# New results for exponential synchronization of linearly coupled ordinary differential systems*

Ping Tong(童评) and Shi-Hua Chen(陈士华)†

*School of Mathematics and Statistics, Wuhan University, Wuhan* 430072, *China*

(Received 14 November 2016; revised manuscript received 3 February 2017; published online 30 March 2017)

This paper investigates the exponential synchronization of linearly coupled ordinary differential systems. The intrinsic nonlinear dynamics may not satisfy the QUAD condition or weak-QUAD condition. First, it gives a new method to analyze the exponential synchronization of the systems. Second, two theorems and their corollaries are proposed for the local or global exponential synchronization of the coupled systems. Finally, an application to the linearly coupled Hopfield neural networks and several simulations are provided for verifying the effectiveness of the theoretical results.

**Keywords:** exponential synchronization, linearly coupled, ordinary differential systems, QUAD condition

**PACS:** 05.45.Xt, 05.45.–a **DOI:** 10.1088/1674-1056/26/5/050503

## 1. Introduction

Synchronization of linearly coupled ordinary differential equations (LCODEs) has been widely applied for decades to different fields, such as neuroscience,[1–4] economics,[5] biology,[6,7] ecology,[8] computer science,[9,10] and so on. Meanwhile, the synchronization technique for LCODEs has been favored by several researchers. For example, several researchers proposed an array of resistive coupled Chua circuits for implementing spatial recognition,[11] some presented various architectures of the coupled chaotic system for secure communication,[11–13] and numerous presented an architecture of the coupled neural networks to store and retrieve complex oscillatory patterns as synchronous states.[14] Therefore, the study of synchronization of LCODEs is significant and necessary.

So far, the synchronization of LCODEs has attracted a great deal of attention from researchers and has been investigated in various ways. The master stability function method based on the transverse Lyapunov exponents is introduced to analyze the local synchronization of coupled oscillators and general coupled dynamical systems.[15–17] In addition, the distance between the state of a coupled dynamical system and synchronization manifold is used to discuss global synchronization of the coupled system.[18,19] Furthermore, two special vectors are used to decompose the entire state space of a coupled dynamical system into a direct sum of the transverse space and synchronization manifold for studying local and global synchronizations of the coupled system.[20]

In this paper, the synchronization of LCODEs is analyzed in a different way. Synchronizing the coupled system means that the states of any two nodes converge to agreement as time $t \to \infty$. Then the synchronization of the coupled system can be achieved, making sure that the states of all the nodes converge to the state of a certain one among the nodes as time $t \to \infty$. It can be found that the coupled system can be transformed into a new one with state variables, which are the differences of the states between the certain node and the other ones. Thus, the exponential convergence of the latter implies the exponential synchronization of the former. Through the exponential convergence analysis of the latter, our results can be obtained.

Generally, the coupled dynamical system described by LCODEs is governed by the following two mechanisms: the intrinsic nonlinear dynamics at each node and the coupling configuration among the nodes. However, most of the existing results of synchronization of LCODEs are based on the assumption that the intrinsic nonlinear dynamics satisfy the QUAD condition or even the Lipschitz condition. The QUAD condition is used in numerous researches on synchronization.[20–23] For example, together with the dissipativity property of the coupling configuration among nodes, the QUAD condition is derived to prove the synchronization of linearly coupled Lorenz systems.[21] The QUAD condition is given in Ref. [22] for the synchronization of systems with non-Lipschitz right-hand sides. In Ref. [23], the QUAD condition is given for the synchronization of networks with discontinuous activations in the presence of time delays.

However, as far as we know, the LCODEs in which the individual dynamics do not satisfy the QUAD condition exist in many practical network systems. Such systems have been investigated in some literatures.[24–27] Therefore, the study of the synchronization of LCODEs is significant and necessary. It should be noted that such synchronization criteria may not be easily verified. Therefore, in order to study the synchronization of LCODEs in which the individual dynamics do not

*Project supported by the National Natural Science Foundation of China (Grant No. 61273215).

