Solvability of a class of PT-symmetric non-Hermitian Hamiltonians: Bethe ansatz method

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We use the Bethe ansatz method to investigate the Schrödinger equation for a class of PT-symmetric non-Hermitian Hamiltonians. Elementary exact solutions for the eigenvalues and the corresponding wave functions are obtained in terms of the roots of a set of algebraic equations. Also, it is shown that the problems possess $sl(2)$ hidden symmetry and then the exact solutions of the problems are obtained by employing the representation theory of $sl(2)$ Lie algebra. It is found that the results of the two methods are the same.

Keywords: PT-symmetry, Bethe ansatz method, Lie algebraic approach, quasi-exactly solvable

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1. Introduction

The Hamiltonian operator of a quantum system characterizes the energy of system and its time evolution. The Hermitian property of the Hamiltonian is also a necessary condition that ensures the spectrum to be real, and the time evolution, for conservation of probability, to be unitary. However, quantum systems with non-Hermitian Hamiltonians are of interest in many branches of theoretical physics, including nuclear physics,[1] optical coupled system,[2] condensed matter systems,[3] etc. In recent two decades, Bender and others[4–16] have studied a class of one-dimensional non-Hermitian Hamiltonians giving rise to a real energy spectrum. These Hamiltonians are distinguished by the fact that they are invariant under the simultaneous space (P) and time (T) reflection (the so-called PT-symmetry). Therefore, it was suggested that instead of Hermiticity, PT-invariance is a sufficient condition for the reality of the energy spectrum.[4,17,18] Complex potentials with real eigenvalues have attracted considerable attention and have been studied by using various techniques such as the perturbation method,[19,20] numerical method,[21–23] asymptotic iteration method (AIM),[24,25] complex Lie algebraic method,[26] Nikiforov–Uvarov method,[27] etc. On the other hand, studies on the PT-symmetry have extended into many areas including supersymmetry,[28–30] quantum field theory[31–33] and also quasi-exact solvability.[5,13,14,34–36] A quantum system is exactly solvable (ES) if all the eigenvalues and eigenfunctions are known analytically. In the literature, various methods and techniques have been proposed and used to study the ES models.[37–48] In contrast, a quantum model is quasi-exactly solvable (QES) if only a certain finite number of eigenvalues and eigenfunctions, but not the whole spectrum, can be obtained algebraically.[49–52] The basic idea behind the phenomenon of (quasi-)exact solvability is the existence of a hidden Lie algebraic structure. Currently, there are two main approaches to studying the QES systems: the Lie algebraic approach[49–55] and the analytical approach which is based on the calculation of the Bethe ansatz equations.[56–60] In this paper, we use the Bethe ansatz method (BAM) to solve the Schrödinger equation for a class of PT-symmetric non-Hermitian models. Also, we solve the models via the Lie algebraic approach and illustrate that the results obtained using the two methods coincide with each other.

The rest of this paper is organized as follows. In Section 2, we introduce the three PT-symmetric potentials and solve the corresponding Schrödinger equations by using the BAM. Also, we obtain exact expressions for the eigenvalues and the corresponding wave functions in terms of the roots of the Bethe ansatz equations. In Section 3, after reviewing the connection between $sl(2)$ Lie algebra and the second order differential equations, we solve the same problems by using the Lie algebraic approach and show that the results of the two methods are consistent. Finally, we draw some conclusions in Section 4.

2. BAM for the non-Hermitian PT-symmetric models

In this section, we introduce the three non-Hermitian PT-symmetric models which are the object of the present article. For each model, using the method of factorization introduced by Chiang and Ho,[58,59] we obtain the exact expressions for the energies and the wave functions in terms of the roots of the Bethe ansatz equations.

2.1. QES PT-symmetric quartic potential

First, we consider the two-parametric QES quartic potential proposed by Bender and Boettcher[5]

$$V(z) = -z^4 + 2\mathrm{i}az^3 + (a^2 - 2b)z^2 + 2\mathrm{i}(ab - J)z, \qquad (1)$$

where a and b are real parameters and J is a positive integer. The corresponding Schrödinger equation is given by

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$(m=\hbar=c=1)$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2}-z^4+2\mathrm{i}az^3+(a^2-2b)z^2+2\mathrm{i}(ab-J)z\right)\psi(z) = E\psi(z). \tag{2}$$

Bender and Boettcher have solved this problem through the zeros of the Bender–Dunne polynomials.[5] Here, we intend to solve the problem using the BAM. For this purpose, from the asymptotic behavior of the wave function at infinity, the following gauge transformation is considered:

$$\psi(z)=\exp\left(-\mathrm{i}\frac{z^3}{3}-a\frac{z^2}{2}-\mathrm{i}bz\right)\varphi(z). \tag{3}$$

