Multi-copy entanglement purification with practical spontaneous parametric down conversion sources*

Shuai-Shuai Zhang(张帅帅)[1,3], Qi Shu(祁舒)[1,3], Lan Zhou(周澜)[2], and Yu-Bo Sheng(盛宇波)[1,†]

[1] *Key Laboratory of Broadband Wireless Communication and Sensor Network Technology of Ministry of Education, Nanjing University of Posts and Telecommunications, Nanjing* 210003, *China*

[2] *College of Mathematics & Physics, Nanjing University of Posts and Telecommunications, Nanjing* 210003, *China*

[3] *Bell Honors School, Nanjing University of Posts and Telecommunications, Nanjing* 210003, *China*

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Entanglement purification is to distill the high quality entanglement from the low quality entanglement with local operations and classical communications. It is one of the key technologies in long-distance quantum communication. We discuss an entanglement purification protocol (EPP) with spontaneous parametric down conversion (SPDC) sources, in contrast to previous EPP with multi-copy mixed states, which requires ideal entanglement sources. We show that the SPDC source is not an obstacle for purification, but can benefit the fidelity of the purified mixed state. This EPP works for linear optics and is feasible in current experiment technology.

Keywords: quantum communication, entanglement, entanglement purification

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1. Introduction

The absolute security of quantum communication makes it have great value in future communication technologies. Many important protocols have been proposed, such as quantum teleportation,[1] quantum secure direct communication (QSDC),[2–5] quantum secret sharing (QSS),[6] quantum key distribution (QKD),[7–9] and some other interesting and important quantum communication and quantum information processing protocols.[10–18] In order to realize the long-distance quantum communication, high quality entanglement is the indispensable resource. However, entanglement can only be prepared locally and then be distributed in the free space or in a fiber. In a practical transmission, the distributed photons will inevitably suffer from the noise in environment. If the entangled quantum system degrades seriously, the quantum communication tasks cannot be well achieved. Moreover, the quantum communication will become insecure.

Entanglement purification is an important quantum technique to obtain the high quality entanglement from the low quality entanglement.[19] In 1996, Bennett *et al.* firstly proposed the concept of entanglement purification which attracted much attention.[19] Generally, existing entanglement purification protocols (EPPs) can be divided into three groups. The first group is to use one copy of degraded mixed state.[20–27] They are the deterministic EPPs which exploit the hyperentanglement or robust entanglement encoded in other degree of freedom. The second group which contains the most EPPs is to use two copies of degraded mixed states.[19,28–44] The third group is to use the multi-copy degraded mixed states.[45–50] In the third group, they usually exploit the controlled-not (CNOT) gates and controlled-controlled-not (CCNOT) gates to complete the purification. However, CNOT and CCNOT are not feasible in current experimental condition, which makes this kind of protocols be far away from current application. In 2015, Cai *et al.* described a fast multi-copy EPP.[50] Their protocol can obtain a high fidelity in one purification step and it does not require the sophisticated CNOT gate or CCNOT gate. The protocol only exploits the feasible polarization beam splitters (PBSs) and single-photon detectors. Similar to Ref. [29], the protocol in Ref. [50] still needs the ideal entanglement source. It is known that the ideal entanglement source is not available. Such obstacle will make this protocol still be far away from practical application.

In this paper, we will investigate the multi-copy EPP with practical spontaneous parametric down conversion (SPDC) sources. Interestingly, the SPDC source is not an obstacle in realization purification, but will benefit the protocol. The SPDC source makes us obtain a higher fidelity than the original protocol.[50] This EPP also requires the linear optics and is feasible in current experiment condition.

This paper is organized as follows. In Section 2, we will describe the basic model of this protocol and discuss the two-copy EPP with SPDC sources. In Section 3, we first discuss the three-copy EPP with SPDC sources and then extend this

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†Corresponding author. E-mail: shengyb@njupt.edu.cn

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EPP to a more general case. In Section 4, we will provide a discussion and conclusion.

2. Two-copy EPP with SPDC sources

The basic model of the protocol is shown in Fig. 1, which is similar to Ref. [29]. Bob and Alice want to share the maximally entangled state $|\Phi^+\rangle_{ab}$ of the form

$$|\Phi^+\rangle_{ab} = \frac{1}{\sqrt{2}}(|H\rangle_a|H\rangle_b + |V\rangle_a|V\rangle_b), \tag{1}$$

where $|H\rangle$ denotes the horizontally polarized photon and $|V\rangle$ denotes the vertically polarized photon, respectively. The subscripts "a" and "b" are the spatial modes belonging to Alice and Bob, respectively. Because of the noise, the state $|\Phi^+\rangle_{ab}$ will become $|\Psi^+\rangle_{ab}$ with the probability of $1-F$, which is called a bit-flip error. Here $|\Psi^+\rangle_{ab}$ can be written as

$$|\Psi^+\rangle_{ab} = \frac{1}{\sqrt{2}}(|H\rangle_a|V\rangle_b + |V\rangle_a|H\rangle_b). \tag{2}$$

Then Alice and Bob will obtain a mixed state of the form

$$\rho_1 = F|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F)|\Psi^+\rangle_{ab}\langle\Psi^+|. \tag{3}$$

