Quasi-periodic solutions and asymptotic properties for the nonlocal Boussinesq equation*

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We construct the Hirota bilinear form of the nonlocal Boussinesq (nlBq) equation with four arbitrary constants for the first time. It is special because one arbitrary constant appears with a bilinear operator together in a product form. A straightforward method is presented to construct quasiperiodic wave solutions of the nlBq equation in terms of Riemann theta functions. Due to the specific dispersion relation of the nlBq equation, relations among the characteristic parameters are nonlinear, then the linear method does not work for them. We adopt the perturbation method to solve the nonlinear relations among parameters in the form of series. In fact, the coefficients of the governing equations are also in series form. The quasiperiodic wave solutions and soliton solutions are given. The relations between the periodic wave solutions and the soliton solutions have also been established and the asymptotic behaviors of the quasiperiodic waves are analyzed by a limiting procedure.

Keywords: nonlocal Boussinesq equation, periodic wave solution, solitary waves, Riemann theta function

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1. Introduction

Analytical study is an important aspect of nonlinear partial differential equations, which plays an important role in describing and explaining natural phenomena. Analytical soliton solutions of nonlinear partial differential equations attract many researchers' attention due to their interesting properties.[1–3] Many different methods have been developed and used to obtain exact soliton solutions of nonlinear partial differential equations, such as the Lie symmetry method,[4] Bäcklund transformation method,[5] Hirota bilinear method,[6–8] inverse scattering transformation,[9] ellipitic function method,[10,11] and other constructing methods.[12–17]

In 1970s, Novikov *et al.*[18–22] developed a method to construct analytical quasiperiodic solutions of the nonlinear evolution equation. Many famous nonlinear evolution equations are proved to have analytical quasiperiodic solutions based on the inverse spectral method[6] and the algebraic geometry method,[23,24] such as the KdV equation,[25–27] the sine-Gordan equation, the Schrödinger equation,[28] and the Camassa–Holm equation. In 1980s, Nakamura developed a direct way to construct a kind of quasiperiodic solution of nonlinear equations explicitly[29,30] based on the Hirota bilinear form.[6,7] This method is easy to determine the characteristic parameters of the waves directly, such as frequencies and phase shifts for given wave numbers and amplitudes. The periodic wave solutions of the KdV equation and the Boussinesq equation were obtained in Refs. [6], [8], and [31]. Fan gave the asymptotic properties of the quasiperiod solution.[32,33] This method has been applied to many famous nonlinear evolution equations.[34–36]

The Boussinesq equation was proposed in the sense of hydrodynamics in its 1870s in original form

$$u_{tt} - u_{xx} = u_{xxxx} + (u^2)_{xx}. \tag{1}$$

Many interesting properties have been revealed, such as the soliton solution and head-on collisions of two solitons. The direct method gave the interaction between solitons exactly. The inverse scattering transform was used to study its initial condition problem. The Bäcklund transformation could deduce a series of solutions to the Boussinesq equation from one solution given previously. In order to overcome the ill-posedness of Eq. (1), two alternative Boussinesq equations were proposed. One is known as the improved Boussinesq equation

$$u_{tt} - u_{xx} = u_{xxtt} + (u^2)_{xx}, \tag{2}$$

and the other one is the good Boussinesq equation

$$u_{tt} - u_{xx} = (u^2)_{xx} - u_{xxxx}. \tag{3}$$

The improved Boussinesq equation (2) and the good Boussinesq equation (3) are both well posed problems.

The nonlocal Boussinesq (nlBq) equation

$$U_y = V_x, \tag{4}$$

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$$V_y = \alpha U_x - U_{3x} - (U^2)_x + \left[\frac{V^2+U_x^2}{U-\alpha/2}\right]_x, \tag{5}$$

where $U = U(x,y)$, $V = V(x,y)$, $U_{mx} = \frac{\partial^m U}{\partial x^m}$, and $V_{ny} = \frac{\partial^n V}{\partial y^n}$, was presented as a resonance-free alternative to the good Boussinesq equation and the behavior of the relevant two-soliton solution as compared to that of the good Boussinesq equation was discussed in detail.[37] The nonlocal Boussinesq equation arose as a compatibility condition for a linear system associated with the bilinear representation of Kaup's higher order wave equation.[37–39] It was shown that the nlBq equation can be regarded as a $pq = c$ reduction of the KP hierarchy. The bilinear form was established in Ref. [38]. Multiple solutions of the nonlocal Boussinesq equation were given in Ref. [36] by the simplified Hirota bilinear method.

We will construct the one- and two-periodic wave solutions of the nlBq equation and give relations between the soliton solutions and the periodic solutions. The organization of this paper is as follows. In Section 2, we will rewrite the bilinear form of the nlBq equation with four arbitrary constants. In Section 3, the one- and two-periodic wave solutions of the nlBq equation are constructed based on its bilinear form. In Section 4, the limiting behaviors of the periodic-wave solutions are analyzed. The periodic-wave solutions degenerate into soliton solutions.

2. The bilinear form of nonlocal Boussinesq equation

The nlBq equation (4) and (5) can be regarded as the lowest order member of a hierarchy of soliton systems[38]

$$\begin{aligned}&\mathrm{i}\Phi_{t_m} + \Phi_{xt_{m-1}} + q_{xt_{m-1}}\Phi = 0,\\ &2\mathrm{i}\Phi_{xt_m} - \mathrm{i}\Phi_{t_2t_{m-1}} + \Phi_{2xt_{m-1}} + (q_{2x}-\alpha)\Phi_{t_{m-1}} + 2q_{xt_{m-1}}\Phi_x\\ &+\mathrm{i}(2q_{xt_m} - q_{t_2t_{m-1}})\Phi = 0 \ \text{ for } \ m \geq 2,\end{aligned}$$

the second member of which is

$$U_t = \alpha U_x - U_{3x} - \frac{3}{2}(U^2)_x + \frac{3}{4}\left[\frac{V^2+U_x^2}{U-\alpha/2}\right]_x, \tag{6}$$

$$V_t = \alpha V_x - V_{3x} - 3(UV)_x + \frac{3}{2}\left[\frac{V_xU_x - VU_{2x}}{U-\alpha/2}\right]_x + \frac{3}{4}\left[\frac{V^3+U_x^2V}{(U-\alpha/2)^2}\right]_x. \tag{7}$$