Substitution of Eq. (3) into Eq. (2) yields

$$\begin{aligned} &H\varphi(z)=0,\\ &H=-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left(2\mathrm{i}z^2+2az+2\mathrm{i}b\right)\frac{\mathrm{d}}{\mathrm{d}z}\\ &\quad+\left(-2\mathrm{i}(J-1)z+a+b^2-E\right). \end{aligned} \tag{4}$$

In applying the BAM to the present problem, we suppose that equation (4) has polynomial solutions as (the Bethe ansatz)

$$\varphi_J(z)=\begin{cases}\prod\limits_{k=2}^{J}(z-z_k), & J\neq 1,\\ 1, & J=1,\end{cases} \tag{5}$$

with unknown roots z_k, which can be explained as the wave function nodes. Now, we want to factorize the operator H as

$$H=A_J+A_J, \tag{6}$$

such that $A_J\phi_J(z)=0$. It is easily seen that operator A_J must have the form

$$A_J=\frac{\mathrm{d}}{\mathrm{d}z}-\sum_{k=2}^{J}\frac{1}{z-z_k}. \tag{7}$$

Then, operator $A_J^\dagger$ can be determined by an educated guess as follows:

$$A_J^\dagger=-\frac{\mathrm{d}}{\mathrm{d}z}+\left(2\mathrm{i}z^2+2az+2\mathrm{i}b\right)-\sum_{k=2}^{J}\frac{1}{z-z_k}. \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6), we obtain

$$\begin{aligned} A_J^\dagger A_J =& -\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left(2\mathrm{i}z^2+2az+2\mathrm{i}b\right)\frac{\mathrm{d}}{\mathrm{d}z}\\ &-\left(2\mathrm{i}z^2+2az+2\mathrm{i}b-\sum_{j\neq k}^{J}\frac{2}{z_j-z_k}\right)\sum_{k=2}^{J}\frac{1}{z-z_k}. \end{aligned} \tag{9}$$

As can be seen, the last term in this equation is a meromorphic function with a simple pole at $z=z_k$ and singularity at $z\to\infty$. Calculating the residues and comparing the results with Eq. (4), we obtain the following relations for the roots z_k (the so-called Bethe ansatz equation) and the energy eigenvalues:

$$\sum_{j\neq k}^{J}\frac{2}{z_j-z_k}-2\mathrm{i}z_k^2-2az_k-2\mathrm{i}b=0, \tag{10}$$

$$E_J=2\mathrm{i}\sum_{k=2}^{J}z_k+a(2J-1)+b^2. \tag{11}$$

Hence, equations (3) and (5), together with Eqs. (10) and (11) yield the exact solutions of the problem. As an example, we give explicit solutions for the first three states. For $J=1$, from Eqs. (3) and (11), we have the following relations for the ground state energy and wave function:

$$E_1=a+b^2, \tag{12}$$

$$\psi_1(z)=\exp\left(-\mathrm{i}\frac{z^3}{3}-a\frac{z^2}{2}-\mathrm{i}bz\right). \tag{13}$$

For the first excited state $J=2$, equations (3), (5), and (11) give

$$E_2=2\mathrm{i}z_2+3a+b^2, \tag{14}$$

$$\psi_2(z)=\exp\left(-\mathrm{i}\frac{z^3}{3}-a\frac{z^2}{2}-\mathrm{i}bz\right)(z-z_2), \tag{15}$$

where the root z_2 is obtained from the Bethe ansatz equation (10) as

$$z_2=\frac{1}{2\mathrm{i}}\left(-a\pm\sqrt{a^2+4b}\right). \tag{16}$$

Analogously, for the second excited state $J=3$, we have

$$E_3=2\mathrm{i}(z_2+z_3)+5a+b^2, \tag{17}$$

$$\psi_3(z)=\exp\left(-\mathrm{i}\frac{z^3}{3}-a\frac{z^2}{2}-\mathrm{i}bz\right)\left(z^2-(z_2+z_3)z+z_2z_3\right), \tag{18}$$

where the distinct roots z_2 and z_3 are obtained from Eq. (10) as

$$(z_2,z_3)=\begin{cases}(-0.40125-1.0527\mathrm{i},\ 0.40125-1.0527\mathrm{i}),\\ (2.3575\mathrm{i},\ 1.5245\mathrm{i}),\\ (-0.88313\mathrm{i},\ 2.1076\mathrm{i}).\end{cases} \tag{19}$$

Here, we have taken the parameters $a=1$ and $b=2$. Our numerical results obtained for the first three states are presented and compared in Table 1. It is observed that our result coincides with those obtained by Bender and Boettcher[5] via a different method.