In order to realize the EPP, some EPPs should exploit the ideal entanglement source.[28,29,50] Unfortunately, such ideal entanglement source is unavailable. Here we use the practical SPDC source to perform the EPP. As shown in Fig. 1, the pump passes through the first beta barium borate (BBO) crystal, named BBO_2, and produces correlated pair of photons in spatial modes a_2b_2 of the form[20,32,51]

$$\rho_{\mathrm{spdc2}} = |0\rangle\langle 0| + p|\Phi^+\rangle_{a_2b_2}\langle\Phi^+| + p^2(|\Phi^+\rangle_{a_2b_2}\langle\Phi^+|)^{\otimes 2} + \cdots. \tag{4}$$

Then the pump pass through the second BBO (BBO_1) and produces a second correlated pair of photons in spatial modes a_1b_1 of the form

$$\rho_{\mathrm{spdc1}} = |0\rangle\langle 0| + p|\Phi^+\rangle_{a_1b_1}\langle\Phi^+| + p^2(|\Phi^+\rangle_{a_1b_1}\langle\Phi^+|)^{\otimes 2} + \cdots. \tag{5}$$

Therefore, state $\rho_{\mathrm{spdc2}} \otimes \rho_{\mathrm{spdc2}}$ can be written as

$$\rho_{\mathrm{spdc2}} \otimes \rho_{\mathrm{spdc1}} = |0\rangle\langle 0| + p(|\Phi^+\rangle_{a_2b_2}\langle\Phi^+| + |\Phi^+\rangle_{a_1b_1}\langle\Phi^+|) + p^2((|\Phi^+\rangle_{a_2b_2}\langle\Phi^+|)^{\otimes 2} + (|\Phi^+\rangle_{a_1b_1}\langle\Phi^+|)^{\otimes 2} + |\Phi^+\rangle_{a_1b_1}\langle\Phi^+| \otimes |\Phi^+\rangle_{a_2b_2}\langle\Phi^+| + \cdots). \tag{6}$$

From Eq. (6), the produced states in spatial modes a_1b_1 and a_2b_2 can be described as follows. With the probability of p, it is in the state $(1/2)(|H\rangle|H\rangle + |V\rangle|V\rangle) \otimes (|a_1\rangle|b_1\rangle + |a_2\rangle|b_2\rangle)$. It is essentially the hyperentanglement in both polarization and spatial mode degrees of freedom.[20,22,23,52] With the probability of p^2, they are in the states $|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$, $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}$ or $|\Phi^+\rangle_{a_2b_2} \otimes |\Phi^+\rangle_{a_1b_1}$. If we consider the errors, the state $|\Phi^+\rangle_{a_1b_1}$ will become $\rho_{a_1b_1}$ and $|\Phi^+\rangle_{a_2b_2}$ will become $\rho_{a_2b_2}$. In this way, the original states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}$ will become four possible cases. States are $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}$ with a probability of p^2F^2. States are $|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}$ with a probability of $p^2(1-F)^2$. States are $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}$ and $|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}$ with an equal probability of $p^2F(1-F)$. Then for $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}$, it becomes states $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}$ with a probability of p^2F and states $|\Psi^+\rangle^{\otimes 2}_{a_1b_1}$ with a probability of $p^2(1-F)$. Similarly, for $|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$, it becomes states $|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ with a probability of p^2F and states $|\Psi^+\rangle^{\otimes 2}_{a_2b_2}$ with a probability of $p^2(1-F)$. Here we should point out that $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}$ cannot become $|\Phi^+\rangle_{a_1b_1} \otimes |\Psi^+\rangle_{a_1b_1}$, for the two pairs are emitted from the BBO simultaneously and they are always in the same position. The two pairs of photons always suffer from the same error. Therefore, they are always the same states. $|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ is analogy with $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}$.

As shown in Fig. 1, PBSs can fully transmit the $|H\rangle$ polarization photon and reflect the $|V\rangle$ polarization photon, respectively. The key step of purification is to select the case that there is only one photon in each output mode which we call it four-mode case.

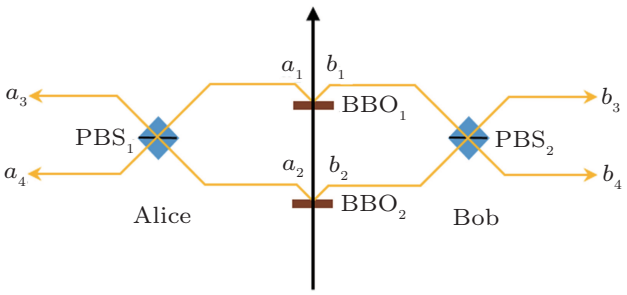


Fig. 1. (color online) The entanglement purification protocol (EPP) with two copies of mixed states using two SPDC sources. PBSs are polarization beam splitters. PBSs can fully transmit the $|H\rangle$ polarized photon and reflect the $|V\rangle$ polarized photon, respectively.