†Corresponding author. E-mail: shcheng@whu.edu.cn

© 2017 Chinese Physical Society and IOP Publishing Ltd 

satisfy the QUAD condition or weak-QUAD condition, some new approaches should be proposed. Compared with the obtained results, we improve the QUAD condition, and propose two weaker conditions for local and global exponential synchronizations of LCODEs respectively on the one hand. On the other hand, our results are suitable for any linearly coupling configuration. This is the requirement that all the existing results for the exponential synchronization of LCODEs can almost never meet.

This paper is organized as follows. Section 2 shows the method of studying the synchronization of LCODEs, and several necessary definitions and lemmas are given. Section 3 presents the main results and their proofs, then several useful corollaries are proposed. The theoretical results are simply applied to the exponential synchronization analysis of linearly coupled Hopfield neural networks, and several simulations are provided in Section 4.

The following notations are useful in this paper. Let $\boldsymbol{R}^n$ and $\boldsymbol{R}^{m\times n}$ denote respectively the $n$-dimensional real vector space and the whole real matrices with $m$ rows and $n$ columns. Let $\boldsymbol{C}^n$ and $\boldsymbol{C}^{m\times n}$ denote respectively the $n$-dimensional complex vector space and the whole complex matrices with $m$ rows and $n$ columns. Let $\boldsymbol{O}_{m\times n}$ and $\boldsymbol{E}_m$ denote respectively a zero matrix with $m$ rows and $n$ columns and the $m$-order identity matrix. We denote the transpose and the Hermitian adjoint of $\boldsymbol{A}\in\boldsymbol{C}^{m\times n}$ by $\boldsymbol{A}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{H}}$, respectively. If $\boldsymbol{A}\in\boldsymbol{R}^{m\times m}$ and $\lambda$ is an eigenvalue of $\boldsymbol{A}$, then we denote the determinant of $\boldsymbol{A}$, the inverse of $\boldsymbol{A}$, and the real part of $\lambda$ by $\det(\boldsymbol{A})$, $\boldsymbol{A}^{-1}$, and $\mathrm{Re}(\lambda)$, respectively. Let $\boldsymbol{A}^{\mathrm{s}}=(\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}})/2$. If $\boldsymbol{A}$ is symmetric, then we denote the minimum eigenvalue and the maximum one of $\boldsymbol{A}$ by $\lambda_{\min}(\boldsymbol{A})$ and $\lambda_{\max}(\boldsymbol{A})$, respectively. Let $\otimes$ denote the Kronecker product of matrices. Let $\|\cdot\|$ denote the Euclidean norm on $\boldsymbol{C}^n$ or the spectral norm on $\boldsymbol{C}^{m\times n}$. Let $o(x)$ denote the infinitely small quantity of higher order than $x\in\boldsymbol{R}$.

## 2. Preliminaries

Generally, the coupled dynamical system described by LCODEs is given as follows:

$$\dot{\boldsymbol{x}}^i=\boldsymbol{f}(t,\boldsymbol{x}^i)+\sum_{j=1,j\neq i}^{m}a_{ij}\boldsymbol{\Gamma}(\boldsymbol{x}^j-\boldsymbol{x}^i),\quad i=1,2,\ldots,m,\quad(1)$$

where $\boldsymbol{x}^i=[x_1^i,x_2^i,\cdots,x_n^i]^{\mathrm{T}}\in\boldsymbol{R}^n$ is the state variable of the $i$-th node only related with the time $t$, $\boldsymbol{f}:[0,+\infty)\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ is a continuous map, $\boldsymbol{\Gamma}\in\boldsymbol{R}^{n\times n}$ and $a_{ij}\in R$ with assumption that $\boldsymbol{\Gamma}\neq\boldsymbol{O}_{n\times n}$ and $a_{ij}\geq 0$, $i\neq j$.