2.2. QES *PT*-symmetric Khare–Mandal potential

Here, we consider the PT-symmetric partner of the one-dimensional Khare–Mandal potential defined by[35]

$$V(x)=-\left(\xi\sinh(2x)-\mathrm{i}N\right)^2, \tag{20}$$

where ξ is a real parameter and N is a positive integer. This potential has been studied by different approaches such as the algebraic method[35,61] and the asymptotic iteration method.[24] The Schrödinger equation with potential (20) is

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\left(\xi\sinh(2x)-\mathrm{i}N\right)^2\right)\psi(x)=E\psi(x). \tag{21}$$

Making the change of variable $z = \mathrm{e}^{2x}$ and considering the following transformation:

$$\psi(z) = z^{(1-N)/2}\exp\left(\frac{\mathrm{i}\xi}{4}\left(z-\frac{1}{z}\right)\right)\varphi(z), \tag{22}$$

we obtain

$$\begin{aligned} &H\varphi(z) = 0,\\ &H = -4z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \left(2\mathrm{i}\xi z^2 - 4(N-2)z + 2\mathrm{i}\xi\right)\frac{\mathrm{d}}{\mathrm{d}z}\\ &\quad + \left(2\mathrm{i}\xi(N-1)z + \xi^2 + 2N - 1 - E\right). \end{aligned} \tag{23}$$

By assuming the following polynomial solutions

$$\varphi_N(z) = \begin{cases} \prod_{k=2}^{N}(z-z_k), & N \neq 1,\\ 1, & N = 1, \end{cases} \tag{24}$$

where z_k are the unknown parameters to be determined, we can obtain the Bethe ansatz equations. In this case, defining the operators A_N and $A_N^\dagger$ as

$$\begin{aligned} A_N &= \frac{\mathrm{d}}{\mathrm{d}z} - \sum_{k=2}^{N}\frac{1}{z-z_k},\\ A_N^\dagger &= -\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\mathrm{i}\xi}{2} + \frac{N-2}{z} - \frac{\mathrm{i}\xi}{2z^2} - \sum_{k=2}^{N}\frac{1}{z-z_k}, \end{aligned} \tag{25}$$

the operator H can be factorized as

$$\begin{aligned} A_N^\dagger A_N \;=\; &-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{-\mathrm{i}\xi z^2 + 2(N-2)z - \mathrm{i}\xi}{2z^2}\frac{\mathrm{d}}{\mathrm{d}z}\\ &+ \left(\frac{\mathrm{i}\xi}{2} - \frac{N-2}{z} + \frac{\mathrm{i}\xi}{2z^2} + \sum_{j\neq k}^{N}\frac{2}{z_j - z_k}\right)\\ &\times \sum_{k=2}^{N}\frac{1}{z-z_k}. \end{aligned} \tag{26}$$

The last term in Eq. (26) is a meromorphic function with two simple poles at $z = z_k$ and $z = 0$. Comparing Eq. (26) with Eq. (23), we obtain the following relations:

$$\sum_{j\neq k}^{N}\frac{2}{z_j - z_k} + \frac{\mathrm{i}\xi}{2z_k^2} - \frac{N-2}{z_k} + \frac{\mathrm{i}\xi}{2} = 0, \tag{27}$$

$$E_N = \xi^2 + 2N - 1 - 2\mathrm{i}\xi\sum_{k=2}^{N} z_k, \tag{28}$$

for the roots z_k and the energy eigenvalues E_N. For example, for $N = 1$, from Eqs. (28) and (22), we have

$$E_1 = \xi^2 + 1, \tag{29}$$

$$\psi_1(z) = \exp\left[\frac{\mathrm{i}\xi}{4}\left(z-\frac{1}{z}\right)\right]. \tag{30}$$

For the first excited state, that is, $N = 2$, we obtain

$$E_2 = \xi^2 + 3 - 2\mathrm{i}\xi z_2, \tag{31}$$

$$\psi_2(z) = z^{-1/2}\exp\left[\frac{\mathrm{i}\xi}{4}\left(z-\frac{1}{z}\right)\right](z-z_2), \tag{32}$$

where $z_2 = \pm\mathrm{i}$. Similarly, for the second excited state, that is, $N = 3$, we have

$$E_3 = \xi^2 + 5 - 2\mathrm{i}\xi(z_2 + z_3), \tag{33}$$

$$\psi_3(z) = z^{-1/2}\exp\left[\frac{\mathrm{i}\xi}{4}\left(z-\frac{1}{z}\right)\right]\left(z^2 - (z_2+z_3)z + z_2z_3\right), \tag{34}$$

where the roots z_2 and z_3 are obtainable from the Bethe ansatz equation (27) as follows:

$$\begin{pmatrix} z_2\\ z_3 \end{pmatrix} = \left\{ \begin{pmatrix} \pm\mathrm{i}\\ \mp\mathrm{i} \end{pmatrix}, \begin{pmatrix} \dfrac{\sqrt{\dfrac{-8\mathrm{i}\xi^2 - 2\mathrm{i} - 2\sqrt{-4\xi^2-1}}{\sqrt{-4\xi^2-1}}}\left(4\mathrm{i}\xi^2 + \mathrm{i} - \sqrt{-4\xi^2-1}\right)\left(\mathrm{i} + \sqrt{-4\xi^2-1} \pm \sqrt{\dfrac{-8\mathrm{i}\xi^2 - 2\mathrm{i} - 2\sqrt{-4\xi^2-1}}{\sqrt{-4\xi^2-1}}}\right)}{8\xi\left(\left(\mathrm{i}\xi^2 + 1/4\mathrm{i} + 3/4\sqrt{-4\xi^2-1}\right)\sqrt{\dfrac{-8\mathrm{i}\xi^2 - 2\mathrm{i} - 2\sqrt{-4\xi^2-1}}{\sqrt{-4\xi^2-1}}} \pm (\mathrm{i}\sqrt{-4\xi^2-1} - 4\xi^2 - 1)\right)}\\ \dfrac{1}{2\xi}\left(\mathrm{i} + \sqrt{-4\xi^2-1} \pm \sqrt{\dfrac{-8\mathrm{i}\xi^2 - 2\sqrt{-4\xi^2-1} - 2\mathrm{i}}{\sqrt{-4\xi^2-1}}}\right) \end{pmatrix}, \begin{pmatrix} \dfrac{\sqrt{\dfrac{8\mathrm{i}\xi^2 - 2\sqrt{-4\xi^2-1} + 2\mathrm{i}}{\sqrt{-4\xi^2-1}}}\left(4\mathrm{i}\xi^2 + \mathrm{i} + \sqrt{-4\xi^2-1}\right)\left(\mathrm{i} - \sqrt{-4\xi^2-1} \pm \sqrt{\dfrac{8\mathrm{i}\xi^2 - 2\sqrt{-4\xi^2-1} + 2\mathrm{i}}{\sqrt{-4\xi^2-1}}}\right)}{8\xi\left(\left(\mathrm{i}\xi^2 + 1/4\mathrm{i} - 3/4\sqrt{-4\xi^2-1}\right)\sqrt{\dfrac{8\mathrm{i}\xi^2 - 2\sqrt{-4\xi^2-1} + 2i}{\sqrt{-4\xi^2-1}}} \mp (\mathrm{i}\sqrt{-4\xi^2-1} + 4\xi^2 + 1)\right)}\\ \dfrac{1}{2\xi}\left(\mathrm{i} - \sqrt{-4\xi^2-1} \pm \sqrt{\dfrac{8\mathrm{i}\xi^2 - 2\sqrt{-4\xi^2-1} + 2i}{\sqrt{-4\xi^2-1}}}\right) \end{pmatrix} \right\}. \tag{35}$$

The numerical results of the first three states are displayed and compared in Table 2. It is seen that the results accord with those obtained by Bagchi *et al.*[35]

2.3. ES *PT*-symmetric Pöschl–Teller potential

In the two previous subsections, we have considered two QES models. Here, we consider an ES model, the Pöschl–Teller potential in its *PT*-symmetric form[62]

$$V(x) = \left(\frac{\alpha^2+\beta^2}{2}-\frac{1}{4}\right)\operatorname{csch}^2(x+\mathrm{i}\varepsilon) + \left(\frac{\alpha^2-\beta^2}{2}\right)\coth(x+\mathrm{i}\varepsilon)\operatorname{csch}(x+\mathrm{i}\varepsilon), \quad (36)$$

with the corresponding Schrödinger equation given by

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\left(\frac{\alpha^2+\beta^2}{2}-\frac{1}{4}\right)\operatorname{csch}^2(x+\mathrm{i}\varepsilon) + \left(\frac{\alpha^2-\beta^2}{2}\right)\coth(x+\mathrm{i}\varepsilon)\operatorname{csch}(x+\mathrm{i}\varepsilon)\right)\psi(x) = E\psi(x). \quad (37)$$

The parameters α, β, and ε are real-valued constants. Using the change of variable $z=\cosh(x+\mathrm{i}\varepsilon)$ and applying the following transformation:

$$\psi(z) = (1-\cosh(x+\mathrm{i}\varepsilon))^{(2\alpha+1)/4} \times (1+\cosh(x+\mathrm{i}\varepsilon))^{(2\beta+1)/4}\varphi(z), \quad (38)$$

we obtain

$$H\varphi(z) = 0,$$
$$H = 4(z^2-1)\frac{\mathrm{d}^2\varphi(z)}{\mathrm{d}z^2}+4((\alpha+\beta+2)z+\alpha-\beta)\frac{\mathrm{d}\varphi(z)}{\mathrm{d}z} + \left(4E+(\alpha+\beta+1)^2\right). \quad (39)$$

By assuming

$$\varphi_n(z) = \begin{cases} \prod_{k=1}^{n}(z-z_k), & n\neq 0 \\ 1, & n=0, \end{cases} \quad (40)$$

and defining the operators A_n and $A_n^\dagger$ as

$$A_n = \frac{\mathrm{d}}{\mathrm{d}z}-\sum_{k=1}^{n}\frac{1}{z-z_k},$$
$$A_n^\dagger = -\frac{\mathrm{d}}{\mathrm{d}z}-\left(\frac{(\alpha+\beta+2)z+\alpha-\beta}{z^2-1}\right)-\sum_{k=1}^{n}\frac{1}{z-z_k}, \quad (41)$$

the operator H can be expressed as

$$A_n^\dagger A_n = -\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\left(\frac{(\alpha+\beta+2)z+\alpha-\beta}{z^2-1}\right)\frac{\mathrm{d}}{\mathrm{d}z} + \left(\frac{(\alpha+\beta+2)z+\alpha-\beta}{z^2-1}+\sum_{j\neq k}^{n}\frac{2}{z_j-z_k}\right) \times \sum_{k=1}^{n}\frac{1}{z-z_k}. \quad (42)$$