For example, with the probability of p^2F^2, states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}$ can be written as

$$\begin{aligned}|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2} &= \frac{1}{\sqrt{2}}(|H_{a_1}H_{b_1}\rangle + |V_{a_1}V_{b_1}\rangle) \otimes \frac{1}{\sqrt{2}}(|H_{a_2}H_{b_2}\rangle + |V_{a_2}V_{b_2}\rangle) \\ &= \frac{1}{2}(|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2} + |H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2} + |V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2} + |V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}).\end{aligned} \tag{7}$$

After passing through PBS_1 and PBS_2, only items $|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}$ and $|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}$ can make each of the output modes a_3b_3 and a_4b_4 contain one photon. Similarly, with the probability of $p^2(1-F)^2$, states $|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}$ can be written as

$$\begin{aligned}|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2} &= \frac{1}{\sqrt{2}}(|H_{a_1}V_{b_1}\rangle+|V_{a_1}H_{b_1}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_2}V_{b_2}\rangle+|V_{a_2}H_{b_2}\rangle)\\ &= \frac{1}{2}(|H\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}+|H\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}+|V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}+|V\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}).\end{aligned}\tag{8}$$

After passing through PBS_1 and PBS_2 , only items $|H\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}$ and $|V\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}$ can make each of the output modes a_3b_3 and a_4b_4 contain one photon. With the probability of p^2F, states $|\Phi^+\rangle^{\otimes2}_{a_1b_1}$ can be written as

$$\begin{aligned}|\Phi^+\rangle^{\otimes2}_{a_1b_1} &= \frac{1}{\sqrt{2}}(|H_{a_1}H_{b_1}\rangle+|V_{a_1}V_{b_1}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_1}H_{b_1}\rangle+|V_{a_1}V_{b_1}\rangle)\\ &= \frac{1}{2}(|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_1}|H\rangle_{b_1}+|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_1}|V\rangle_{b_1}+|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_1}|H\rangle_{b_1}+|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_1}|V\rangle_{b_1}).\end{aligned}\tag{9}$$

After passing through PBS_1 and PBS_2, only items $|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_1}|V\rangle_{b_1}$ and $|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_1}|H\rangle_{b_1}$ can make each of the output modes a_3b_3 and a_4b_4 contain one photon. With the probability of $p^2(1-F)$, states $|\Psi^+\rangle^{\otimes2}_{a_1b_1}$ can be written as

$$\begin{aligned}|\Psi^+\rangle^{\otimes2}_{a_1b_1} &= \frac{1}{\sqrt{2}}(|H_{a_1}V_{b_1}\rangle+|V_{a_1}H_{b_1}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_1}V_{b_1}\rangle+|V_{a_1}H_{b_1}\rangle)\\ &= \frac{1}{2}(|H\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_1}|V\rangle_{b_1}+|H\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_1}|H\rangle_{b_1}+|V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_1}|V\rangle_{b_1}+|V\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_1}|H\rangle_{b_1}).\end{aligned}\tag{10}$$

After passing through PBS_1 and PBS_2, only items $|H\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_1}|H\rangle_{b_1}$ and $|V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_1}|V\rangle_{b_1}$ can make each of the output modes a_3b_3 and a_4b_4 contain one photon. States $|\Phi^+\rangle^{\otimes2}_{a_2b_2}$ and $|\Psi^+\rangle^{\otimes2}_{a_2b_2}$ can also have the probability to cause the four-mode cases, similar to states $|\Phi^+\rangle^{\otimes2}_{a_1b_1}$ and $|\Psi^+\rangle^{\otimes2}_{a_1b_1}$, respectively, i.e., items $|H\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_2}|V\rangle_{b_2}$, $|V\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_2}|H\rangle_{b_2}$, and items $|H\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_2}|H\rangle_{b_2}$, $|V\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_2}|V\rangle_{b_2}$ can satisfy four-mode case.

On the other hand, states $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}$ and $|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}$ cannot satisfy the four-mode case. By selecting the four-mode case, one can obtain

$$\begin{aligned}|\phi_1\rangle = \frac{1}{\sqrt{2}}&(|H\rangle_{a_3}|H\rangle_{b_3}|H\rangle_{a_4}|H\rangle_{b_4}\\ &+|V\rangle_{a_3}|V\rangle_{b_3}|V\rangle_{a_4}|V\rangle_{b_4}),\end{aligned}\tag{11}$$

with the probability of $p^2F^2/2$, and

$$\begin{aligned}|\phi_2\rangle = \frac{1}{\sqrt{2}}&(|H\rangle_{a_3}|V\rangle_{b_3}|H\rangle_{a_4}|V\rangle_{b_4}\\ &+|V\rangle_{a_3}|H\rangle_{b_3}|V\rangle_{a_4}|H\rangle_{b_4}),\end{aligned}\tag{12}$$

with the probability of $p^2(1-F)^2/2$. For the rest of the states, which satisfy the four-mode case, the following states $|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_1}|V\rangle_{b_1}$, $|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_1}|H\rangle_{b_1}$, $|H\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_1}|H\rangle_{b_1}$, and $|V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_1}|V\rangle_{b_1}$ all become $|V\rangle_{a_3}|V\rangle_{b_3}|H\rangle_{a_4}|H\rangle_{b_4}$ and the total probability is $p^2/2$. Similarly $|H\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_2}|V\rangle_{b_2}$, $|V\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_2}|H\rangle_{b_2}$, $|H\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_2}|H\rangle_{b_2}$, and $|V\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_2}|V\rangle_{b_2}$ all become $|H\rangle_{a_3}|H\rangle_{b_3}|V\rangle_{a_4}|V\rangle_{b_4}$, and the probability is $p^2/2$. Therefore, one can also obtain the state

$$\begin{aligned}|\phi_3\rangle = \frac{1}{\sqrt{2}}&(|H\rangle_{a_3}|H\rangle_{b_3}|V\rangle_{a_4}|V\rangle_{b_4}\\ &+|V\rangle_{a_3}|V\rangle_{b_3}|H\rangle_{a_4}|H\rangle_{b_4}),\end{aligned}\tag{13}$$

