Fully nonlinear (2+1)-dimensional displacement shallow water wave equation*

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Recently, a new (2+1)-dimensional displacement shallow water wave equation (2DDSWWE) was constructed by applying the variational principle of analytic mechanics in the Lagrange coordinates. However, the simplification of the nonlinear term related to the incompressibility of the shallow water in the 2DDSWWE is a disadvantage of this approach. Applying the theory of nonlinear continuum mechanics, we add some new nonlinear terms to the 2DDSWWE and construct a new fully nonlinear (2+1)-dimensional displacement shallow water wave equation (FN2DDSWWE). The presented FN2DDSWWE contains all nonlinear terms related to the incompressibility of shallow water. The exact travelling-wave solution of the proposed FN2DDSWWE is also obtained, and the solitary-wave solution can be deduced from the presented travelling-wave solution under a special selection of integral constants.

Keywords: shallow water system, Hamilton variational principle, displacement, solitary wave

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1. Introduction

Shallow water waves near the ocean shore are of great interest in ocean engineering. The general characteristic of such waves is that the scale of vertical motion is much smaller than that of horizontal motion, allowing a considerable simplification in the governing equation and numerical solution.[1–4] The first equation for the shallow water wave was proposed by Saint-Venant in 1871. The Saint-Venant equation uses two assumptions: (i) the pressure distribution is hydrostatic, and (ii) the horizontal velocity is independent of the vertical coordinate.[2] While Saint-Venant only derived the one-dimensional equation, the basic principle and assumptions of his equation were extended to the two-dimensional shallow water wave problems by many scholars, for example, Lamb[5] and Hendershott.[6] Both the one-dimensional and two-dimensional equations are commonly referred to as the Saint-Venant equation. It is generally accepted that the Saint-Venant equation is applicable for the case where the ratio of the water depth to the wave length is less than approximately 0.05.[7]

Because of the hydrostatic pressure assumption, the solitary wave solution cannot be derived from the Saint-Venant equation. The solitary water wave was first observed by John Scott Russell in 1834. Since then it has attracted the attention of many researchers from different areas such as fluid mechanics, solid mechanics, quantum mechanics, electromagnetics, coastal engineering, and the study of sand dunes.[8–14] Nevertheless, the mathematical foundation for the solitary water wave was lacking for many years after Russell's observation. To find the solitary wave solution, Boussinesq took into account the effect of the vertical velocity on the pressure and derived the shallow water equation, the Boussinesq equation.[15] By simplifying the Boussinesq equation, he derived the famous Korteweg de Vries (KdV) equation that was rediscovered in 1877 by Diederik Korteweg and Gustav de Vries. They were the first to obtain the solitary wave solution in terms of the KdV equation.[16] It is generally accepted that the Boussines and KdV equations are valid for the case that the ratio of the water depth to the wave length is less than approximately 0.1. There have been many subsequent works devoted to the extensions of the Boussinesq equation in different directions, such as improved frequency dispersion, varying bathymetry, and the inclusion of wave breaking, among other applications.[17–23] All the extended Boussinesq equations are referred to as Boussinesq-type equations. Reviews of the Boussinesq-type equations can be found in Ref. [24].

Currently, the derivation of the Boussinesq-type equations is an important issue for the shallow water problems. One approach is the use of the perturbation approximation to the Navier–Stokes equation. Another approach is based on the Hamilton variational principle. The advantage of the second approach is that the equation obtained through the Hamilton variational principle can maintain the symplectic structure and conservation properties of the shallow water systems.[25,26] Luke[27] was the first to propose the least ac-

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tion principle for the water waves with an even bottom. Subsequently, Whitham[28] obtained the Lagrangian functions of the Boussinesq and KdV equations. Zakharov[29] firstly derived the Hamilton canonical equation for the water wave problem. Lu and Dai[30] dealt with the Hamiltonian formulation of the nonlinear water waves in a two-fluid system. Reviews of the Hamilton variational principle of water waves can be found in Refs. [25], [26], and [31].