The construction of this higher order flow is proved to be crucial in the bilinearization of the nlBq equations (4) and (5). In fact, equation (5) cannot be written in bilinear form directly, but it can be written in bilinear form by introducing a new variable t with the aid of Eq. (6). If we set q satisfying

$$U = u_0 + q_{2x}, \quad V = v_0 + q_{xy}, \tag{8}$$

then equation (4) is satisfied automatically. Equation (5) can be changed into the following form:

$$q_{yy} = (\alpha - 2u_0)q_{xx} - q_{xxxx} - q_{xx}^2 + \frac{(v_0+q_{xy})^2 + q_{xxx}^2}{u_0 + q_{xx} - \alpha/2} + c_1, \tag{9}$$

and equation (6) can be regarded as an auxiliary equation in terms of t and rewritten as

$$q_{xt} = (\alpha - 3u_0)q_{2x} - q_{xxxx} - \frac{3}{2}q_{xx}^2 + \frac{3}{4}\frac{(v_0+q_{xy})^2 + q_{xxx}^2}{u_0 + q_{xx} - \alpha/2} + c_2, \tag{10}$$

where c_1 and c_2 are integrable constants. The fractal terms can be cancelled by subtracting three times Eq. (9) from four times Eq. (10)

$$4q_{xt} - (\alpha - 6u_0)q_{xx} + q_{xxxx} + 3q_{xx}^2 - 3q_{yy} + C_1 = 0, \tag{11}$$

where $C_1 = 4c_2 - 3c_1$. On the other hand, equation (9) can be written as

$$\begin{aligned}&\left(q_{xx} - \frac{\alpha}{2} + u_0\right)\left(q_{yy} - (\alpha - 2u_0)\,q_{xx} + q_{xxxx} + q_{xx}^2 - c_1\right)\\ &= v_0^2 + 2v_0q_{xy} + q_{xy}^2 + q_{xxx}^2.\end{aligned} \tag{12}$$

With the aid of Eq. (11), equation (12) is equivalent to the following equation:

$$\begin{aligned}&4(q_{xxxt} + 3q_{xx}q_{xt}) - (\alpha - 6u_0)(q_{xxxx} + 3q_{xx}^2)\\ &+ (q_{xxxxxx} + 15q_{xx}q_{xxxx} + 15q_{xx}^3)\\ &- 3(q_{xxyy} + q_{xx}q_{yy} + 2q_{xy}^2) + 12(\alpha - 2u_0)\\ &\times (q_{xt} - q_{yy} + u_0q_{xx}) + 3(C_1 - 2c_1)q_{xx}\\ &- 12v_0q_{xy} - 6v_0^2 + 3(\alpha - 2u_0)(c_1 + C_1) = 0,\end{aligned} \tag{13}$$

which can be verified directly. This equation looks more complex than its original version, but it can be converted into the bilinear form easily. In fact, if we assume

$$q = 2\ln f, \tag{14}$$

then by using the standard identities for the Hirota D-operators[6]

$$\begin{aligned}&D_x^mD_y^nD_t^kf(x,y,t)g(x,y,t)\\ &= (\partial_x - \partial_{x'})^m(\partial_y - \partial_{y'})^n(\partial_t - \partial_{t'})^kf(x,y,t)\\ &\times g(x',y',t')|_{x'=x,y'=y,t'=t},\end{aligned} \tag{15}$$

we can obtain the bilinear form of Eqs. (11) and (13) as follows:

$$\mathcal{H} := [4D_xD_t - 3D_y^2 + D_x^4 - (\alpha - 6u_0)D_x^2 + C_1]f\cdot f = 0, \tag{16}$$

$$\begin{aligned}\mathcal{G} := [&4D_x^3D_t - (\alpha - 6u_0)D_x^4 + D_x^6 - 3D_x^2D_y^2 + 12(\alpha - 2u_0)\\ &\times (D_xD_t - D_y^2 + u_0D_x^2) + 3(C_1 - 2c_1)D_x^2\\ &- 12v_0D_xD_y - 6v_0^2\\ &+ 3(\alpha - 2u_0)(c_1 + C_1)]f\cdot f = 0,\end{aligned} \tag{17}$$

where u_0, v_0, c_1, and C_1 are arbitrary integration constants. They are crucial in constructing the periodic solutions, while

in the process of constructing the soliton solutions, they are often put to zero.

It can be seen that equations (11) and (13) are equivalent to equations (5) and (7), respectively. Now we have obtained the bilinear form of the nonlocal Boussinesq equation by introducing new variable t and auxiliary equation (7). These bilinear equations are different from many existing bilinear equations for searching for Riemann theta function periodic wave solutions, such as in Refs. [32], [33], [35], and [36], where the arbitrary constants appeared separately. The arbitrary constant c_1 appears in a nonlinear form in Eq. (17). Bilinear equations (16) and (17) degenerate into Eq. (30) in Ref. [38] when $u_0 = v_0 = 0, c_1 = C_1 = 0$.

3. Periodic wave solutions

3.1. One-periodic wave solution

The Riemann theta function with genus $N = 1$[40,41] takes the form

$$\vartheta(\xi) = \vartheta(\xi,\tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} n\xi - \pi n^2\tau}, \tag{18}$$

where phase variable $\xi = px + \omega y + \sigma t + \xi^0$, $\tau > 0$.

To make the theta function (18) be a solution to the bilinear equation (16), equations (16) and (17) should be vanished by substituting $f = \vartheta(\xi,\tau)$

$$\begin{aligned}
&\mathcal{H}(D_x,D_y,D_t)f\cdot f\\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \mathcal{H}(D_x,D_y,D_t)\,\mathrm{e}^{2\pi \mathrm{i} n\xi - \pi n^2\tau}\\
&\quad\times \mathrm{e}^{2\pi \mathrm{i} m\xi - \pi m^2\tau} = 0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\mathcal{G}(D_x,D_y,D_t)f\cdot f\\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \mathcal{G}(D_x,D_y,D_t)\,\mathrm{e}^{2\pi \mathrm{i} n\xi - \pi n^2\tau}\\
&\quad\times \mathrm{e}^{2\pi \mathrm{i} m\xi - \pi m^2\tau} = 0.
\end{aligned} \tag{20}$$