Let $\boldsymbol{x}=[\boldsymbol{x}^{1\mathrm{T}},\boldsymbol{x}^{2\mathrm{T}},\cdots,\boldsymbol{x}^{m\mathrm{T}}]^{\mathrm{T}}\in\boldsymbol{R}^{mn}$, $a_{ii}=-\sum_{j=1,j\neq i}^{m}a_{ij}$, $i=1,2,\cdots,m$, and $\boldsymbol{A}=(a_{ij})\in\boldsymbol{R}^{m\times m}$. Then system (1) can be written as

$$\dot{\boldsymbol{x}}^i=\boldsymbol{f}(t,\boldsymbol{x}^i)+\sum_{j=1}^{m}a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j,\ i=1,2,\cdots,m.$$

Moreover,

$$\dot{\boldsymbol{x}}^i=\begin{bmatrix}\boldsymbol{f}(t,\boldsymbol{x}^1)\\ \vdots\\ \boldsymbol{f}(t,\boldsymbol{x}^m)\end{bmatrix}+(\boldsymbol{A}\otimes\boldsymbol{\Gamma})\boldsymbol{x}.\quad(2)$$

$\boldsymbol{A}$ is the coupling matrix with zero-sum rows, which is determined by the topological structure of the LCODEs.

**Definition 1** If there exists a constant $\delta>0$ such that for any initial state $\boldsymbol{x}(0)$ satisfying $\|\boldsymbol{x}^i(0)-\boldsymbol{x}^j(0)\|\leq\delta$, $i,j=1,2,\ldots,m$, there exist $\varepsilon>0$, $M\geq 0$, and $T\geq 0$ such that

$$\|\boldsymbol{x}^i(t)-\boldsymbol{x}^j(t)\|\leq M\mathrm{e}^{-\varepsilon t},\quad i,j=1,2,\cdots,m,$$

for all $t\geq T$, then system (1) is locally exponentially synchronized. Moreover, if for any initial state $\boldsymbol{x}(0)$, there exist $\varepsilon>0$, $M\geq 0$, and $T\geq 0$ such that the above formulas hold for all $t\geq T$, then system (1) is globally exponentially synchronized.

Definition 1 describes the local and global exponential synchronizations of the LCODEs.

**Definition 2** Suppose that $\boldsymbol{P}\in\boldsymbol{R}^{n\times n}$ is a symmetric and positive definite matrix and $\boldsymbol{\Delta}\in\boldsymbol{R}^{n\times n}$. $\mathscr{F}_{\mathrm{quad}}(\boldsymbol{\Delta},\boldsymbol{P})$ denotes a class of continuous maps $\boldsymbol{f}:[0,+\infty)\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ satisfying

$$(\boldsymbol{u}-\boldsymbol{v})^{\mathrm{T}}\boldsymbol{P}[(\boldsymbol{f}(t,\boldsymbol{u})-\boldsymbol{f}(t,\boldsymbol{v}))-\boldsymbol{\Delta}(\boldsymbol{u}-\boldsymbol{v})]\leq 0,\quad(3)$$

for all $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{R}^n$ and all $t\geq 0$.

**Definition 3** Suppose that $\boldsymbol{P}\in\boldsymbol{R}^{n\times n}$ is a symmetric and positive definite matrix and $\boldsymbol{\Delta}\in\boldsymbol{R}^{n\times n}$. $\mathscr{F}_{\mathrm{pard}}(\boldsymbol{\Delta},\boldsymbol{P})$ denotes a class of continuous maps $\boldsymbol{f}:[0,+\infty)\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ satisfying that for any $t\geq 0$ and any $\boldsymbol{v}\in\boldsymbol{R}^n$, there exists a matrix $\boldsymbol{D}(t,\boldsymbol{v})\in\boldsymbol{R}^{n\times n}$ such that

$$\lim_{\|\boldsymbol{u}-\boldsymbol{v}\|\to 0}\frac{\|\boldsymbol{f}(t,\boldsymbol{u})-\boldsymbol{f}(t,\boldsymbol{v})-\boldsymbol{D}(t,\boldsymbol{v})(\boldsymbol{u}-\boldsymbol{v})\|}{\|\boldsymbol{u}-\boldsymbol{v}\|}=0,\quad(4)$$

$$(\boldsymbol{u}-\boldsymbol{v})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{D}(t,\boldsymbol{v})-\boldsymbol{\Delta})(\boldsymbol{u}-\boldsymbol{v})\leq 0,\quad(5)$$

for all $\boldsymbol{u}\in\boldsymbol{R}^n$.