Comparing the residues at the simple poles $z=\pm 1$ and $z=z_k$ in Eqs. (39) and (42), the closed form of the energy and the Bethe ansatz equations are obtained as

$$E_n = -n(\alpha+\beta+n+1)-\frac{1}{4}(\alpha+\beta+1)^2, \quad (43)$$
$$\sum_{j\neq k}^{n}\frac{2}{z_j-z_k}+\frac{(\alpha+\beta+2)z_k+\alpha-\beta}{z_k^2-1}=0, \quad (44)$$

respectively. In the following, we obtain explicit solutions for the first three states. For $n=0$, from Eqs. (43) and (38), we have

$$E_0 = -\frac{(\alpha+\beta+1)^2}{4}, \quad (45)$$
$$\psi_0(z) = (1-z)^{(2\alpha+1)/4}(1+z)^{(2\beta+1)/4}. \quad (46)$$

For the first excited state, $n=1$, we have

$$E_1 = -(\alpha+\beta+2)-\frac{(\alpha+\beta+1)^2}{4}, \quad (47)$$
$$\psi_1(z) = (1-z)^{(2\alpha+1)/4}(1+z)^{(2\beta+1)/4}(z-z_1), \quad (48)$$

where z_1 is obtainable from Bethe ansatz equation (44) as

$$z_1 = -\frac{\alpha-\beta}{\alpha+\beta+2}. \quad (49)$$

Similarly, for the second excited state, $n=2$, we have

$$E_2 = -2(\alpha+\beta+3)-\frac{(\alpha+\beta+1)^2}{4}, \quad (50)$$
$$\psi_2(z) = (1-z)^{(2\alpha+1)/4}(1+z)^{(2\beta+1)/4} \times \left(z^2-(z_1+z_2)z+z_1z_2\right), \quad (51)$$

where the roots z_1 and z_2 are obtainable from Eq. (44) as

$$(z_1,z_2) = \begin{cases} \left(\dfrac{(\beta-\alpha)\sqrt{(\alpha+2)(\beta+2)(\alpha+\beta+3)}+\beta^2+\alpha^2+3\beta+3\alpha+4}{(\beta+\alpha+2)\sqrt{(\alpha+2)(\beta+2)(\alpha+\beta+3)}+\beta^2-\alpha^2+3\beta-3\alpha}, \dfrac{\beta^2-\alpha^2+3\beta-3\alpha-2\sqrt{(\alpha+2)(\beta+2)(\alpha+\beta+3)}}{(\beta+\alpha+4)(\alpha+\beta+3)}\right) \\ \left(\dfrac{(-\beta+\alpha)\sqrt{(\alpha+2)(\beta+2)(\beta+\alpha+3)}+\beta^2+\alpha^2+3\beta+3\alpha+4}{(-\beta-\alpha-2)\sqrt{(\alpha+2)(\beta+2)(\beta+\alpha+3)}+\beta^2-\alpha^2+3\beta-3\alpha}, \dfrac{\beta^2-\alpha^2+3\beta-3\alpha+2\sqrt{(\alpha+2)(\beta+2)(\beta+\alpha+3)}}{(\beta+\alpha+4)(\beta+\alpha+3)}\right). \end{cases} \quad (52)$$

Our numerical results obtained for the ground, first, and second excited states are presented and compared in Table 3. It is seen that our results accord with those obtained in Ref. [25]. In the next section, we try to reproduce these results through the $sl(2)$ algebraization.

3. Lie algebraic approach for the non-Hermitian *PT*-symmetric models

In the previous section, we have obtained the solutions of the three models in terms of the roots of the Bethe ansatz equations. Here, we illustrate how the relation with the Lie algebra $sl(2)$ underlies the exact and quasi-exact solvability of the models. For each model, we show that the underlying differential equation can be represented as an element of the universal enveloping algebra of $sl(2)$. To this aim, we first recall the Lie algebraic approach to quasi-exact solvability of the second order differential equations introduced in Ref. [49]. A differential equation is said to be Lie algebraic if it is an element of the universal enveloping algebra of a finite-dimensional Lie algebra. In particular, the most general one-dimensional second order differential equation H can be expressed as a quadratic combination[49]