There have been numerous works published on the shallow water wave problem. However, during the research, two disadvantages of the KdV solitary-wave solution have been observed by the researchers:[32,33] 1) the KdV soliton can move in only one direction (right or left) while the real solitary waves can move in both directions; 2) the velocity of the KdV soliton is slightly larger than that actually observed for the shallow water solitary waves. The Boussinesq equation can overcome the first disadvantage. However, the second disadvantage remains. In Ref. [34], Zhong and Yao applied the Lagrange coordinates and the variational principle of analytic mechanics for the shallow water waves. They derived a new (1+1)-dimensional displacement shallow water wave equation (1DDSWWE)

$$q_{tt}-\frac{h^2}{3}q_{xxtt}+gh\left(3q_x-1\right)q_{xx}=0, \tag{1}$$

where $q(x,t)$ denotes the horizontal displacement of the shallow water system, g is the gravity acceleration, and $z=-h$ denotes the water bottom.

Interestingly, the solitary wave solution of the 1DDSWWE can move in both directions and the velocity of the solitary wave matches exactly with the velocity measured by Russell, which is slightly smaller than that of the KdV solution.[33,35] Subsequently, the use of the Lagrange coordinates and the variation principle of analytic mechanics was extended by Liu and Lou[32,33,36] to the (2+1)-dimensional shallow water wave system. They derived a (2+1)-dimensional displacement shallow water wave equation (2DDSWWE)

$$\begin{cases} s_{tt}+gh\left(3s_x+w_y-1\right)s_{xx}+gh\left(s_x+w_y-1\right)w_{xy} \\ -\dfrac{h^2}{3}\left(s_x+w_y\right)_{xtt}=0, \\ w_{tt}+gh\left(3w_y+s_x-1\right)w_{yy}+gh\left(s_x+w_y-1\right)s_{xy} \\ -\dfrac{h^3}{3}\left(s_x+w_y\right)_{ytt}=0, \end{cases} \tag{2}$$

where s and w are the horizontal displacements of the (2+1)-dimensional shallow water system in x-axis and y-axis, respectively. Both s and w are independent of the vertical coordinate z. The incompressibility condition of water is given by

$$\zeta(x,y,t)=-h\frac{s_x+w_y+s_xw_y}{(1+s_x)(1+w_y)}=-h\frac{s_x+w_y+s_xw_y}{1+s_x+w_y+s_xw_y}, \tag{3}$$

where $z=\zeta(x,y,t)$ defines the water free surface. However the incompressibility condition of Eq. (3) is incomplete. In this paper, we derive the following fully nonlinear incompressible condition:

$$\zeta(x,y,t)=-h\frac{\alpha}{1+\alpha},\quad \alpha=s_x+w_y+s_xw_y-s_yw_x, \tag{4}$$

based on continuous mechanics. Unlike Eq. (3), the proposed incompressibility condition (4) contains the nonlinear term $-s_yw_x$. Based on Eq. (4), a fully nonlinear (2+1)-dimensional displacement shallow water wave equation (FN2DDSWWE) is constructed in Section 2. We give the travelling-wave solution of the proposed FN2DDSWWE in Section 3.

2. FN2DDSWWE

We assume that the water is an inviscid fluid of constant density. Water is also assumed to be incompressible and possess a free surface, along which the pressure is constant. However the surface-energy effects are negligible. Let (x,y,z) represent the location of a certain particle P in water at an initial time t_0, and $(X,\ Y,\ Z)$ represent the location of P at time t. The free surface evolution is given by $z=\zeta(x,y,t)\equiv\zeta$ and the bottom plane by $z=-h$. Let $s\equiv s(x,y,z,t)$, $w\equiv w(x,y,z,t)$, and $v\equiv v(x,y,z,t)$ represent water displacements in x, y, and z directions, respectively. Obviously, we have

$$\begin{aligned} X(x,y,z,t)&=x+s(x,y,z,t),\\ Y(x,y,z,t)&=y+w(x,y,z,t),\\ Z(x,y,z,t)&=z+v(x,y,z,t). \end{aligned} \tag{5}$$

Based on the theory of nonlinear continuum mechanics, the incompressibility condition of water is given by

$$A=\begin{vmatrix} \dfrac{\partial X}{\partial x} & \dfrac{\partial X}{\partial y} & \dfrac{\partial X}{\partial z} \\ \dfrac{\partial Y}{\partial x} & \dfrac{\partial Y}{\partial y} & \dfrac{\partial Y}{\partial z} \\ \dfrac{\partial Z}{\partial x} & \dfrac{\partial Z}{\partial y} & \dfrac{\partial Z}{\partial z} \end{vmatrix}=1. \tag{6}$$