It is not difficult to see that $\mathscr{F}_{\mathrm{quad}}(\boldsymbol{\Delta},\boldsymbol{P})$ and $\mathscr{F}_{\mathrm{pard}}(\boldsymbol{\Delta},\boldsymbol{P})$ have the following connection. If for any $t\geq 0$ and any $\boldsymbol{v}\in\boldsymbol{R}^n$, there exists a matrix $\boldsymbol{D}(t,\boldsymbol{v})\in\boldsymbol{R}^{n\times n}$ such that formula (4) holds, then $\boldsymbol{f}\in\mathscr{F}_{\mathrm{quad}}(\boldsymbol{\Delta},\boldsymbol{P})$implies $\boldsymbol{f}\in\mathscr{F}_{\mathrm{pard}}(\boldsymbol{\Delta},\boldsymbol{P})$. When the map $\boldsymbol{f}(t,\boldsymbol{v})$ is differential with respect to $\boldsymbol{f}$ on its domain, formula (4) can be obtained.

To analyze the exponential synchronization of system (1), let $\boldsymbol{y}^i=\boldsymbol{x}^{i+1}-\boldsymbol{x}^1$, $i=1,2,\ldots,m-1$ and $\boldsymbol{y}=[\boldsymbol{y}^{1\mathrm{T}},\boldsymbol{y}^{2\mathrm{T}},\ldots,\boldsymbol{y}^{m-1\mathrm{T}}]^{\mathrm{T}}\in\boldsymbol{R}^{(m-1)n}$. For every $i=1,2,\ldots,m$, it is not difficult to calculate that

$$\begin{aligned}&\sum_{j=1,j\neq i}^{m}a_{ij}\boldsymbol{\Gamma}(\boldsymbol{x}^j-\boldsymbol{x}^i)\\&=\sum_{j=2,j\neq i}^{m}a_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^{j-1}-\sum_{j=1,j\neq i}^{m}a_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^{i-1}=\sum_{j=1}^{m-1}c_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j,\end{aligned}$$

where

$$c_{ij}=\begin{cases} a_{i,j+1}, & j\neq i-1,\\ -\sum\limits_{k=1,k\neq i}^{m} a_{ik}, & j=i-1.\end{cases}$$

Suppose $\boldsymbol{C}_i=[c_{i1},c_{i2},\cdots,c_{i,m-1}]\in\boldsymbol{R}^{1\times(m-1)}$, then

$$\sum_{j=1,j\neq i}^{m} a_{ij}\boldsymbol{\Gamma}(\boldsymbol{x}^j-\boldsymbol{x}^i)=(\boldsymbol{C}_i\otimes\boldsymbol{\Gamma})\boldsymbol{y}^i.$$

And suppose

$$\boldsymbol{C}=\begin{bmatrix}\boldsymbol{C}_2-\boldsymbol{C}_1\\ \vdots\\ \boldsymbol{C}_m-\boldsymbol{C}_1\end{bmatrix}\in\boldsymbol{R}^{(m-1)\times(m-1)}.$$

Then for every $i=1,2,\ldots,m-1$,

$$\begin{aligned}\dot{\boldsymbol{y}}^i &= \dot{\boldsymbol{x}}^{i+1}-\dot{\boldsymbol{x}}^1\\ &= \boldsymbol{f}(t,\boldsymbol{x}^{i+1})-\boldsymbol{f}(t,\boldsymbol{x}^1)+[(\boldsymbol{C}_{i+1}-\boldsymbol{C}_1)\otimes\boldsymbol{\Gamma}]\boldsymbol{y}^i.\end{aligned}$$

Moreover,

$$\dot{\boldsymbol{y}}=\begin{bmatrix}\boldsymbol{f}(t,\boldsymbol{x}^2)-\boldsymbol{f}(t,\boldsymbol{x}^1)\\ \vdots\\ \boldsymbol{f}(t,\boldsymbol{x}^m)-\boldsymbol{f}(t,\boldsymbol{x}^1)\end{bmatrix}+(\boldsymbol{C}\otimes\boldsymbol{\Gamma})\boldsymbol{y}. \tag{6}$$

System (1) is exponentially synchronized when the trajectories of system (6) exponentially converge to the origin.