$$H=\sum_{a,b=0,\pm} C_{ab}T^aT^b+\sum_{a=0,\pm} C_aT^a+C, \tag{53}$$

with the following generators

$$T_n^+=-z^2\frac{\mathrm{d}}{\mathrm{d}z}+nz,\quad T_n^0=z\frac{\mathrm{d}}{\mathrm{d}z}-\frac{n}{2},\quad T_n^-=\frac{\mathrm{d}}{\mathrm{d}z}, \tag{54}$$

which obey the $sl(2)$ commutation relations

$$[T_n^+,T_n^-]=2T_n^0,\quad [T_n^0,T_n^\pm]=\pm T_n^\pm, \tag{55}$$

and leave invariant the finite-dimensional space

$$P_{n+1}=\left\langle 1,z,z^2,\ldots,z^n\right\rangle. \tag{56}$$

Introducing the grading of generators (54) as

$$\mathrm{Deg}(T_n^+)=+1\,,\ \mathrm{Deg}(T_n^0)=0\,,\ \mathrm{Deg}(T_n^-)=-1\ , \tag{57}$$

it is seen that the generators (54) map a monomial z^n into a monomial z^{n+D} with $D=-1,0,+1$. According to Ref. [49], a QES operator always possesses terms of positive degree, while an ES operator has no terms of positive degree. Therefore, the most general forms of the QES and ES operators are given as

$$\begin{aligned}H_{\mathrm{QES}}&=C_{++}T^+T^++C_{+0}T^+T^0+C_{+-}T^+T^-+C_{0-}T^0T^-\\&\quad+C_{--}T^-T^-+C_+T^++C_0T^0+C_-T^-+C,\end{aligned} \tag{58}$$

$$\begin{aligned}H_{\mathrm{ES}}&=C_{+-}T^+T^-+C_{0-}T^0T^-+C_{--}T^-T^-\\&\quad+C_0T^0+C_-T_n^-+C,\end{aligned} \tag{59}$$

respectively. In the following, we apply the above results to obtain the general exact solutions of the models via the $sl(2)$ algebraization.

3.1. QES *PT*-symmetric quartic potential

Comparing Eq. (4) with Eq. (58), it is easy to verify that Eq. (4) can be expressed as an element of the universal enveloping algebra of $sl(2)$ as

$$\begin{aligned}&H\varphi(z)=0,\\&H=-T^-T^--2\mathrm{i}T^++2\mathrm{i}bT^-+2aT^0+\left(b^2+aJ-E\right),\end{aligned} \tag{60}$$

where $J=n+1=1,2,3,\ldots$. Accordingly, the operator H preserves the finite-dimensional invariant subspace spanned by $\left\langle 1,z,z^2,\ldots,z^{J-1}\right\rangle$, which allows us to determine the J states algebraically. Here, we calculate the exact solutions for the first three states which can be generalized to arbitrary J. For $J=1$, from Eq. (60), we have

$$\left(a+b^2-E_1\right)a_0=0, \tag{61}$$

and so the energy is obtained as

$$E_1=a+b^2. \tag{62}$$

For $J=2$, from Eq. (60), we obtain

$$\begin{pmatrix}3a+b^2-E_2 & -2\mathrm{i}\\ 2\mathrm{i}b & a+b^2-E_2\end{pmatrix}\begin{pmatrix}a_1\\a_0\end{pmatrix}=0, \tag{63}$$

which yields the energy as

$$E_2=2a+b^2\pm\sqrt{a^2+4b}. \tag{64}$$

For $J=3$, we obtain the following matrix equation

$$\begin{pmatrix}5a+b^2-E_3 & -2\mathrm{i} & 0\\ 4\mathrm{i}b & 3a+b^2-E_3 & -4\mathrm{i}\\ -2 & 2\mathrm{i}b & a+b^2-E_3\end{pmatrix}\begin{pmatrix}a_2\\a_1\\a_0\end{pmatrix}=0. \tag{65}$$

Accordingly, we arrive at the following general equation for arbitrary J:

$$\begin{pmatrix}(2J-1)a+b^2-E & -2\mathrm{i} & 0 & 0 & \cdots & 0\\ 2(J-1)\mathrm{i}b & \ddots & \ddots & \ddots & \ddots & \vdots\\ -(J-1)(J-2) & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -2(J-2)\mathrm{i} & 0\\ 0 & \cdots & -6 & 4\mathrm{i}b & 3a+b^2-E & -2(J-1)\mathrm{i}\\ 0 & \cdots & 0 & -2 & 2\mathrm{i}b & a+b^2-E\end{pmatrix}\begin{pmatrix}a_{J-1}\\a_{J-2}\\a_{J-3}\\\vdots\\a_1\\a_0\end{pmatrix}=0, \tag{66}$$

whose non-trivial solution condition (Cramer's rule) gives the exact solutions of the system. Also, from Eq. (3), the wave function is given by

$$\psi_J(z) = \exp\left(-\mathrm{i}\frac{z^3}{3} - a\frac{z^2}{2} - \mathrm{i}bz\right)\sum_{m=0}^{J-1} a_m z^m, \tag{67}$$

where the expansion coefficient a_m's satisfy the following four-term recurrence relation

$$\begin{aligned}&(-m(m-1))\,a_m + (2\mathrm{i}(m-1)b)\,a_{m-1}\\ &+ \left((2m-3)a + b^2 - E\right)a_{m-2}\\ &+ (-2(m-1)\mathrm{i})\,a_{m-3} = 0,\end{aligned} \tag{68}$$

with boundary conditions $a_{-1} = 0$, $a_{-2} = 0$, and $a_J = 0$. As can be seen, the results are identical with the results of BAM in Eq. (11). The numerical results are presented and compared in Table 1.

