled and analyzed the complex-order elements. Some interesting phenomena are found: the real part of the order affects the phase of output signal, and the imaginary part affects the amplitude.

The complex derivative is a generalization of the concept of integer derivative and fractional derivative, which is used in modeling electric elements for the first time in this paper. By applying the complex derivative, a high-dimensional parameter space is obtained. A simple complex-order memristor model is given, and the models of the complex-order memcapacitor and meminductor still need research. The application of complex-order elements (especially memory elements) in circuits will be a research direction. Complex derivative is a new concept in the field of engineering, and we will continue to focus on it and its application in the next work.

6. Conclusions

In this paper, the complex-order electric elements concept is proposed for the first time, and the complex-order elements are modeled and analyzed. By applying the concept of complex derivative, a high-dimensional parameter space is obtained. Some interesting phenomena are found: the real part of the order affects the phase of the output signal, and the imaginary part affects the amplitude for both complex-order capacitor and complex-order memristor. A more interesting thing is that the complex-order capacitor can do well at the time of fitting electrochemistry impedance spectra. The complex-order memristor was also analyzed, the area inside the hysteresis loops increases with the increasing of the imaginary part of the order and decreases with the increasing of the real part. Some complex cases of complex-order memristors hysteresis loops are analyzed at last. There are still a lot of interesting phenomena about the complex-order electric elements worthy of studying, and we will continue to investigate them in future.

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