The following marks and lemmas are useful in the rest of the development.

Suppose

$$\boldsymbol{L}=\begin{bmatrix}-1&1&0&0&\cdots&0&0\\ -1&0&1&0&\cdots&0&0\\ -1&0&0&1&\cdots&0&0\\ \vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\ -1&0&0&0&\cdots&0&1\end{bmatrix}\in\boldsymbol{R}^{(m-1)\times m}.$$

Let $\boldsymbol{L}=[\boldsymbol{L}_1,\boldsymbol{L}_2]$, where $\boldsymbol{L}_1\in\boldsymbol{R}^{(m-1)\times1}$ and $\boldsymbol{L}_2\in\boldsymbol{R}^{(m-1)\times(m-1)}$, and

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{A}_{11}&\boldsymbol{A}_{12}\\ \boldsymbol{A}_{21}&\boldsymbol{A}_{22}\end{bmatrix},$$

where $\boldsymbol{A}_{11}\in\boldsymbol{R}^{1\times1}$, $\boldsymbol{A}_{12}\in\boldsymbol{R}^{1\times(m-1)}$, $\boldsymbol{A}_{21}\in\boldsymbol{R}^{(m-1)\times1}$, and $\boldsymbol{A}_{22}\in\boldsymbol{R}^{(m-1)\times(m-1)}$. Then $\boldsymbol{L}_2=\boldsymbol{E}_{m-1}$.

**Lemma 1** The coupling matrix $\boldsymbol{A}$ has at least an eigenvalue 0. If $\lambda$ is an eigenvalue of $\boldsymbol{A}$ and $\lambda\neq0$, then $\mathrm{Re}(\lambda)<0$.

By the Gerschgorin theorem,[28] we can prove Lemma 1. The results related with Lemma 1 can be found in Ref. [20].

**Lemma 2** $\boldsymbol{C}$ and $\boldsymbol{A}$ have the following relationships:

(a) $\boldsymbol{CL}=\boldsymbol{LA}$,

(b) $\boldsymbol{C}=\boldsymbol{L}_1\boldsymbol{A}_{12}+\boldsymbol{A}_{22}$,

(c) $\lambda_1=0,\lambda_2,\ldots,\lambda_m$ are the $m$ eigenvalues of $\boldsymbol{A}$ if and only if $\lambda_2,\ldots,\lambda_m$ are the $m-1$ eigenvalues of $\boldsymbol{C}$.

**Proof** It can be shown that $\boldsymbol{y}=(\boldsymbol{L}\otimes\boldsymbol{E}_n)\boldsymbol{x}$. By formulas (2) and (6), we have

$$(\boldsymbol{L}\otimes\boldsymbol{E}_n)(\boldsymbol{A}\otimes\boldsymbol{\Gamma})\boldsymbol{x}=(\boldsymbol{C}\otimes\boldsymbol{\Gamma})(\boldsymbol{L}\otimes\boldsymbol{E}_n)\boldsymbol{x},$$

namely,

$$(\boldsymbol{LA}\otimes\boldsymbol{\Gamma})\boldsymbol{x}=(\boldsymbol{CL}\otimes\boldsymbol{\Gamma})\boldsymbol{x},$$

for all $\boldsymbol{x}\in\boldsymbol{R}^{mn}$. Because $\boldsymbol{\Gamma}$ is a non-zero matrix, we have $\boldsymbol{CLu}=\boldsymbol{LAu}$ for all $\boldsymbol{u}\in\boldsymbol{R}^n$. Thus, $\boldsymbol{CL}=\boldsymbol{LA}$. It implies that $\boldsymbol{C}=\boldsymbol{L}_1\boldsymbol{A}_{12}+\boldsymbol{A}_{22}$. Thus (a) and (b) hold.