Table 1. Solutions of the first three states for the PT-symmetric quartic potential and comparison among them, with $a = 1$ and $b = 2$.

J	Energy (BAM) Eq. (11)	Energy (QES) Eq. (66)	Energy (Bender–Dunne) Ref. [5]	Wave function (BAM)
1	5	5	5	$\psi_1(z) = \exp\left(-\mathrm{i}\frac{z^3}{3} - \frac{z^2}{2} - 2\mathrm{i}z\right)$
2	3 9	3 9	3 9	$\psi_2(z) = \begin{cases}\exp\left(-\mathrm{i}\frac{z^3}{3} - \frac{z^2}{2} - 2\mathrm{i}z\right)(z-2\mathrm{i})\\ \exp\left(-\mathrm{i}\frac{z^3}{3} - \frac{z^2}{2} - 2\mathrm{i}z\right)(z+\mathrm{i})\end{cases}$
3	1.235958910 6.553075856 13.21096523	1.235958910 6.553075858 13.21096524	1.235958910 6.553075858 13.21096524	$\psi_3(z) = \begin{cases}\exp\left(-\mathrm{i}\frac{z^3}{3} - \frac{z^2}{2} - 2\mathrm{i}z\right)(z^2 - 3.8820\mathrm{i}z - 3.5940)\\ \exp\left(-\mathrm{i}\frac{z^3}{3} - \frac{z^2}{2} - 2\mathrm{i}z\right)(z^2 - 1.2235\mathrm{i}z + 1.8633)\\ \exp\left(-\mathrm{i}\frac{z^3}{3} - \frac{z^2}{2} - 2\mathrm{i}z\right)(z^2 + 2.1055\mathrm{i}z - 1.2693)\end{cases}$

3.2. QES *PT*-symmetric Khare–Mandal potential

Comparing Eq. (23) with Eq. (58), it is seen that the operator H can be expressed in the Lie-algebraic form

$$\begin{aligned}H\varphi(z) &= 0,\\ H &= -T^+T^- - \frac{\mathrm{i}\xi}{2}T^+ + \frac{\mathrm{i}\xi}{2}T^- + T^0\end{aligned}$$

$$+\left(\frac{(N-1)}{2} + \frac{1-\xi^2-2N+E}{4}\right), \tag{69}$$

where $N = n+1 = 1,2,3,\ldots$. This algebraization enables us to use the representation theory of $sl(2)$. As a result, we obtain the following matrix equation:

$$\begin{pmatrix} \chi & -\frac{\mathrm{i}\xi}{2} & 0 & 0 & 0\\ \frac{\mathrm{i}(N-1)\xi}{2} & \chi - N + 2 & -\mathrm{i}\xi & 0 & 0\\ 0 & & & \ddots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \mathrm{i}\xi & \chi - N + 2 & \frac{-\mathrm{i}(N-1)\xi}{2}\\ 0 & 0 & 0 & \frac{\mathrm{i}\xi}{2} & \chi \end{pmatrix}\begin{pmatrix} a_{N-1}\\ a_{N-2}\\ \vdots\\ \vdots\\ a_1\\ a_0 \end{pmatrix} = 0, \tag{70}$$

where

$$\chi = \frac{E_N - \xi^2 - 2N + 1}{4}. \tag{71}$$

Following the derivation of the previous section, one can easily check that the non-trivial solutions of Eq. (70) are the same as those obtained by BAM in Eq. (28). Also, from Eq. (22), the wave function of the system is given by

$$\psi_N(z) = z^{(1-N)/2}\exp\left(\frac{\mathrm{i}\xi}{4}\left(z - \frac{1}{z}\right)\right)\sum_{m=0}^{N-1} a_m z^m, \tag{72}$$

where the expansion coefficients a_m's satisfy the following recursion relation

$$\begin{aligned}&\left(\mathrm{i}\frac{m\xi}{2}\right)a_m + \left(\frac{E - \xi^2 - 6m + 3}{4}\right)a_{m-1}\\ &- (\mathrm{i}\xi)\,a_{m-2} = 0,\end{aligned} \tag{73}$$

with initial conditions $a_N = 0$ and $a_{-1} = 0$. The numerical results are reported and compared in Table 2.

Table 2. Solutions of the first three states for the PT-symmetric Khare–Mandal potential and comparison among them, with $\xi = 2$.