According to the previous results in Refs. [29], [30], [32], and [34], if the following four equations are satisfied:

$$\sum_{n=-\infty}^{\infty} \mathcal{H}[2\pi\mathrm{i}\cdot 2np, 2\pi\mathrm{i}\cdot 2n\omega, 2\pi\mathrm{i}\cdot 2n\sigma]\,\mathrm{e}^{-2\pi n^2\tau} = 0, \tag{21}$$

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \mathcal{H}[2\pi\mathrm{i}\cdot (2n-1)p, 2\pi\mathrm{i}\cdot (2n-1)\omega, 2\pi\mathrm{i}\cdot (2n-1)\sigma]\\
&\quad\times \mathrm{e}^{-\pi[n^2+(1-n)^2]\tau} = 0,
\end{aligned} \tag{22}$$

$$\sum_{n=-\infty}^{\infty} \mathcal{G}[2\pi\mathrm{i}\cdot 2np, 2\pi\mathrm{i}\cdot 2n\omega, 2\pi\mathrm{i}\cdot 2n\sigma]\,\mathrm{e}^{-2\pi n^2\tau} = 0, \tag{23}$$

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \mathcal{G}[2\pi\mathrm{i}\cdot (2n-1)p, 2\pi\mathrm{i}\cdot (2n-1)\omega, 2\pi\mathrm{i}\cdot (2n-1)\sigma]\\
&\quad\times \mathrm{e}^{-\pi[n^2+(1-n)^2]\tau} = 0,
\end{aligned} \tag{24}$$

then equations (19) and (20) hold. Combining the above four equations and the form of bilinear equations (16) and (17), we obtain

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} [-64\pi^2n^2p\sigma + 48\pi^2n^2\omega^2 + 256\pi^4n^4p^4\\
&\quad +16(\alpha-6u_0)\pi^2n^2p^2 + C_1]\,\mathrm{e}^{-2\pi n^2\tau} = 0,\\
&\sum_{n=-\infty}^{\infty} [-16\pi^2(2n-1)^2p\sigma + 12\pi^2(2n-1)^2\omega^2\\
&\quad +16\pi^4(2n-1)^4p^4 + 4(\alpha-6u_0)\pi^2(2n-1)^2p^2 + C_1]\\
&\quad\times \mathrm{e}^{-\pi(2n^2-2n+1)\tau} = 0,\\
&\sum_{n=-\infty}^{\infty} [(2^{10}\pi^4n^4p^3 - 3\cdot 2^6(\alpha-2u_0)\pi^2n^2p)\sigma - 3(2^8\pi^4n^4p^2\\
&\quad -2^6(\alpha-2u_0)\pi^2n^2)\omega^2 + 192v_0\pi^2n^2p\omega - 4096\pi^6n^6p^6\\
&\quad -256(\alpha-6u_0)\pi^4n^4p^4 - 48(4\alpha u_0 - 8u_0^2\\
&\quad +C_1 - 2c_1)\pi^2n^2p^2 - 6v_0^2\\
&\quad +3(\alpha-2u_0)(c_1+C_1)]\,\mathrm{e}^{-2\pi n^2\tau} = 0,\\
&\sum_{n=-\infty}^{\infty} [(64\pi^4(2n-1)^4p^3 - 48(\alpha-2u_0)\pi^2(2n-1)^2p)\sigma\\
&\quad -(48\pi^4(2n-1)^4p^2 - 48(\alpha-2u_0)\pi^2(2n-1)^2)\omega^2\\
&\quad +48v_0\pi^2(2n-1)^2p\omega - 64\pi^6(2n-1)^6p^6\\
&\quad -16(\alpha-6u_0)\pi^4(2n-1)^4p^4\\
&\quad -12(4\alpha u_0 - 8u_0^2 + C_1 - 2c_1)\pi^2(2n-1)^2p^2\\
&\quad -6v_0^2 + 3(\alpha-2u_0)(c_1+C_1)]\,\mathrm{e}^{-\pi(2n^2-2n+1)\tau} = 0.
\end{aligned} \tag{25}$$

Then theta function $\vartheta(\xi,\tau)$ (18) is an exact solution of Eq. (16).

Equation system (25) determines the relations among all the parameters. We suppose that c_1, C_1, σ, and ω are in terms of p, α, u_0, and v_0. Different from many existing references, ω appears in a nonlinear form in Eq. (25). We will adopt the perturbation method to find the series solution of equation system (25). In fact, by assuming $\lambda = \mathrm{e}^{-\pi\tau}$, we have $\mathrm{e}^{-2\pi n^2\tau} = \lambda^{2n^2}$ and $\mathrm{e}^{-\pi(2n^2-2n+1)\tau} = \lambda^{2n^2-2n+1}$, then each equation of system (25) is in the form of a series of λ. So we naturally use the perturbation method to obtain the series solution. Furthermore, the perturbation method is valid to the linear equation.[32,35,36]

In order to obtain the undetermined parameters, we can also suppose that c_1, C_1, σ, and ω are in a series of λ

$$\begin{aligned}
c_1 &= \sum_{i=0}^{\infty} c_{1,i}\lambda^i, \quad C_1 = \sum_{i=0}^{\infty} C_{1,i}\lambda^i,\\
\sigma &= \sum_{i=0}^{\infty} \sigma_i\lambda^i, \quad \omega = \sum_{i=0}^{\infty} \omega_i\lambda^i.
\end{aligned} \tag{26}$$

Then substituting Eq. (26) into Eq. (25), collecting the coefficients of λ^i, and setting to zero one by one, we can obtain an equation system about σ_i, ω_i, $c_{1,i}$, and $C_{1,i}$, $(i = 0,1,2,\ldots)$. One can solve the equations corresponding to each λ^i, $i = 0,1,\ldots$ to solve the equation system step by step. As a result, we obtain the following series solution:

$$\omega = p\Psi - \frac{2pv_0}{\Phi}$$

$$
\begin{aligned}
&-\frac{16\pi^2p^3\left(48\pi^2p^2\Phi+3\Phi^2+128\pi^4p^4\right)}{\Psi\Phi^2}\lambda^2\\
&-\frac{1}{\Psi^3}\Bigg(144\pi^2p^3\Phi+6336p^5\pi^4\\
&+\frac{79872\pi^6p^7}{\Phi}+\frac{491520p^9\pi^8}{\Phi^2}\\
&+\frac{1572864p^{11}\pi^{10}}{\Phi^3}+\frac{2097152p^{13}\pi^{12}}{\Phi^4}\Bigg)\lambda^4+\mathrm{O}(\lambda^6),\\
\sigma\ =\ &\frac{3p\Phi}{2}+\frac{p\left(-\alpha+8\pi^2p^2\right)}{2}-3\frac{pv_0\Psi}{\Phi}+3\frac{pv_0{}^2}{\Phi^2}\\
&-\Bigg(96\pi^2p^3+\frac{144\pi^2p^3\left(8\pi^2p^2\Psi-v_0\right)}{\Psi\Phi}\\
&-\frac{48\pi^2p^3\left(48\pi^2v_0p^2-64\Psi\pi^4p^4\right)}{\Psi\Phi^2}-\frac{6144\pi^6p^7v_0}{\Psi\Phi^3}\Bigg)\lambda^2\\
&-\Bigg(\frac{288\pi^2p^3\Phi}{\Phi+4\pi^2p^2}+\frac{144\pi^2p^3\left(-3v_0+56\pi^2p^2\Psi\right)}{\Psi^3}\\
&+\frac{576p^5\pi^4(112\pi^2p^2\Psi-33v_0)}{\Psi^3\Phi}\\
&+\frac{18432\pi^6p^7\left(-13v_0+8\pi^2p^2\Psi\right)}{\Psi^3\Phi^2}\\
&-\frac{1474560p^9\pi^8v_0}{\Psi^3\Phi^3}-\frac{4718592p^{11}\pi^{10}v_0}{\Psi^3\Phi^4}\\
&-\frac{6291456p^{13}\pi^{12}v_0}{\Psi^3\Phi^5}\Bigg)\lambda^4+\mathrm{O}(\lambda^6),\\
c_1\ =\ &\frac{2v_0^2}{\alpha-2u_0}+\frac{768(32\pi^2p^2+3\alpha-6u_0)\pi^4p^4}{\alpha-2u_0}\lambda^4+\mathrm{O}(\lambda^8),\\
C_1\ =\ &-384\pi^4p^4\lambda^2-2304\pi^4p^4\lambda^4+\mathrm{O}(\lambda^6),
\end{aligned}
\tag{27}
$$

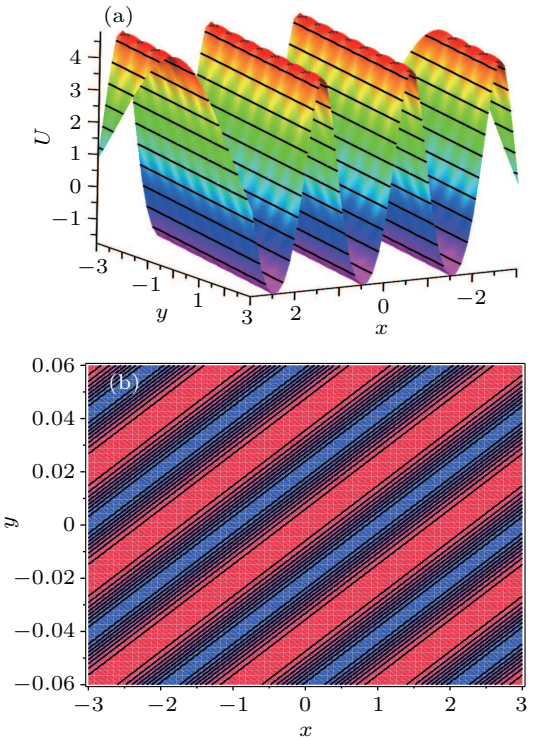


Fig. 1. (color online) A one-periodic wave U of the nlBq equations (4) and (5) via expression (28) with parameters $p=0.5$, $\tau=0.8$, $\alpha=0.1$, $u_0=1$, $v_0=1.2$, and $\xi^0=0$. (a) Perspective view of the periodic wave U. (b) The corresponding contour plot.

where $\Phi=\alpha-2u_0$ and $\Psi=\sqrt{\Phi+4\pi^2p^2}$. So far we have obtained the one-periodic wave solution $f(x,y,t)=\vartheta(\xi,\tau)$ to the bilinear equations (16) and (17) with the parameters c_1,C_1,σ, and ω given in Eq. (27). Then from Eqs. (8) and (14), we obtain the solution of the nlBq equation as

$$U=u_0+2p^2\frac{\partial^2\ln\vartheta(\xi,\tau)}{\partial\xi^2}=u_0+2p^2\frac{\vartheta\vartheta_{2\xi}-\vartheta_\xi^2}{\vartheta^2},\tag{28}$$

$$V=v_0+2p\omega\frac{\partial^2\ln\vartheta(\xi,\tau)}{\partial\xi^2}=v_0+2p\omega\frac{\vartheta\vartheta_{2\xi}-\vartheta_\xi^2}{\vartheta^2}.\tag{29}$$

The U and its contour are shown in Fig. 1.

The one-periodic wave solution has the following characters. (i) It has two fundamental periods 1 and τ in the phase variable ξ. (ii) It is actually a kind of one-dimensional cnoidal waves, i.e., there is a single phase variable ξ. (iii) It has only one wave pattern for all time and can be viewed as a parallel superposition of overlapping one-solitary waves placed one period apart; see Fig. 1.

3.2. Two-periodic wave solution

We construct the two-periodic wave solution of the nlBq equation in this subsection. By taking the genus $N=2$, the Riemann theta function takes the following form:

$$\vartheta(\boldsymbol{\xi})=\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})=\sum_{\boldsymbol{n}\in Z^2}\mathrm{e}^{-\pi\langle\boldsymbol{\tau n},\boldsymbol{n}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{n}\rangle},\tag{30}$$

where $\boldsymbol{n}=(n_1,n_2)^{\mathrm{T}}\in Z^2$, $\boldsymbol{\xi}=(\xi_1,\xi_2)^{\mathrm{T}}\in C^2$, $\xi_i=p_ix+\omega_iy+\sigma_it+\xi_i^0$, $i=1,2$, and $\boldsymbol{\tau}$ is a positive definite and real-valued symmetric 2×2 matrix which takes the form

$$\boldsymbol{\tau}=\begin{pmatrix}\tau_{11}&\tau_{12}\\ \tau_{21}&\tau_{22}\end{pmatrix},\ \tau_{11}>0,\ \tau_{22}>0,\ \tau_{21}=\tau_{12},\ \tau_{11}\tau_{22}-\tau_{12}^2>0.$$