Next, it is easy to verify that

$$\begin{aligned}&\begin{bmatrix}1&\boldsymbol{O}_{1\times(m-1)}\\ \boldsymbol{L}_1&\boldsymbol{E}_{m-1}\end{bmatrix}\boldsymbol{A}\\ &=\begin{bmatrix}0&\boldsymbol{A}_{12}\\ \boldsymbol{O}_{(m-1)\times1}&\boldsymbol{C}\end{bmatrix}\begin{bmatrix}1&\boldsymbol{O}_{1\times(m-1)}\\ \boldsymbol{L}_1&\boldsymbol{E}_{m-1}\end{bmatrix}.\end{aligned}$$

Suppose that

$$\tilde{\boldsymbol{L}}=\begin{bmatrix}1&\boldsymbol{O}_{1\times(m-1)}\\ \boldsymbol{L}_1&\boldsymbol{E}_{m-1}\end{bmatrix},\quad\tilde{\boldsymbol{C}}=\begin{bmatrix}0&\boldsymbol{A}_{12}\\ \boldsymbol{O}_{(m-1)\times1}&\boldsymbol{C}\end{bmatrix}.$$

Obviously, $\tilde{\boldsymbol{L}}$ is reversible. Thus, we have

$$\begin{aligned}\det(\lambda\boldsymbol{E}_m-\boldsymbol{A}) &= \det(\lambda\boldsymbol{E}_m-\tilde{\boldsymbol{L}}^{-1}\tilde{\boldsymbol{C}}\tilde{\boldsymbol{L}})\\ &= \det(\lambda\boldsymbol{E}_m-\tilde{\boldsymbol{C}})\\ &= \lambda\det(\lambda\boldsymbol{E}_m-\boldsymbol{C}).\end{aligned}$$

This implies that (c) holds.

Lemma 2 gives the representation of $\boldsymbol{C}$ by the coupling matrix $\boldsymbol{A}$.

## 3. Main results

In this section, we investigate the exponential synchronization of system (1) with $\boldsymbol{f}$, $\boldsymbol{\Gamma}$, and $\boldsymbol{A}$, and present several results different from the existing ones.

**Theorem 1** Assume that there exists a symmetric and positive definite matrix $\boldsymbol{P}\in\boldsymbol{R}^{n\times n}$ and a matrix $\boldsymbol{\Delta}\in\boldsymbol{R}^{n\times n}$ such that $\boldsymbol{f}\in\mathscr{F}_{\mathrm{quad}}(\boldsymbol{\Delta},\boldsymbol{P})$. If there exists a real positive definite diagonal matrix $\boldsymbol{\Xi}=\mathrm{diag}\{\xi_1,\xi_2,\cdots,\xi_{m-1}\}$ such that $\boldsymbol{\Xi}\otimes\boldsymbol{P\Delta}+\boldsymbol{\Xi}C\otimes\boldsymbol{P\Gamma}$ is negative definite, then system (1) is globally exponentially synchronized.

**Proof** Let

$$V(t)=\frac{1}{2}\boldsymbol{y}(t)^{\mathrm{T}}(\boldsymbol{\Xi}\otimes\boldsymbol{P})\boldsymbol{y}(t). \tag{7}$$