Substituting Eq. (5) into Eq. (6) gives

$$\begin{vmatrix} 1+s_x & s_y & s_z \\ w_x & 1+w_y & w_z \\ v_x & v_y & 1+v_z \end{vmatrix}=1. \tag{7}$$

For the shallow water system, the horizontal displacements s and w are assumed to be independent of the vertical coordinate z; hence, equation (7) can be expressed as

$$v_z=\frac{-\alpha}{1+\alpha}, \tag{8}$$

where α is defined in Eq. (4). Integrating Eq. (8) with respect to z and noting that the vertical displacement $v(x,y,z,t)=0$ at the water bottom $z=-h$, we obtain

$$v(x,y,z,t)=-\frac{\alpha(z+h)}{1+\alpha}. \tag{9}$$

Obviously, the water free surface can be defined by the vertical displacement at the surface, i.e., $z=\zeta=v(x,y,0,t)$. Substitution of Eq. (9) into $z=\zeta=v(x,y,0,t)$ yields Eq. (4), which is the fully nonlinear incompressibility condition for the shallow water system.

The shallow water wave equation can be derived by using the Hamilton variational principle, i.e., $\delta S=0$, where δ represents the variational operator and

$$S=\int_0^t (T-U)\,\mathrm{d}\tau \tag{10}$$

represents the action of the shallow water system. T and U represent the kinetic and potential energies, respectively. The kinetic energy is given by

$$\begin{aligned} T &= \int_{-L_1}^{L_1}\int_{-L_2}^{L_2}\int_{-h}^{0}\frac{1}{2}\rho\left(s_t^2+w_t^2+v_t^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_{-L_1}^{L_1}\int_{-L_2}^{L_2}\frac{1}{2}\rho h\left(s_t^2+w_t^2\right)\mathrm{d}x\mathrm{d}y \\ &\quad+\int_{-L_1}^{L_1}\int_{-L_2}^{L_2}\int_{-h}^{0}\frac{1}{2}\rho v_t^2\,\mathrm{d}x\mathrm{d}y\mathrm{d}z, \end{aligned} \tag{11}$$

where ρ is the mass density, and the water is assumed to be contained in a rectangular box of $-L_1<x<L_1$ and $-L_2<y<L_2$. For the travelling wave solution, L_1 and L_2 may be considered to be infinite. In terms of Eq. (9), the vertical velocity is given by

$$v_t=-\frac{(z+h)\,\alpha_t}{(1+\alpha)^2}=-(z+h)\,\alpha_t, \tag{12}$$

where the approximation of $1/(1+\alpha)^2\approx 1$ is used. Substitution of Eq. (12) into Eq. (11) yields

$$T=\frac{1}{2}\int_{-L_1}^{L_1}\int_{-L_2}^{L_2}\rho\left(hs_t^2+hw_t^2+\frac{h^3}{3}\alpha_t^2\right)\mathrm{d}x\mathrm{d}y. \tag{13}$$

The potential energy is

$$\begin{aligned} U &= \int_{-L_1}^{L_1}\int_{-L_2}^{L_2}\int_{-h}^{0}\rho g\,(z+v)\,\mathrm{d}z\mathrm{d}x\mathrm{d}y \\ &= C-\int_{-L_1}^{L_1}\int_{-L_2}^{L_2}\frac{\rho g h^2}{2}\frac{\alpha}{1+\alpha}\,\mathrm{d}x\mathrm{d}y \\ &= C-\int_{-L_1}^{L_1}\int_{-L_2}^{L_2}\frac{\rho g h^2}{2}\alpha\left(1-\alpha+\alpha^2+\cdots\right)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{14}$$

where $C=\int_{-L_1}^{L_1}\int_{-L_2}^{L_2}\int_{-h}^{0}\rho g z\,\mathrm{d}z\mathrm{d}x\mathrm{d}y$ is constant. Substituting Eqs. (13) and (14) into Eq. (10) and taking the first variation of the action, we obtain

$$\begin{aligned} \delta S &= -\int_0^t\int_\Omega \rho h\left(\delta s s_{tt}+\delta w w_{tt}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \\ &\quad+\int_0^t\int_\Omega \rho\beta\delta\alpha\,\mathrm{d}x\mathrm{d}y\mathrm{d}\tau, \end{aligned} \tag{15}$$

where $\Omega=[-L_1,\,L_1]\times[-L_2,\,L_2]$, and

$$\beta=\frac{gh^2}{2}\left(1-2\alpha+3\alpha^2+\cdots\right)-\frac{h^3}{3}\alpha_{tt}. \tag{16}$$