with the probability of p^2. In order to get $|\Phi^+\rangle_{ab}$, one should measure the states above in spatial modes a_4b_4 in the basis $|\pm\rangle = (1/\sqrt{2})(|H\rangle \pm |V\rangle)$. Similar to Refs. [29], [30], and [49], one can ultimately obtain a new mixed state

$$\rho'_{ab} = F_2|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F_2)|\Psi^+\rangle_{ab}\langle\Psi^+|,\tag{14}$$

and the fidelity is

$$F_2 = \frac{F^2+2}{F^2+(1-F)^2+2}.\tag{15}$$

Here the subscription "2" means that the EPP with two copies of the mixed states.

3. Multi-copy EPP with SPDC sources

In this section, we will discuss the EPP with multi-copy mixed states. As shown in Fig. 2, the pump passes through three BBOs, and produces the entangled photon pairs ρ_{spdc1}, ρ_{spdc2}, and ρ_{spdc3} in spatial modes a_1b_1, a_2b_2, and a_3b_3, respectively. Here ρ_{spdc1}, ρ_{spdc2}, and ρ_{spdc3} have the same form as shown in Eq. (4) and Eq. (5). In order to realize the EPP, we should select the six-mode case. That is each of the output modes a_4, b_4, a_5, b_5, a_6, and b_6 exactly registers one photon. Therefore, for the whole system $\rho_{spdc1}\otimes\rho_{spdc2}\otimes\rho_{spdc3}$,

only items $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$, $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$, $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_3b_3}$, $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}|\Phi^+\rangle_{a_2b_2}$, $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}|\Phi^+\rangle_{a_3b_3}$, $|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle^{\otimes 2}_{a_3b_3}$, and $|\Phi^+\rangle^{\otimes 2}_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$ can satisfy the six-mode cases.

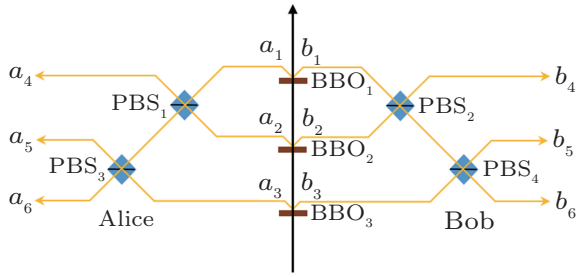


Fig. 2. (color online) The EPP with three copies of mixed states using three SPDC sources.

If the state $|\Phi^+\rangle_{ab}$ suffers from the noise, it will become a mixed state ρ_1 as shown in Eq. (3). In this way, the original state $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$ will become a mixture of the following states as given below.

(i) With the probability of p^3F^3, they are in the states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$.

(ii) With the probability of $p^3(1-F)^3$, they are in the states $|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3}$.

(iii) With the equal probability of $p^3F^2(1-F)$, they are in the states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3}$, $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$, and $|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$.

(iv) With the equal probability of $p^3F(1-F)^2$, they are in the states $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3}$, $|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3}$, and $|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3}$, respectively.

For the states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$, it becomes states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ with the probability of p^3F^2, states $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_2b_2}$ with the probability of $p^3F(1-F)$, states $|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ with the probability of $p^3F(1-F)$, and $|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_2b_2}$ with the probability of $p^3(1-F)^2$, respectively. The other states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_3b_3}$, $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}|\Phi^+\rangle_{a_2b_2}$, $|\Phi^+\rangle^{\otimes 2}_{a_1b_1}|\Phi^+\rangle_{a_3b_3}$, $|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle^{\otimes 2}_{a_3b_3}$, and $|\Phi^+\rangle^{\otimes 2}_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$ can also be described in the same principle.

Similar to Section 2, the key step of this EPP is to select the case that there is only one photon in each output mode and we call it six-mode case. Firstly, we discuss the state $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$.

Because of noise as described in the above discussion, the state $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$ will become eight possible different states as

$$
\begin{aligned}
&|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3},\ |\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3},\\
&|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3},\ |\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3},\\
&|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3},\ |\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3},\\
&|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3},\ |\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3},
\end{aligned}
$$

respectively. Interestingly, as shown in Fig. 2, after passing through the four PBSs, only states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$ and $|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3}$ have the possibility to satisfy the six-mode case. For example, with the probability of p^3F^3, states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}$ can be written as

$$
\begin{aligned}
|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3} &= \frac{1}{\sqrt{2}}(|H_{a_1}H_{b_1}\rangle+|V_{a_1}V_{b_1}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_2}H_{b_2}\rangle+|V_{a_2}V_{b_2}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_3}H_{b_3}\rangle+|V_{a_3}V_{b_3}\rangle)\\
&= \frac{1}{2\sqrt{2}}(|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_3}|H\rangle_{b_3}+|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_3}|H\rangle_{b_3}\\
&\quad+|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_3}|H\rangle_{b_3}+|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_3}|H\rangle_{b_3}\\
&\quad+|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_3}|V\rangle_{b_3}+|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_3}|V\rangle_{b_3}\\
&\quad+|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_3}|V\rangle_{b_3}+|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_3}|V\rangle_{b_3}). \qquad (16)
\end{aligned}
$$