For any right maximally defined solution $\boldsymbol{y}=\boldsymbol{y}(t)$ of Eq. (6) with the initial time 0, let $V(t)=V(\boldsymbol{y}(t))$. Then the derivative $\dot{V}(t)$ of $V$ along the solution $\boldsymbol{y}=\boldsymbol{y}(t)$ of Eq. (6) satisfies

$$\begin{aligned}\dot{V}(t) &= \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Xi}\otimes\boldsymbol{P})\dot{\boldsymbol{y}}\\ &= \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Xi}\otimes\boldsymbol{P})\begin{bmatrix}\boldsymbol{f}(t,\boldsymbol{x}^2)-\boldsymbol{f}(t,\boldsymbol{x}^1)\\ \vdots\\ \boldsymbol{f}(t,\boldsymbol{x}^m)-\boldsymbol{f}(t,\boldsymbol{x}^1)\end{bmatrix}\\ &\quad+\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Xi}\otimes\boldsymbol{P})(\boldsymbol{C}\otimes\boldsymbol{\Gamma})\boldsymbol{y}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m-1} \xi_i \boldsymbol{y}^{i\mathrm{T}} \boldsymbol{P}(\boldsymbol{f}(t,\boldsymbol{x}^{i+1}) - \boldsymbol{f}(t,\boldsymbol{x}^1)) + \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Xi}\boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma})\boldsymbol{y} \\
&= \sum_{i=1}^{m-1} \xi_i \boldsymbol{y}^{i\mathrm{T}} \boldsymbol{P}[(\boldsymbol{f}(t,\boldsymbol{x}^{i+1}) - \boldsymbol{f}(t,\boldsymbol{x}^1)) - \boldsymbol{\Delta}\boldsymbol{y}^i] \\
&\quad + \sum_{i=1}^{m-1} \xi_i \boldsymbol{y}^{i\mathrm{T}} \boldsymbol{P}\boldsymbol{\Delta}\boldsymbol{y}^i + \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Xi}\boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma})\boldsymbol{y} \\
&\le \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Xi} \otimes \boldsymbol{P}\boldsymbol{\Delta} + \boldsymbol{\Xi}\boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma})\boldsymbol{y}. \qquad (8)
\end{aligned}
$$

Let $\boldsymbol{G} = \boldsymbol{\Xi} \otimes \boldsymbol{P}\boldsymbol{\Delta} + \boldsymbol{\Xi}\boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma}$. Then $\boldsymbol{G}$ is negative definite. By the Rayleigh–Ritz theorem,[28]

$$
\lambda_{\max}(\boldsymbol{G}^{\mathrm{s}}) = \max_{y \neq 0} \frac{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{s}}\boldsymbol{y}}{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}} = \max_{\boldsymbol{y} \neq \boldsymbol{0}} \frac{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{y}}{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}}.
$$

It is easy to see that $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{y} \le \lambda_{\max}(\boldsymbol{G}^{\mathrm{s}})\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}$ and $\lambda_{\max}(\boldsymbol{G}^{\mathrm{s}}) < 0$. Similarly, we obtain

$$
\frac{2V(\boldsymbol{y})}{\lambda_{\max}(\boldsymbol{P}) \max\limits_{1 \le i \le m-1} \xi_i} \le \boldsymbol{y}^{\mathrm{T}}\boldsymbol{y},
$$

where $\lambda_{\max}(\boldsymbol{P}) > 0$ and $\max\limits_{1 \le i \le m-1} \xi_i > 0$. So we have

$$
\dot{V}(t) \le \boldsymbol{y}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{y} \le \lambda_{\max}(\boldsymbol{G}^{\mathrm{s}})\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y} \le \frac{2\lambda_{\max}(\boldsymbol{G}^{\mathrm{s}})}{\lambda_{\max}(\boldsymbol{P}) \max\limits_{1 \le i \le m-1} \xi_i} V(t).
$$

It can be shown that both $\boldsymbol{y}(t)$ and $V(t)$ are defined on $[0, +\infty)$, and moreover, exponentially converge to 0 as $t \to +\infty$. Hence, we prove this theorem.