Because the boundary conditions in the x and y directions are

$$\begin{cases} s(x,y,t)=0, & \text{as } |x|=L_1, \text{ or } |y|=L_2, \\ w(x,y,t)=0, & \text{as } |x|=L_1, \text{ or } |y|=L_2, \end{cases} \tag{17}$$

equation (15) can be rewritten as

$$\begin{aligned} \delta S &= -\int_0^t\int_\Omega \rho\delta s\left[hs_{tt}+\beta_x(1+w_y)-\beta_y w_x\right]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau \\ &\quad-\int_0^t\int_\Omega \rho\delta w\left[hw_{tt}+\beta_y(1+s_x)-\beta_x s_y\right]\mathrm{d}x\mathrm{d}y\mathrm{d}\tau. \end{aligned} \tag{18}$$

According to the Hamilton variational principle, $\delta S=0$. Hence, we obtain

$$\begin{cases} hs_{tt}=\beta_x(1+w_y)-\beta_y w_x=0, \\ hw_{tt}=\beta_y(1+s_y)-\beta_x w_y=0. \end{cases} \tag{19}$$

Making use of the approximation $\beta=\left(1-2\alpha+3\alpha^2\right)gh^2/2-\alpha_{tt}h^3/3$, we can rewrite Eq. (19) as

$$\begin{cases} s_{tt}+\dfrac{h^2}{3}\left[\alpha_{tty}w_x-\alpha_{ttx}(1+w_y)\right] \\ +gh(3\alpha-1)\left[\alpha_x(1+w_y)-\alpha_y w_x\right]=0, \\ w_{tt}+\dfrac{h^2}{3}\left[\alpha_{ttx}s_y-\alpha_{tty}(1+s_x)\right] \\ +gh(3\alpha-1)\left[\alpha_y(1+s_x)-s_y\alpha_x\right]=0, \end{cases} \tag{20}$$

which is called the fully nonlinear (2+1)-dimensional displacement shallow water wave equation (FN2DDSWWE). When $w=0$ and s is independent of y, the FN2DDSWWE will be reduced back to the (1+1)-dimensional displacement shallow water wave equation, i.e., Eq. (1). Making use of the lower-order approximation, we can write Eq. (20) as

$$\begin{cases} s_{tt}+gh(3s_x+w_y-1)s_{xx}+gh(s_x+w_y-1)w_{xy} \\ -\dfrac{h^2}{3}(s_x+w_y)_{xtt}+f_1=0, \\ w_{tt}+gh(3w_y+s_x-1)w_{yy}+gh(s_x+w_y-1)s_{xy} \\ -\dfrac{h^3}{3}(s_x+w_y)_{ytt}+f_2=0, \end{cases} \tag{21}$$

where

$$\begin{aligned} f_1 &= gh\left[(s_y w_x)_x+(s_x w_x+w_y w_x)_y\right], \\ f_2 &= gh\left[(s_y w_y+s_x s_y)_x+(s_y w_x)_y\right]. \end{aligned} \tag{22}$$

Obviously, equation (22) will be reduced back to the 2DDSWWE (2) when the nonlinear terms f_1 and f_2 are neglected. These two nonlinear terms reflect the effect of the nonlinear term $-s_y w_x$ in the proposed fully nonlinear incompressibility condition (4).