After passing through PBS$_1$, PBS$_2$, PBS$_3$, and PBS$_4$, items $|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_3}|H\rangle_{b_3}$ and $|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_3}|V\rangle_{b_3}$ can make each of the output modes a_4b_4, a_5b_5, and a_6b_6 contain one photon. By selecting the six-mode case, they can obtain

$$
|\psi_1\rangle = \frac{1}{\sqrt{2}}(|H\rangle_{a_4}|H\rangle_{b_4}|H\rangle_{a_5}|H\rangle_{b_5}|H\rangle_{a_6}|H\rangle_{b_6}+|V\rangle_{a_4}|V\rangle_{b_4}|V\rangle_{a_5}|V\rangle_{b_5}|V\rangle_{a_6}|V\rangle_{b_6}) \qquad (17)
$$

with the probability of $p^3F^3/4$. On the other hand, with the probability of $p^3(1-F)^3$, states $|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3}$ can be written as

$$
\begin{aligned}
|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle_{a_2b_2}|\Psi^+\rangle_{a_3b_3} &= \frac{1}{\sqrt{2}}(|H_{a_1}V_{b_1}\rangle+|V_{a_1}H_{b_1}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_2}V_{b_2}\rangle+|V_{a_2}H_{b_2}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_3}V_{b_3}\rangle+|V_{a_3}H_{b_3}\rangle)\\
&= \frac{1}{2\sqrt{2}}(|H\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_3}|V\rangle_{b_3}+|H\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_3}|V\rangle_{b_3}\\
&\quad+|V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_3}|V\rangle_{b_3}+|V\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_3}|V\rangle_{b_3}\\
&\quad+|H\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_3}|H\rangle_{b_3}+|H\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_3}|H\rangle_{b_3}
\end{aligned}
$$

$$+|V\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_3}|H\rangle_{b_3}+|V\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_3}|H\rangle_{b_3}). \tag{18}$$

After passing through PBS_1, PBS_2, PBS_3, and PBS_4, items $|H\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_3}|V\rangle_{b_3}$ and $|V\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_3}|H\rangle_{b_3}$ can make each of the output modes a_4b_4, a_5b_5, and a_6b_6 contain one photon. By selecting the six-mode case, they can obtain

$$|\psi_2\rangle=\frac{1}{\sqrt{2}}(|H\rangle_{a_4}|V\rangle_{b_4}|H\rangle_{a_5}|V\rangle_{b_5}|H\rangle_{a_6}|V\rangle_{b_6}+|V\rangle_{a_4}|H\rangle_{b_4}|V\rangle_{a_5}|H\rangle_{b_5}|V\rangle_{a_6}|H\rangle_{b_6}) \tag{19}$$

with the probability $p^3(1-F)^3/4$.

Next, we discuss states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$. If errors occurs, states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ will become states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$, $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_2b_2}$, $|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$, and $|\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_2b_2}$ which we have described before. Interestingly, only two states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ and $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_2b_2}$ have the possibility to satisfy the six-mode case. For example, with the probability of p^3F^2, states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ can be written as

$$\begin{aligned}|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}&=\frac{1}{\sqrt{2}}(|H_{a_1}H_{b_1}\rangle+|V_{a_1}V_{b_1}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_2}H_{b_2}\rangle+|V_{a_2}V_{b_2}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_2}H_{b_2}\rangle+|V_{a_2}V_{b_2}\rangle)\\&=\frac{1}{2\sqrt{2}}(|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_2}|H\rangle_{b_2}+|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_2}|H\rangle_{b_2}\\&\quad+|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_2}|H\rangle_{b_2}+|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_2}|H\rangle_{b_2}\\&\quad+|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_2}|V\rangle_{b_2}+|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_2}|V\rangle_{b_2}\\&\quad+|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_2}|V\rangle_{b_2}+|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_2}|V\rangle_{b_2}).\end{aligned} \tag{20}$$

After passing through PBS_1, PBS_2, PBS_3, and PBS_4, only $|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_2}|V\rangle_{b_2}$ and $|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_2}|H\rangle_{b_2}$ can satisfy six-mode case. By selecting six-mode case, $|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_2}|V\rangle_{b_2}$ and $|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_2}|H\rangle_{b_2}$ will both become the state $|H\rangle_{a_4}|H\rangle_{b_4}|V\rangle_{a_5}|V\rangle_{b_5}|H\rangle_{a_6}|H\rangle_{b_6}$ with the probability of $p^3F^2/4$. With the probability of $p^3F(1-F)$, states $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_2b_2}$ can be written as

$$\begin{aligned}|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_2b_2}&=\frac{1}{\sqrt{2}}(|H_{a_1}H_{b_1}\rangle+|V_{a_1}V_{b_1}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_2}V_{b_2}\rangle+|V_{a_2}H_{b_2}\rangle)\otimes\frac{1}{\sqrt{2}}(|H_{a_2}V_{b_2}\rangle+|V_{a_2}H_{b_2}\rangle)\\&=\frac{1}{2\sqrt{2}}(|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_2}|V\rangle_{b_2}+|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_2}|V\rangle_{b_2}\\&\quad+|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|H\rangle_{a_2}|V\rangle_{b_2}+|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_2}|V\rangle_{b_2}\\&\quad+|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_2}|H\rangle_{b_2}+|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_2}|H\rangle_{b_2}\\&\quad+|V\rangle_{a_1}|V\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_2}|H\rangle_{b_2}+|V\rangle_{a_1}|V\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|V\rangle_{a_2}|H\rangle_{b_2}).\end{aligned} \tag{21}$$