In order to make the theta function (30) satisfy the bilinear equation (16), we substitute Eq. (30) into the left side of Eq. (16) to obtain

$$
\begin{aligned}
&\mathcal{H}(D_x,D_y,D_t)\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})\cdot\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})\\
&=\sum_{\boldsymbol{n}\in Z^2}\sum_{\boldsymbol{m}\in Z^2}\mathcal{H}(D_x,D_y,D_t)\,\mathrm{e}^{-\pi\langle\boldsymbol{\tau n},\boldsymbol{n}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}\\
&\quad\times\mathrm{e}^{-\pi\langle\boldsymbol{\tau m},\boldsymbol{m}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{m}\rangle}=0,
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
&\mathcal{G}(D_x,D_y,D_t)\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})\cdot\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})\\
&=\sum_{\boldsymbol{n}\in Z^2}\sum_{\boldsymbol{m}\in Z^2}\mathcal{G}(D_x,D_y,D_t)\,\mathrm{e}^{-\pi\langle\boldsymbol{\tau n},\boldsymbol{n}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{n}\rangle}\\
&\quad\times\mathrm{e}^{-\pi\langle\boldsymbol{\tau m},\boldsymbol{m}\rangle+2\pi\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{m}\rangle}=0.
\end{aligned}
\tag{32}
$$

We denote $\boldsymbol{s}^1=(0,0)^{\mathrm{T}}$, $\boldsymbol{s}^2=(1,0)^{\mathrm{T}}$, $\boldsymbol{s}^3=(0,1)^{\mathrm{T}}$, and $\boldsymbol{s}^4=(1,1)^{\mathrm{T}}$, the necessary conditions of Eqs. (31) and (32) are the following eight equations:

$$
\begin{aligned}
&\sum_{\boldsymbol{n}\in Z^2}\mathcal{H}[2\pi\mathrm{i}\langle2\boldsymbol{n}-\boldsymbol{s}^j,\boldsymbol{p}\rangle,2\pi\mathrm{i}\langle2\boldsymbol{n}-\boldsymbol{s}^j,\boldsymbol{\omega}\rangle,\\
&2\pi\mathrm{i}\langle2\boldsymbol{n}-\boldsymbol{s}^j,\boldsymbol{\sigma}\rangle]\varepsilon_j(\boldsymbol{n})=0,\\
&\sum_{\boldsymbol{n}\in Z^2}\mathcal{G}[2\pi\mathrm{i}\langle2\boldsymbol{n}-\boldsymbol{s}^j,\boldsymbol{p}\rangle,2\pi\mathrm{i}\langle2\boldsymbol{n}-\boldsymbol{s}^j,\boldsymbol{\omega}\rangle,
\end{aligned}
$$

$$2\pi \mathrm{i}\langle 2\boldsymbol{n}-\boldsymbol{s}^j,\boldsymbol{\sigma}\rangle]\varepsilon_j(\boldsymbol{n})=0, \tag{33}$$

where $\varepsilon_j(\boldsymbol{n})=\mathrm{e}^{-\pi[\langle(\boldsymbol{s}^j-\boldsymbol{n})\boldsymbol{\tau},\boldsymbol{s}^j-\boldsymbol{n}\rangle+\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{n}\rangle]}$, $\boldsymbol{s}^j=(s_1^j,s_2^j)^{\mathrm{T}}$, $j=1,2,3,4$. Let $\lambda_1=\mathrm{e}^{-\pi\tau_{11}}$, $\lambda_2=\mathrm{e}^{-\pi\tau_{22}}$, $\lambda_3=\mathrm{e}^{-2\pi\tau_{12}}$, then we have

$$\varepsilon_j(\boldsymbol{n})=\lambda_1^{n_1^2+(n_1-s_1^j)^2}\lambda_2^{n_2^2+(n_2-s_2^j)^2}\lambda_3^{n_1n_2+(n_1-s_1^j)(n_2-s_2^j)}. \tag{34}$$

From Eq. (33), we find that the coefficients of ω_k, σ_k, $k=1,2$ and the constants c_1, C_1, u_0, v_0 are all series of λ_1, λ_2, so to obtain the undetermined parameters, we can also suppose them to be series of λ_1,λ_2. The concrete forms are

$$\sigma_1=\sum_{i,j\geq 0}\sigma_{1,i,j}\lambda_1^i\lambda_2^j,\quad \sigma_2=\sum_{i,j\geq 0}\sigma_{2,i,j}\lambda_1^i\lambda_2^j,$$

$$\omega_1=\sum_{i,j\geq 0}\omega_{1,i,j}\lambda_1^i\lambda_2^j,\quad \omega_2=\sum_{i,j\geq 0}\omega_{2,i,j}\lambda_1^i\lambda_2^j,$$
$$c_1=\sum_{i,j\geq 0}c_{1,i,j}\lambda_1^i\lambda_2^j,\quad C_1=\sum_{i,j\geq 0}C_{1,i,j}\lambda_1^i\lambda_2^j,$$
$$u_0=\sum_{i,j\geq 0}u_{0,i,j}\lambda_1^i\lambda_2^j,\quad v_0=\sum_{i,j\geq 0}v_{0,i,j}\lambda_1^i\lambda_2^j. \tag{35}$$

Then substituting (35) into (33), collecting all the coefficients of $\lambda_1^i\lambda_2^j$ occurring in the above equations, and setting to zero one by one, we can obtain a system equation about $\omega_{k,i,j}$, $\sigma_{k,i,j}$, $c_{1,i,j}$, $C_{1,i,j}$, $u_{0,i,j}$, $v_{0,i,j}$ for $k=1,2$, $i,j=0,1,\ldots$ The idea of perturbation can be used to solve the complicated equation system and the following approximate values can be obtained:

$$\omega_k = p_k\left(-2\frac{v_{0,0,0}}{Q}+R\right)-p_k\left(\frac{u_{0,1,0}}{R}+\frac{2v_{0,1,0}}{Q}+\frac{4u_{0,1,0}v_{0,0,0}}{Q^2}\right)\lambda_1-p_k\left(\frac{u_{0,0,1}}{R}+\frac{2v_{0,0,1}}{Q}+\frac{4u_{0,0,1}v_{0,0,0}}{Q^2}\right)\lambda_2+\cdots, \tag{36}$$