N	Energy (BAM) Eq. (28)	Energy (QES) Eq. (70)	Energy (AIM) Ref. [24]	Wave function (BAM)
1	5	5	5	$\psi_1(z) = z^{-1/2}\exp\left(\frac{\mathrm{i}}{2}\left(z-\frac{1}{z}\right)\right)$
2	3 11	3 11	3 11	$\psi_2(z) = \begin{cases} z^{-1/2}\exp\left(\frac{\mathrm{i}}{2}\left(z-\frac{1}{z}\right)\right)(z+\mathrm{i}) \\ z^{-1/2}\exp\left(\frac{\mathrm{i}}{2}\left(z-\frac{1}{z}\right)\right)(z-\mathrm{i}) \end{cases}$
3	2.753788748 9 19.24621125	2.753788748 9 19.24621125	2.753788748 9 19.24621125	$\psi_3(z) = \begin{cases} z^{-1}\exp\left(\frac{\mathrm{i}}{2}\left(z-\frac{1}{z}\right)\right)\left(z^2+(1.5616i)z-1\right) \\ z^{-1}\exp\left(\frac{\mathrm{i}}{2}\left(z-\frac{1}{z}\right)\right)\left(z^2+1\right) \\ z^{-1}\exp\left(\frac{\mathrm{i}}{2}\left(z-\frac{1}{z}\right)\right)\left(z^2-(2.5616i)z-1\right) \end{cases}$

3.3. ES PT-symmetric Pöschl–Teller potential

In this case, comparing Eq. (39) with Eq. (58), the Lie algebraic form of the Hamiltonian is obtained as

$$\begin{aligned} H\varphi(z) &= 0, \\ H &= 4T^+T^- - 4T^-T^- + 4(\alpha-\beta)T^- \\ &\quad + 4(\alpha+\beta+n+2)T^0 + \sigma, \end{aligned} \tag{74}$$

where

$$\sigma = 4E + 2n(\alpha+\beta+n+2) + (\alpha+\beta+1)^2. \tag{75}$$

As can be seen, operator (74) has no terms of positive degree and so it is an ES operator (Turbiner's theorem[49]). In this case, the general matrix equation has the following form:

$$\begin{pmatrix} \sigma+2n(\alpha+\beta+n) & 0 & 0 & 0 & 0 \\ 4n(\alpha-\beta) & \sigma+(2n-4)(\alpha+\beta+n)-4n & 0 & 0 & 0 \\ -4n(n-1) & 4(n-1)(\alpha-\beta) & \ddots & 0 & 0 \\ 0 & -4(n-1)(n-2) & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ \vdots \\ a_0 \end{pmatrix} = 0. \tag{76}$$

The energy eigenvalues can be easily calculated from the diagonal elements of the coefficient matrix, which exactly coincide with those given in Eq. (43). The numerical results are reported and compared in Table 3. Also, from Eq. (38), the wave function is given as

$$\begin{aligned} \psi_n(z) &= (1-z)^{(2\alpha+1)/4}(1+z)^{(2\beta+1)/4} \\ &\quad \times \sum_{m=0}^{n} a_m z^m, \end{aligned} \tag{77}$$

where the expansion coefficients a_m's satisfy the recursion relation

$$\begin{aligned} &(-4m(m+1))\,a_{m+1} + (4m(\alpha-\beta))\,a_m \\ &+ (\sigma + (2m-6)(\alpha+\beta+m+1) - 8m)\,a_{m-1} = 0, \end{aligned} \tag{78}$$

with boundary conditions $a_{n+1} = 0$ and $a_{-1} = 0$.

Table 3. Solutions of the first three states for the PT-symmetric Pöschl–Teller potential and comparison among them, with $\alpha = 2$ and $\beta = 1$.

n	Energy (BAM) Eq. (43)	Energy (QES) Eq. (76)	Energy (AIM) Ref. [25]	Wave function (BAM)
0	−4	−4	−4	$\psi_0(z) = (1-z)^{5/4}(1+z)^{3/4}$
1	−9	−9	−9	$\psi_1(z) = (1-z)^{5/4}(1+z)^{3/4}(z+0.2)$
2	−16	−16	−16	$\psi_2(z) = \begin{cases} (1-z)^{5/4}(1+z)^{3/4}(z^2+1.0938z+0.2991) \\ (1-z)^{5/4}(1+z)^{3/4}(z^2-0.5224z+0.0682) \end{cases}$

4. Conclusions

In this paper, using the Bethe ansatz method, we solve the Schrödinger equation for a class of PT-symmetric potentials and present exact expressions for the energies and the corresponding wave functions in terms of the roots of the Bethe ansatz equations. In addition, we show that the models possess the $sl(2)$ hidden algebraic structure. Then, we obtain the general solutions of the systems by using the representation theory of $sl(2)$ Lie algebra. As expected, the real energy spectra are obtained for each model, for each of parameter values. Also, we show that our results are consistent with those obtained by other methods.

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