3. Travelling-wave solutions to FN2DDSWWE

In this section, exact travelling-wave solutions are constructed for the proposed FN2DDSWWE. To find the exact travelling-wave solution, we suppose that

$$s = e(\xi), \quad w = f(\xi), \quad \xi = k_1 x + k_2 y + t, \tag{23}$$

where k_1 and k_2 are independent of x, y, and t. In terms of Eq. (23), we have

$$\begin{aligned}
&s_{tt} = \frac{\mathrm{d}^2 e}{\mathrm{d}\xi^2}, \quad s_x = \frac{\mathrm{d}e}{\mathrm{d}\xi}k_1, \quad s_y = \frac{\mathrm{d}e}{\mathrm{d}\xi}k_2,\\
&s_{xx} = \frac{\mathrm{d}^2 e}{\mathrm{d}\xi^2}k_1^2, \quad s_{xy} = \frac{\mathrm{d}^2 e}{\mathrm{d}\xi^2}k_1 k_2, \quad s_{yy} = \frac{\mathrm{d}^2 e}{\mathrm{d}\xi^2}k_2^2,\\
&w_{tt} = \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}, \quad w_x = \frac{\mathrm{d}f}{\mathrm{d}\xi}k_1, \quad w_y = \frac{\mathrm{d}f}{\mathrm{d}\xi}k_2,\\
&w_{xx} = \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}k_1^2, \quad w_{xy} = \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}k_1 k_2, \quad w_{yy} = \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}k_2^2, \qquad (24)\\
&\alpha = s_x + w_y + s_x w_y - s_y w_x = \frac{\mathrm{d}e}{\mathrm{d}\xi}k_1 + \frac{\mathrm{d}f}{\mathrm{d}\xi}k_2. \qquad (25)
\end{aligned}$$

The substitution of Eqs. (24) and (25) into Eq. (20) yields

$$\begin{cases}
\dfrac{\mathrm{d}^2 e}{\mathrm{d}\xi^2} - \dfrac{h^2}{3}k_1\dfrac{\mathrm{d}^3\alpha}{\mathrm{d}\xi^3} + ghk_1\left[1.5\dfrac{\mathrm{d}}{\mathrm{d}\xi}\alpha^2 - \dfrac{\mathrm{d}\alpha}{\mathrm{d}\xi}\right] = 0,\\
\dfrac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} - \dfrac{h^2}{3}k_2\dfrac{\mathrm{d}^3\alpha}{\mathrm{d}\xi^3} + ghk_2\left[1.5\dfrac{\mathrm{d}}{\mathrm{d}\xi}\alpha^2 - \dfrac{\mathrm{d}\alpha}{\mathrm{d}\xi}\right] = 0.
\end{cases} \tag{26}$$

Integrating Eq. (26) once over ξ, we obtain

$$\begin{cases}
\dfrac{\mathrm{d}e}{\mathrm{d}\xi} - \dfrac{h^2}{3}k_1\dfrac{\mathrm{d}^2\alpha}{\mathrm{d}\xi^2} + ghk_1\left(1.5\alpha^2 - \alpha\right) = C_1,\\
\dfrac{\mathrm{d}f}{\mathrm{d}\xi} - \dfrac{h^2}{3}k_2\dfrac{\mathrm{d}^2\alpha}{\mathrm{d}\xi^2} + ghk_2\left(1.5\alpha^2 - \alpha\right) = C_2,
\end{cases} \tag{27}$$

where C_1 and C_2 are two integral constants. Letting the two equations in Eq. (27) be divided by k_1 and k_2, respectively, followed by the subtraction of the first equation from the second one gives

$$\frac{1}{k_1}\frac{\mathrm{d}e}{\mathrm{d}\xi} - \frac{1}{k_1}C_1 = \frac{1}{k_2}\frac{\mathrm{d}f}{\mathrm{d}\xi} - \frac{1}{k_2}C_2. \tag{28}$$

Combining Eq. (25) with Eq. (28) yields

$$\begin{cases}
\dfrac{\mathrm{d}e}{\mathrm{d}\xi} = \dfrac{k_1}{k_1^2+k_2^2}\alpha + \dfrac{k_2^2 C_1}{k_1^2+k_2^2} - \dfrac{k_1 k_2}{k_1^2+k_2^2}C_2,\\
\dfrac{\mathrm{d}f}{\mathrm{d}\xi} = \dfrac{k_2}{k_1^2+k_2^2}\alpha - \dfrac{k_1 k_2}{k_1^2+k_2^2}C_1 + \dfrac{k_1^2}{k_1^2+k_2^2}C_2.
\end{cases} \tag{29}$$