$$\begin{aligned}\sigma_k = {}& \frac{3p_kQ}{2}+\frac{p_k\left(-\alpha+8\pi^2{p_k}^2\right)}{2}-\frac{3p_kv_{0,0,0}R}{Q}+\frac{3p_k{v_{0,0,0}}^2}{Q^2}-3p_k\left(\frac{u_{0,1,0}R+v_{0,1,0}}{R}+\frac{4v_{0,1,0}\pi^2{p_k}^2+u_{0,1,0}v_{0,0,0}}{RQ}\right.\\ &\left.+\frac{8u_{0,1,0}v_{0,0,0}\pi^2{p_k}^2-2v_{0,0,0}v_{0,1,0}R}{RQ^2}-\frac{4u_{0,1,0}{v_{0,0,0}}^2}{Q^3}\right)\lambda_1-3p_k\left(\frac{u_{0,0,1}R+v_{0,0,1}}{R}+\frac{4v_{0,0,1}\pi^2{p_k}^2+u_{0,0,1}v_{0,0,0}}{RQ}\right.\\ &\left.+\frac{8u_{0,0,1}v_{0,0,0}\pi^2{p_k}^2-2v_{0,0,0}v_{0,0,1}R}{RQ^2}-\frac{4u_{0,0,1}{v_{0,0,0}}^2}{Q^3}\right)\lambda_2+\cdots,\end{aligned} \tag{37}$$

$$\begin{aligned}c_1 = {}& \frac{2v_{0,0,0}^2}{\alpha-2u_{0,0,0}}+\frac{4v_{0,0,0}(v_{0,1,0}\alpha-2v_{0,1,0}u_{0,0,0}+u_{0,1,0}v_{0,0,0})}{(\alpha-2u_{0,0,0}^2)^2}\lambda_1+\frac{4v_{0,0,0}(v_{0,0,1}\alpha-2v_{0,0,1}u_{0,0,0}+u_{0,0,1}v_{0,0,0})}{(\alpha-2u_{0,0,0}^2)^2}\lambda_2\\ &+\left[384\pi^4{p_1}^4+\frac{2(1024\pi^6{p_1}^6+2v_{0,0,0}v_{0,2,0}+{v_{0,1,0}}^2)}{Q}+\frac{4v_{0,0,0}(2u_{0,1,0}v_{0,1,0}+u_{0,2,0}v_{0,0,0})}{Q^2}+\frac{8{u_{0,1,0}}^2{v_{0,0,0}}^2}{Q^3}\right]\lambda_1^2\\ &+\left[384\pi^4{p_2}^4+\frac{2(1024\pi^6{p_2}^6+2v_{0,0,0}v_{0,0,2}+{v_{0,0,1}}^2)}{Q}+\frac{4v_{0,0,0}(2u_{0,0,1}v_{0,0,1+u_{0,0,2}v_{0,0,0}})}{Q^2}+\frac{8{u_{0,0,1}}^2{v_{0,0,0}}^2}{Q^3}\right]\lambda_2^2\\ &+\left[\frac{4(v_{0,0,1}v_{0,1,0}+v_{0,0,0}v_{0,1,1})}{Q}+\frac{16u_{0,1,0}{v_{0,0,0}}^2u_{0,0,1}}{Q^3}+\frac{4v_{0,0,0}(2u_{0,0,1}v_{0,1,0}+2u_{0,1,0}v_{0,0,1}+u_{0,1,1}v_{0,0,0})}{Q^2}\right]\lambda_1\lambda_2\\ &+\cdots,\end{aligned} \tag{38}$$

$$C_1 = -384\pi^4p_1^4\lambda_1^2-384\pi^4p_2^4\lambda_2^2+\cdots, \tag{39}$$

$$u_0 = u_{0,0,0}+u_{0,1,0}\lambda_1+u_{0,0,1}\lambda_2+u_{0,1,1}\lambda_1\lambda_2+u_{0,2,0}\lambda_1^2+u_{0,0,2}\lambda_2^2+\cdots, \tag{40}$$

$$v_0 = v_{0,0,0}+v_{0,1,0}\lambda_1+v_{0,0,1}\lambda_2+v_{0,1,1}\lambda_1\lambda_2+v_{0,2,0}\lambda_1^2+v_{0,0,2}\lambda_2^2+\cdots, \tag{41}$$

where $Q=\alpha-2u_{0,0,0}$ and $R=\sqrt{Q+4\pi^2{p_k}^2}$ for $k=1,2$.

Up to now, we have obtained the two-periodic wave solution $f(x,y,t)=\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})$ of the bilinear equations (16) and (17), the parameters ω_k, σ_k, c_1, C_1, u_0, and v_0 for $k=1,2$ are given by Eqs. (36)–(41). Then from Eqs. (8) and (14), we can obtain the corresponding solution of the nlBq equation as

$$U=u_0+2\frac{\partial^2\ln\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})}{\partial x^2}, \tag{42}$$

$$V=v_0+2\frac{\partial^2\ln\vartheta(\xi_1,\xi_2,\boldsymbol{\tau})}{\partial x\partial y}. \tag{43}$$

The U and its contour are shown in Fig. 2.

The two-periodic wave solution has the following characters. (i) It is a direct generalization of the one-periodic waves, its surface pattern is two-dimensional, i.e., there are two phase variables ξ_1 and ξ_2, which has two independent spatial periods in the two independent horizontal directions. (ii) It has $2N$ fundamental periods $\{\zeta_i,i=1,2,\ldots,N\}$ and $\{\tau_i,i=1,2,\ldots,N\}$ in (ξ_1,ξ_2) with $\zeta_1=(1,0,\ldots,0)^{\mathrm{T}}$, $\ldots$, $\zeta_N=(0,0,\ldots,1)^{\mathrm{T}}$. (iii) If the parameters satisfy the ratio relation

$$\omega_2\sim d\omega_1,\ \ \xi_2\sim d\xi_1,\ \ \vartheta(\xi_1,\xi_2)\sim\vartheta(\xi_1,d\xi_2), \tag{44}$$

the two-periodic wave is actually one-dimensional and it degenerates to the one-periodic wave.