After passing through PBS_1, PBS_2, PBS_3, and PBS_4, only items

$$|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_2}|H\rangle_{b_2}$$

and

$$|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_2}|V\rangle_{b_2}$$

can satisfy the six-mode case. Interestingly, by selecting the six-mode case,

$$|H\rangle_{a_1}|H\rangle_{b_1}|H\rangle_{a_2}|V\rangle_{b_2}|V\rangle_{a_2}|H\rangle_{b_2}$$

and

$$|H\rangle_{a_1}|H\rangle_{b_1}|V\rangle_{a_2}|H\rangle_{b_2}|H\rangle_{a_2}|V\rangle_{b_2}$$

both become $|H\rangle_{a_4}|H\rangle_{b_4}|V\rangle_{a_5}|V\rangle_{b_5}|H\rangle_{a_6}|H\rangle_{b_6}$ with the probability $p^3F(1-F)/4$.

Therefore, from the states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ and $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_2b_2}$, they can obtain the states

$$|H\rangle_{a_4}|H\rangle_{b_4}|V\rangle_{a_5}|V\rangle_{b_5}|H\rangle_{a_6}|H\rangle_{b_6}$$

with the total probability $p^3F/4$.

Similarly, the states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_3b_3}$ can be described in the same way. Because of errors, it will become four different states as

$$\begin{aligned}&|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_3b_3},\quad |\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_3b_3},\\&|\Psi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_3b_3},\quad |\Psi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_3b_3}.\end{aligned}$$

Only the states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_3b_3}$ and $|\Phi^+\rangle_{a_1b_1}|\Psi^+\rangle^{\otimes 2}_{a_3b_3}$ have the possibility to satisfy six-mode case. By selecting six-mode case, they can totally obtain $|V\rangle_{a_4}|V\rangle_{b_4}|H\rangle_{a_5}|H\rangle_{b_5}|V\rangle_{a_6}|V\rangle_{b_6}$ with the probability of $p^3F/4$.

Moreover, if considering the states $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_3b_3}$ and $|\Phi^+\rangle_{a_1b_1}|\Phi^+\rangle^{\otimes 2}_{a_2b_2}$ together, they can obtain the state

$$\begin{aligned}|\psi_3\rangle=\frac{1}{\sqrt{2}}&(|H\rangle_{a_4}|H\rangle_{b_4}|V\rangle_{a_5}|V\rangle_{b_5}|H\rangle_{a_6}|H\rangle_{b_6}\\&+|V\rangle_{a_4}|V\rangle_{b_4}|H\rangle_{a_5}|H\rangle_{b_5}|V\rangle_{a_6}|V\rangle_{b_6})\end{aligned} \tag{22}$$

with the probability of $p^3F/2$.

The other states

$$|\Phi^+\rangle_{a_1b_1}^{\otimes 2}|\Phi^+\rangle_{a_2b_2}, \quad |\Phi^+\rangle_{a_1b_1}^{\otimes 2}|\Phi^+\rangle_{a_3b_3},$$
$$|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}^{\otimes 2}, \quad |\Phi^+\rangle_{a_2b_2}^{\otimes 2}|\Phi^+\rangle_{a_3b_3}$$

can be discussed with the same principle. If considering the states $|\Phi^+\rangle_{a_1b_1}^{\otimes 2}|\Phi^+\rangle_{a_2b_2}$ and $|\Phi^+\rangle_{a_2b_2}|\Phi^+\rangle_{a_3b_3}^{\otimes 2}$ together, they can obtain the state

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}(|H\rangle_{a_4}|H\rangle_{b_4}|H\rangle_{a_5}|H\rangle_{b_5}|V\rangle_{a_6}|V\rangle_{b_6} + |V\rangle_{a_4}|V\rangle_{b_4}|V\rangle_{a_5}|V\rangle_{b_5}|H\rangle_{a_6}|H\rangle_{b_6}) \quad (23)$$

with the probability of $p^3F/2$. If considering the states $|\Phi^+\rangle_{a_1b_1}^{\otimes 2}|\Phi^+\rangle_{a_3b_3}$ and $|\Psi^+\rangle_{a_2b_2}^{\otimes 2}|\Phi^+\rangle_{a_3b_3}$ together, they can obtain the state

$$|\psi_5\rangle = \frac{1}{\sqrt{2}}(|H\rangle_{a_4}|H\rangle_{b_4}|V\rangle_{a_5}|V\rangle_{b_5}|V\rangle_{a_6}|V\rangle_{b_6} + |V\rangle_{a_4}|V\rangle_{b_4}|H\rangle_{a_5}|H\rangle_{b_5}|H\rangle_{a_6}|H\rangle_{b_6}) \quad (24)$$

with the probability of $p^3F/2$.

From the states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$, $|\psi_4\rangle$, and $|\psi_5\rangle$, after measuring the photons in spatial modes a_5, b_5, a_6, and b_6 in the basis $|\pm\rangle$, states $|\psi_1\rangle$, $|\psi_3\rangle$, $|\psi_4\rangle$, and $|\psi_5\rangle$ will become $|\Phi^+\rangle_{ab}$ and $|\psi_2\rangle$ will become $|\Psi^+\rangle_{ab}$.