Multiplication of the two equations in (27) by $k_1\mathrm{d}\alpha/\mathrm{d}\xi$ and $k_2\mathrm{d}\alpha/\mathrm{d}\xi$, respectively, followed by the addition of these two equations gives

$$\begin{aligned}
&\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}\alpha - \frac{h^2\left(k_2^2+k_1^2\right)}{3}\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}\frac{\mathrm{d}^2\alpha}{\mathrm{d}\xi^2}\\
&\quad + gh\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}\left(1.5\alpha^2 - \alpha\right)\left(k_2^2+k_1^2\right)\\
&= \frac{\mathrm{d}\alpha}{\mathrm{d}\xi}\left(k_2 C_2 + k_1 C_1\right). \qquad (30)
\end{aligned}$$

Integrating Eq. (30) once over ξ, we can obtain

$$\begin{aligned}
&\frac{1}{2}\alpha^2 - \frac{h^2\left(k_2^2+k_1^2\right)}{6}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}\right)^2 + \frac{gh\left(k_2^2+k_1^2\right)}{2}\left(\alpha^3 - \alpha^2\right)\\
&= \alpha\left(k_2 C_2 + k_1 C_1\right) + C_3, \qquad (31)
\end{aligned}$$

which can be reformulated as

$$\left(\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}\right)^2 = \frac{3g}{4h}\left(4\alpha^3 + 4a\alpha^2 + b\alpha + c\right), \tag{32}$$

where

$$\begin{aligned}
&a = \frac{1}{gh\left(k_2^2+k_1^2\right)} - 1, \quad b = -\frac{8\left(k_2 C_2 + k_1 C_1\right)}{gh\left(k_2^2+k_1^2\right)},\\
&c = -\frac{8C_3}{gh\left(k_2^2+k_1^2\right)}. \qquad (33)
\end{aligned}$$

The transformation is then made to

$$\alpha = \frac{4h}{3g}\gamma - \frac{a}{3}. \tag{34}$$

Equation (32) can be rewritten as

$$\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\xi}\right)^2 = 4\gamma^3 - g_2\gamma - g_3, \tag{35}$$

where

$$\begin{aligned}
&g_2 = \left(\frac{3g}{4h}\right)^2\left(\frac{4a^2}{3} - b\right),\\
&g_3 = -\left(\frac{3g}{4h}\right)^3\left(\frac{8a^3}{27} - \frac{ab}{3} + c\right). \qquad (36)
\end{aligned}$$

Equation (35) is the famous Weierstrass elliptic equation, the solution of which is the Weierstrass elliptic function $\gamma = \wp(\xi - \xi_0;\, g_2,\, g_3)$, where ξ_0 is an arbitrary integral constant. Hence the solution of Eq. (32) is

$$\alpha = \frac{4h}{3g}\wp(\xi - \xi_0;\, g_2,\, g_3) - \frac{a}{3}. \tag{37}$$

Equation (37) can also be expressed by the Jacobi elliptic cn function as

$$\alpha = \frac{4h}{3g}\left[e_2 - (e_2 - e_3)\,\mathrm{cn}^2\left(\sqrt{e_1 - e_3}\,(\xi - \xi_0);\, m\right)\right] - \frac{a}{3}, \tag{38}$$

where $m^2 = (e_2 - e_3)/(e_1 - e_3)$, and e_i $(i = 1, 2, 3,\ e_1 \geq e_2 \geq e_3)$ are the three roots of $4\gamma^3 - g_2\gamma - g_3 = 0$. Substitution of Eq. (38) into Eq. (29) yields

$$\begin{cases}
\dfrac{\mathrm{d}e}{\mathrm{d}\xi} = -\dfrac{k_1 4h\left(e_2 - e_3\right)}{\left(k_1^2+k_2^2\right)3g}\mathrm{cn}^2\left(\sqrt{e_1 - e_3}\,(\xi - \xi_0);\, m\right)\\
\qquad\qquad + \dfrac{1}{k_1^2+k_2^2}\left(\dfrac{4hk_1 e_2}{3g} - \dfrac{ak_1}{3} + k_2^2 C_1 - k_1 k_2 C_2\right),\\
\dfrac{\mathrm{d}f}{\mathrm{d}\xi} = -\dfrac{k_2 4h\left(e_2 - e_3\right)}{\left(k_1^2+k_2^2\right)3g}\mathrm{cn}^2\left(\sqrt{e_1 - e_3}\,(\xi - \xi_0);\, m\right)\\
\qquad\qquad + \dfrac{1}{k_1^2+k_2^2}\left(\dfrac{4hk_2 e_2}{3g} - \dfrac{ak_2}{3} - k_1 k_2 C_1 + k_1^2 C_2\right).
\end{cases} \tag{39}$$