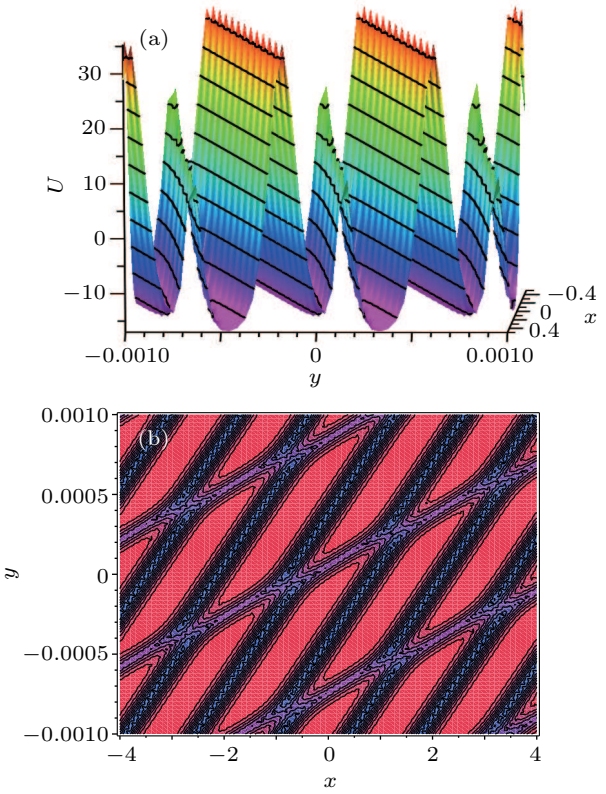


Fig. 2. (color online) A two-periodic wave U of the nlBq equations (4) and (5) via expression (42) with parameters $p_1 = 0.5$, $p_2 = 0.3$, $\tau_{1,1} = 0.8$, $\tau_{2,2} = 1.6$, $\alpha = 0.1$, $u_{0,i,j} = 0$, $v_{0,i,j} = 0$, $i, j = 0, 1, \ldots$, and $\xi_1^0 = \xi_2^0 = 0$. (a) Perspective view of the periodic wave U. (b) The corresponding contour plot.

4. Asymptotic properties of the periodic wave solutions

In this section, we will show the relations between the one- and two-periodic wave solutions and the corresponding one- and two-soliton solutions.

4.1. Limiting behavior of the one-periodic wave solution

In this subsection, we will prove that the one-periodic wave solution of the nlBq equation can degenerate into its one-soliton solution with $\lambda = \mathrm{e}^{-\pi\tau} \to 0$.

First of all, based on the bilinear equations (16) and (17) with the integration constants both being 0, we can easily obtain the one-soliton solution of the nlBq equation

$$f_1 = 1 + \mathrm{e}^{\eta}, \tag{45}$$

with $\eta = \beta x + \gamma y + \delta t + \eta^0$, where $\gamma = \beta\sqrt{\alpha - \beta^2}$ and $\delta = \alpha\beta - \beta^3$.

If the parameters ω, σ, c_1, and C_1 satisfy Eq. (27), theta function (18) will be a solution to Eqs. (16) and (17). In order to prove that the one-periodic wave solution can degenerate into the one-soliton solution, we will examine the limiting behavior of $\vartheta(\xi, \tau)$

$$\begin{aligned}\vartheta(\xi,\tau) \;=\; & 1 + \mathrm{e}^{2\pi\mathrm{i}\xi - \pi\tau} + \mathrm{e}^{-2\pi\mathrm{i}\xi - \pi\tau} + \mathrm{e}^{4\pi\mathrm{i}\xi - 4\pi\tau} \\ & + \mathrm{e}^{-4\pi\mathrm{i}\xi - 4\pi\tau} + \cdots. \end{aligned} \tag{46}$$

When $u_0 = v_0 = 0$ and $\lambda = \mathrm{e}^{-\pi\tau} \to 0$, from Eq. (27), we know that the parameters have the following degenerating values:

$$\begin{aligned} & c_1, \quad C_1 \to 0, \\ & \sigma \to p(4\pi^2 p^2 + \alpha), \\ & \omega \to p\sqrt{4\pi^2 p^2 + \alpha}. \end{aligned} \tag{47}$$

Let us denote

$$\rho = 2\pi\mathrm{i}p, \quad \theta^0 = 2\pi\mathrm{i}\xi^0 - \pi\tau, \tag{48}$$

then we have

$$\begin{aligned} & 2\pi\mathrm{i}\omega \to 2\pi\mathrm{i}p\sqrt{4\pi^2 p^2 + \alpha} = \rho\sqrt{\alpha - \rho^2} \equiv \mu, \\ & 2\pi\mathrm{i}\sigma \to 2\pi\mathrm{i}p(4\pi^2 p^2 + \alpha) = \rho(\alpha - \rho^2) \equiv \nu. \end{aligned} \tag{49}$$

Assuming $\theta = \rho x + \mu y + \nu t + \theta^0$, from Eqs. (45), (48), and (49), we can see that $1 + \mathrm{e}^{\theta}$ is a one-soliton solution to the nlBq equation, the following result can also be obtained:

$$\theta = 2\pi\mathrm{i}\xi - \pi\tau, \tag{50}$$

where $\xi = px + \omega y + \sigma t + \xi^0$ is just the same as in Eq. (18). Therefore, equation (46) can be changed into

$$\begin{aligned}\vartheta(\xi,\tau) \;=\; & 1 + \mathrm{e}^{\theta} + \lambda^2(\mathrm{e}^{2\theta} + \mathrm{e}^{-\theta}) \\ & + \lambda^6(\mathrm{e}^{3\theta} + \mathrm{e}^{-2\theta}) + \cdots \to 1 + \mathrm{e}^{\theta} \end{aligned} \tag{51}$$

as $\lambda \to 0$.

Up to now, we have proved that the one-periodic wave solution of the nlBq equation actually degenerates into its one-soliton solution when $\lambda = \mathrm{e}^{-\pi\tau} \to 0$, which gives a further validation for the correctness of this solution to the nlBq equation.

4.2. Limiting behavior of the two-periodic wave solution

In this subsection, we will prove that the two-periodic wave solution of the nlBq equation can degenerate into its two-soliton solution with $\lambda_1 = \mathrm{e}^{-\pi\tau_{11}}$ and $\lambda_2 = \mathrm{e}^{-\pi\tau_{22}} \to 0$.