Finally, they can ultimately obtain a new mixed state

$$\rho''_{ab} = F_3|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F_3)|\Psi^+\rangle_{ab}\langle\Psi^+|, \quad (25)$$

and the fidelity is

$$F_3 = \frac{F^3+6F}{F^3+(1-F)^3+6F}. \quad (26)$$

It is straightforward to extend this protocol to the EPP with N pairs of mixed states.

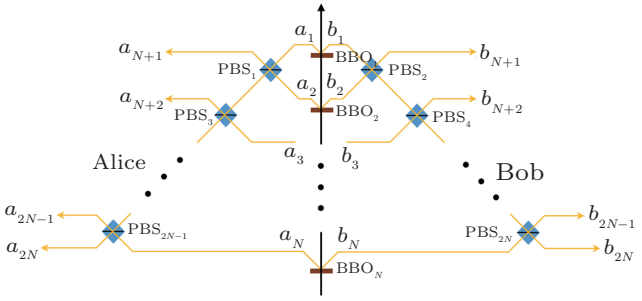


Fig. 3. (color online) The EPP with N copies of mixed states using N SPDC sources.

As shown in Fig. 3, the pump passes through N BBOs and creates the correlated photon pairs in spatial modes a_1b_1, a_2b_2, …, a_Nb_N, respectively. Similar to the EPP with two pairs of mixed states and three pairs of mixed states described above, for the EPP with four pairs of mixed states, they select the eight-mode cases and obtain the new fidelity as

$$F_4 = \frac{F^4+12F^2+4}{F^4+(1-F)^4+12F^2+4}. \quad (27)$$

Generally, we can also obtain

$$F_5 = \frac{F^5+20F^3+20F}{F^5+(1-F)^5+20F^3+20F},$$
$$\cdots,$$
$$F_N = \frac{F^N+\alpha_1*F^{N-2}+\cdots+\alpha_k 2^{k-1}F^{N-2k}+\cdots}{F^N+(1-F)^N+\alpha_1*F^{N-2}+\cdots+\alpha_k 2^{k-1}F^{N-2k}+\cdots}. \quad (28)$$

Here $k=1,2,\ldots,[N/2]$. If N is odd, $[N/2]$ equals $(N-1)/2$. Otherwise, it equals $N/2$. Here $\alpha_1 = N(N-1)$ and α_k $(k\geq 2)$ is

$$\alpha_k = \sum_{i=1}^{N-2k+1} i(i+1)\Big(\sum_{m_1+m_2+\cdots+m_{k-1}=N-k-i}(m_1m_2\cdots m_{k-1})\Big). \quad (29)$$

Here $m_1,m_2,\ldots,m_{k-1}\geq 1$ and they are all different positive integers which satisfy the condition $m_1+m_2+\cdots+m_{k-1} = N-k-i$. Here we should iterate over all possible m_i $(i=1,2,\ldots,k-1)$ and sum over all possible cases.

We can also calculate the total success probability as

$$P_2 = \frac{p^2}{2}(F^2+(1-F)^2+2),$$
$$P_3 = \frac{p^3}{4}(F^3+(1-F)^3+6F),$$
$$\cdots,$$
$$P_N = \frac{p^N}{2^{N-1}}\big[F^N+(1-F)^N + \alpha_1*F^{N-2}+\cdots+\alpha_k 2^{k-1}F^{N-2k}+\cdots\big]. \quad (30)$$

Here P_2, P_3, P_N are the success probabilities with two-copy, three-copy, and N-copy purification.

4. Discussion and conclusion

So far, we have completely discussed this EPP. We mainly discussed the EPP for bit-flip error. Certainly, similar to Refs. [29] and [50], the phase-flip error can also be purified with the same principle, by adding the Hadamard operations to convert the phase-flip error to bit-flip error. In linear optics, the Hadamard operation can be implemented by quarter

wave plate (QWP). It is interesting to calculate the fidelity of this EPP. As shown in Fig. 4, we calculated the new fidelity F_N altered with the initial fidelity F. Here we let $F \in (0.5, 1)$ and $N = 2, 3, 4$ and 5 respectively. For example, if $F = 0.55$, we can obtain that $F_2 = 0.919$ and $F_4 = 0.9947$. In conventional EPPs,[29,50] we require the fidelity of initial mixed state to $F > 0.5$. In this EPP, we show that if $F < 0.5$, we can also obtain a higher fidelity of mixed state.

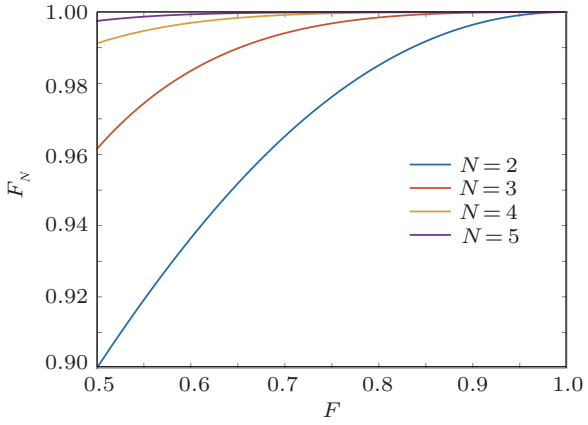


Fig. 4. (color online) The increased fidelity F_N altered with the initial fidelity F. N is the number of the degraded photon pairs.