Integrating Eq. (39) once over ξ, we obtain

$$\begin{cases} s = e(\xi) = -\dfrac{k_1 4h(e_2-e_3)}{(k_1^2+k_2^2)3g}\lambda(\xi) \\ \quad + \dfrac{\xi}{k_1^2+k_2^2}\left(\dfrac{4hk_1e_2}{3g} - \dfrac{ak_1}{3} + k_2^2C_1 - k_1k_2C_2\right) + e_0, \\ v = f(\xi) = -\dfrac{k_2 4h(e_2-e_3)}{(k_1^2+k_2^2)3g}\lambda(\xi) \\ \quad + \dfrac{\xi}{k_1^2+k_2^2}\left(\dfrac{4hk_2e_2}{3g} - \dfrac{ak_2}{3} - k_1k_2C_1 + k_1^2C_2\right) + f_0, \end{cases} \tag{40}$$

where e_0 and f_0 are arbitrary integral constants, and

$$\begin{aligned} \lambda(\xi) &= \int \mathrm{cn}^2\left(\sqrt{e_1-e_3}(\xi-\xi_0); m\right)\mathrm{d}\xi \\ &= \frac{E\left(\mathrm{am}\left(\sqrt{e_1-e_3}(\xi-\xi_0), m\right), m\right)}{\sqrt{e_1-e_3}m^2} \\ &\quad - \frac{1-m^2}{m^2}(\xi-\xi_0), \end{aligned} \tag{41}$$

with am being the Jacobi amplitude function and $E(x; m)$ the elliptic integral of the second kind.

Substituting Eq. (38) into Eq. (4), we can obtain the periodic water surface evolution, i.e.,

$$\zeta(x,y,t) = -h\frac{\dfrac{4h}{3g}\left[e_2-(e_2-e_3)\,\mathrm{cn}^2\left(\sqrt{e_1-e_3}(\xi-\xi_0); m\right)\right] - \dfrac{a}{3}}{1+\dfrac{4h}{3g}\left[e_2-(e_2-e_3)\,\mathrm{cn}^2\left(\sqrt{e_1-e_3}(\xi-\xi_0); m\right)\right] - \dfrac{a}{3}}. \tag{42}$$

When $C_1 = C_2 = C_3 = 0$,

$$e_1 = e_2 = \frac{ga}{4h}, \quad e_3 = -\frac{ga}{2h}, \quad m = 1, \tag{43}$$

solution (42) will be degenerated to a solitary-wave solution

$$\zeta(x,y,t) = \frac{ha\,\mathrm{sech}^2\left(\sqrt{\dfrac{3ga}{4h}}(\xi-\xi_0)\right)}{1-a\,\mathrm{sech}^2\left(\sqrt{\dfrac{3ga}{4h}}(\xi-\xi_0)\right)}. \tag{44}$$

In this case, solution (40) will be simplified as

$$\begin{cases} s = \dfrac{-ak_1}{k_1^2+k_2^2}\sqrt{\dfrac{4h}{3ga}}\tanh\left(\sqrt{\dfrac{3ga}{4h}}(\xi-\xi_0)\right) + e_0, \\ v = \dfrac{-ak_2}{k_1^2+k_2^2}\sqrt{\dfrac{4h}{3ga}}\tanh\left(\sqrt{\dfrac{3ga}{4h}}(\xi-\xi_0)\right) + f_0. \end{cases} \tag{45}$$

Figure 1 represents the periodic evolution of the water surface where the parameters are chosen as $\{k_1 = k_2 = 0.07, C_1 = C_2 = 0, C_3 = 10^{-6}, h = 10, g = 9.81\}$. When C_3 is chosen to be zero , the periodic evolution will be degenerated to a solitary wave evolution, as displayed in Fig. 2.