At the beginning, by use of Hirota's method in Ref. [6], we can easily find the two-soliton solution of the nlBq equation based on its bilinear form (16) and (17) with the integration constants both being 0, its concrete form is as follows:

$$f_2 = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + a_{12}\,\mathrm{e}^{\eta_1 + \eta_2}, \tag{52}$$

with $\eta_k = \beta_k x + \gamma_k y + \delta_k t + \eta_k^0$, where $\gamma_k = \beta_k\sqrt{\alpha - \beta_k^2}$, $\delta_k = \alpha\beta_k - \beta_k^3$, $k = 1, 2$, and

$$a_{12} = \frac{\alpha - \beta_1\beta_2 - \sqrt{\alpha - \beta_1{}^2}\sqrt{\alpha - \beta_2{}^2}}{\alpha + \beta_1\beta_2 - \sqrt{\alpha - \beta_1{}^2}\sqrt{\alpha - \beta_2{}^2}}. \tag{53}$$

When ω_k, σ_k, c_1, C_1, u_0, and v_0 for $k = 1, 2$ satisfy Eqs. (36)–(41), the theta function (30) is a solution to the bilinear equations (16) and (17).

In order to prove that the two-periodic wave solution can degenerate into the two-soliton solution (52), we will examine the limiting behavior of $\vartheta(\xi_1,\xi_2,\tau)$,

$$\begin{aligned}&\vartheta(\xi_1,\xi_2,\tau)\\ &=1+(\mathrm{e}^{2\pi\mathrm{i}\xi_1}+\mathrm{e}^{-2\pi\mathrm{i}\xi_1})\mathrm{e}^{-\pi\tau_{11}}+(\mathrm{e}^{2\pi\mathrm{i}\xi_2}+\mathrm{e}^{-2\pi\mathrm{i}\xi_2})\mathrm{e}^{-\pi\tau_{22}}\\ &\quad+(\mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2)}+\mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2)})\mathrm{e}^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})}+\cdots. \quad (54)\end{aligned}$$

If we denote

$$\begin{aligned}&\rho_k=2\pi\mathrm{i}p_k,\quad \theta_k^0=2\pi\mathrm{i}\xi_k^0-\pi\tau_{kk},\\ &A_{12}=\frac{\alpha+4\pi^2p_1p_2-\sqrt{\alpha+4\pi^2p_1^2}\sqrt{\alpha+4\pi^2p_2^2}}{\alpha-4\pi^2p_1p_2-\sqrt{\alpha+4\pi^2p_1^2}\sqrt{\alpha+4\pi^2p_2^2}},\\ &\tau_{12}=-\frac{\ln A_{12}}{2\pi}, \quad (55)\end{aligned}$$

and take $u_{0,i,j}=v_{0,i,j}=0$ for $i,j\geq 0$, then $Q\to\alpha$, $R\to\sqrt{\alpha+4\pi^2p_k^2}$ for $k=1,2$. So we also have

$$\begin{aligned}&A_{12}=\frac{\alpha-\rho_1\rho_2-\sqrt{\alpha-\rho_1^2}\sqrt{\alpha-\rho_2^2}}{\alpha+\rho_1\rho_2-\sqrt{\alpha-\rho_1^2}\sqrt{\alpha-\rho_2^2}},\\ &2\pi\mathrm{i}\omega_k\to 2\pi\mathrm{i}p_k\sqrt{\alpha+4\pi^2p_k^2}=\rho_k\sqrt{\alpha-\rho_k^2}\equiv\mu_k,\\ &2\pi\mathrm{i}\sigma_k\to 2\pi\mathrm{i}p_k(\alpha+4\pi^2p_k^2)=\rho_k(\alpha-\rho_k^2)\equiv\nu_k, \quad (56)\end{aligned}$$

for $k=1,2$.

Assuming $\theta_k=\rho_kx+\mu_ky+\nu_kt+\theta_k^0$, from Eqs. (52), (53), (55), and (56), we can see that $1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+A_{12}\mathrm{e}^{\theta_1+\theta_2}$ is a two-soliton solution of the nlBq equation, the following result can also be obtained:

$$\theta_k=2\pi\mathrm{i}\xi_k-\pi\tau_{kk}, \quad (57)$$

where $\xi_k=p_kx+\omega_ky+\sigma_kt+\xi_k^0$ is just the same as in Eq. (30). Then equation (54) can be changed into

$$\begin{aligned}\vartheta(\xi_1,\xi_2,\tau)&=1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+A_{12}\mathrm{e}^{\theta_1+\theta_2}+\lambda_1^2\mathrm{e}^{-\theta_1}\\ &\quad+\lambda_2^2\mathrm{e}^{-\theta_2}+\lambda_1^2\lambda_2^2\mathrm{e}^{-\theta_1-\theta_2-2\pi\tau_{12}}\\ &\quad+\cdots\to 1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+A_{12}\mathrm{e}^{\theta_1+\theta_2} \quad (58)\end{aligned}$$

as $\lambda_1,\ \lambda_2\to 0$. From Eqs. (38) and (39), we can also see that the integration constants c_1 and C_1 both degenerate into 0 when $\lambda_1,\lambda_2\to 0$.

Up to now, we have proved that the two-periodic wave solution of the nlBq equation actually degenerates into its two-soliton solution when $\lambda_1=\mathrm{e}^{-\pi\tau_{11}}$, $\lambda_2=\mathrm{e}^{-\pi\tau_{22}}\to 0$. When $t=0$, this two-soliton solution is equivalent to the two-soliton solution presented in Ref. [36].

5. Conclusion

We have established a bilinear form to the nlBq equation, which consists of four arbitrary constants. This new bilinear form not only will degenerate to the known form in Ref. [38], but can also be used to construct periodic and quasiperiodic solutions to the nlBq equation in terms of the Riemman theta function.

The algebraic equation system about useful parameters is nonlinear, which is greatly different from the existing linear algebraic equation system in reference about constructing a quasiperiodic solution by the Riemman theta function. According to the series form of the coefficients of the governing equation, we have solved the nonlinear governing equation in series form by the perturbation method. In fact, our perturbation method is also suitable for the existing linear equation system appearing in reference about constructing a quasiperiodic solution by Riemman theta function.

We also show that the Riemann theta solutions degenerate to the soliton solutions. When $t=0$, the degenerated soliton solutions are equivalent to Wazwaz's multiple soliton solution[36] by a simplified bilinear method.

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