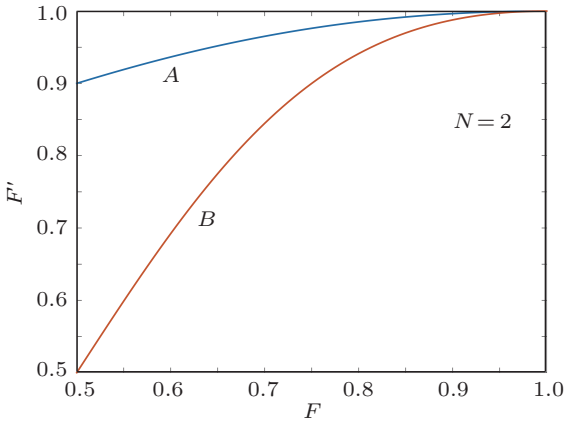


Fig. 5. (color online) The increased fidelity F' altered with the initial fidelity F. The curve A is this two-copy EPP with SPDC sources. Curve B is the EPP in Ref. [29].

In Figs. 5 and 6, we calculated the purified fidelity in EPPs with two copies and three copies in Refs. [29] and [50], respectively. In Fig. 5, curve A represents the purified fidelity F' altered with the initial fidelity F in this EPP with two copies. Curve B represents the purified fidelity F' altered with the initial fidelity F in Ref. [29]. In Fig. 6, curve A represents the purified fidelity F' altered with the initial fidelity F in this EPP with two copies. Curve B represents the purified fidelity F' altered with the initial fidelity F in Ref. [50]. From both Fig. 5 and Fig. 6, we show that the SPDC source is not the obstacle but the advantage. Actually, in Ref. [30], they have realized the EPP with SPDC sources. In their experiment, if the initial fidelity is $3/4$, they can obtain a higher fidelity $13/14$, which is also larger than the theory value $9/10$ with ideal entanglement sources.

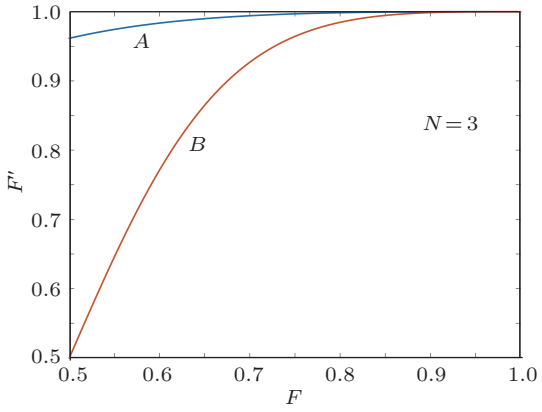


Fig. 6. (color online) The increased fidelity F' altered with the initial fidelity F. The curve A is this three-copy EPP with SPDC sources. Curve B is the EPP in Ref. [50].

In Fig. 7, we calculated the total success probability altered with the initial fidelity F. We let $p = 0.1$. Curve a is the case of two-copy purification. Curve b is the case of three-copy purification, and curve c is the case of four-copy purification, respectively. We show that the success probability $P_N \sim p^N/2^{N-1}$ and it decreases with N. In EPP with linear optics, they should exploit the post-selection principle. After the success purification, the purified mixed state should also be detected and they cannot improve the fidelity by repeating the EPP. With multi-copy purification, though the total success probability is low, we can obtain a higher fidelity for perform the EPP for one step. Therefore, it is a trade off between success probability and fidelity. In order to realize the multi-copy EPP, we should let the pump pass through N BBOs to generate N pairs of mixed states simultaneously. Such cascaded entanglement sources have been widely realized in current experiment, such as multi-photon entanglement generation,[53] quantum teleportation of multiple degrees of freedom of a single photon,[54] experimental Greenberger–Horne–Zeilinger-type six-photon quantum nonlocality,[55] and so on.

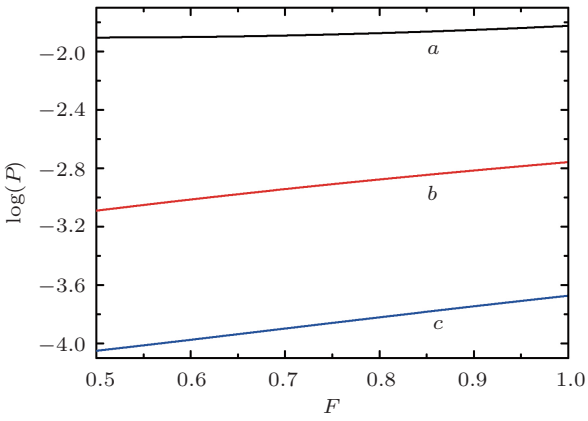


Fig. 7. (color online) The success probability P altered with the initial fidelity F. The curve a is the success probability for two-copy EPP. Curve b is success probability for three-copy EPP, and curve c is success probability for four-copy EPP, respectively. Here $p = 0.1$.

In conclusion, we have discussed the multi-copy EPP with practical SPDC sources. This EPP requires linear optics

to complete the task and it is feasible in current experiment condition. Moreover, compared with the EPP with ideal entanglement sources, we show that the SPDC source is not an obstacle but an advantage and can benefit the purification. As the ideal entanglement sources are not available, this EPP may be useful in long-distance quantum communication.

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