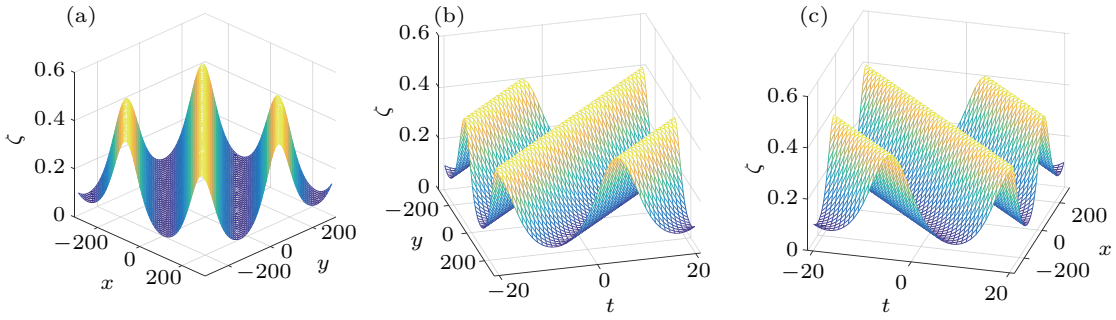


Fig. 1. (color online) Periodic evolution of water surface (a) along x and y at $t = 1$, (b) along y and t at $x = 1$, (c) along x and t at $y = 1$. The parameters are $k_1 = k_2 = 0.07$, $C_1 = C_2 = 0$, $C_3 = 10^{-6}$, $g = 9.81$, and $h = 10$.

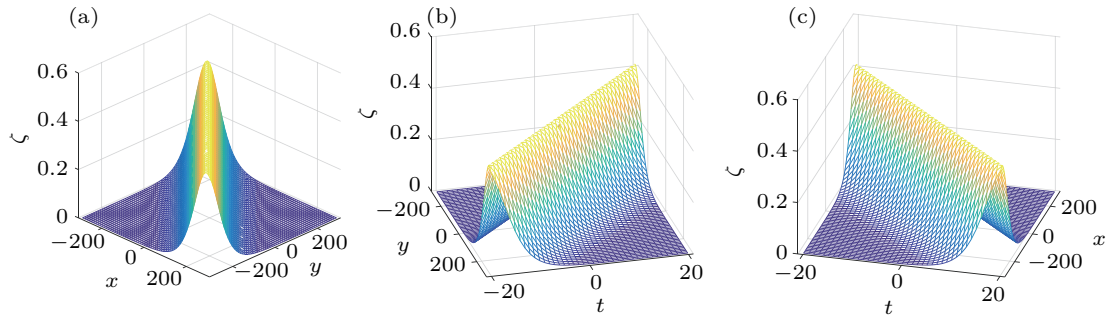


Fig. 2. (color online) Periodic evolution of water surface (a) along x and y at $t = 1$, (b) along y and t at $x = 1$, (c) along x and t at $y = 1$. The parameters are $k_1 = k_2 = 0.07$, $C_1 = C_2 = C_3 = 0$, $g = 9.81$, and $h = 10$.

4. Conclusion

A new fully nonlinear (2+1)-dimensional shallow water wave equation called FN2DSWWE is constructed by using the Hamilton variation principle of analytic mechanics under the Lagrange coordinate. In the new FN2DSWWE, the horizontal displacements $s(x,y,t)$ and $w(x,y,t)$ in the x and y directions are the two unknown variables. Once $s(x,y,t)$ and $w(x,y,t)$ are obtained, the water free surface $\zeta(x,y,t)$ can be determined by using the fully nonlinear incompressibility condition for shallow water systems. The general travelling wave solution of the proposed FN2DSWWE is obtained by means of the Jacobi elliptic function, the Jacobi amplitude function, and the elliptic integral of the second kind. Especially under a special selection of integral constants, the solution is degenerated to a solitary-wave solution. The FN2DSWWE can be reduced back to the (1+1)-dimensional displacement shallow water wave equation proposed by Zhong when one horizontal displacement, i.e., $s(x,y,t)$ or $w(x,y,t)$, is set to be zero. It can also be reduced back to the (2+1)-dimensional shallow water wave equation proposed by Liu and Lou when the effect of the nonlinear term $-s_y w_x$ in the incompressibility condition is neglected.

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