**Theorem 2** Assume that there exists a symmetric and positive definite matrix $\boldsymbol{P} \in \boldsymbol{R}^{n \times n}$ and a matrix $\boldsymbol{\Delta} \in \boldsymbol{R}^{n \times n}$ such that $\boldsymbol{f} \in \mathscr{F}_{\mathrm{pard}}(\boldsymbol{\Delta}, \boldsymbol{P})$. If there exists a symmetric and positive definite matrix $\boldsymbol{\Pi} \in \boldsymbol{R}^{(m-1)\times(m-1)}$ such that $\boldsymbol{\Pi} \otimes \boldsymbol{P}\Delta + \boldsymbol{\Pi}\boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma}$ is negative definite, then system (1) is locally exponentially synchronized.

**Proof** Suppose that

$$
V_1(t) = \frac{1}{2}\boldsymbol{y}(t)^{\mathrm{T}}(\boldsymbol{\Pi} \otimes \boldsymbol{P})\boldsymbol{y}(t). \qquad (9)
$$

Then the derivative $\dot{V}_1(t)$ of $V_1$ along the solution $\boldsymbol{y} = \boldsymbol{y}(t)$ of (6) satisfies

$$
\begin{aligned}
\dot{V}_1(t) &= \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Pi} \otimes \boldsymbol{P})\dot{\boldsymbol{y}} \\
&= \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Pi} \otimes \boldsymbol{P}) \begin{bmatrix} \boldsymbol{f}(t,\boldsymbol{x}^2) - \boldsymbol{f}(t,\boldsymbol{x}^1) \\ \vdots \\ \boldsymbol{f}(t,\boldsymbol{x}^m) - \boldsymbol{f}(t,\boldsymbol{x}^1) \end{bmatrix} \\
&\quad + \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Pi} \otimes \boldsymbol{P})(\boldsymbol{C} \otimes \boldsymbol{\Gamma})\boldsymbol{y} \\
&= \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Pi} \otimes \boldsymbol{P}) \begin{bmatrix} \boldsymbol{D}(t,\boldsymbol{x}^1)\boldsymbol{y}^1 \\ \vdots \\ \boldsymbol{D}(t,\boldsymbol{x}^1)\boldsymbol{y}^{m-1} \end{bmatrix} \\
&\quad + o(\| \boldsymbol{y} \|^2) + \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Pi}\boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma})\boldsymbol{y} \\
&= \boldsymbol{y}^{\mathrm{T}}[\boldsymbol{\Pi} \otimes \boldsymbol{P}(\boldsymbol{D}(t,\boldsymbol{x}^1) - \boldsymbol{\Delta})]\boldsymbol{y} \\
&\quad + \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Pi} \otimes \boldsymbol{P}\boldsymbol{\Delta} + \boldsymbol{\Pi}\boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma})\boldsymbol{y} + o(\| \boldsymbol{y} \|^2). \qquad (10)
\end{aligned}
$$

Let $\boldsymbol{Q}_1(t,\boldsymbol{x}^1) = \boldsymbol{P}(\boldsymbol{D}(t,\boldsymbol{x}^1))$, and $\boldsymbol{Q} = \boldsymbol{\Pi} \otimes \boldsymbol{P}\boldsymbol{\Delta} + \boldsymbol{\Pi}\boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma}$, where $\boldsymbol{D}(t,\boldsymbol{x}^1)$ is given in Definition 3 with $\boldsymbol{v} = \boldsymbol{x}^1$. Then formula (5) implies that for any $t \ge 0$ and any $\boldsymbol{x}^1 \in \boldsymbol{R}^n$, $\boldsymbol{Q}_1(t,\boldsymbol{x}^1)$ is semi-negative definite, and thus, $\boldsymbol{\Pi} \otimes \boldsymbol{Q}_1(t,\boldsymbol{x}^1)$ is also semi-negative definite. Since $\boldsymbol{Q}$ is negative definite, then similarly to the proof of Theorem 1, we have $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{y} \le \lambda_{\max}(\boldsymbol{Q}^{\mathrm{s}})\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}$ and $\lambda_{\max}(\boldsymbol{Q}^{\mathrm{s}}) < 0$. Hence,

$$
\dot{V}_1(t) \le \lambda_{\max}(\boldsymbol{Q}^{\mathrm{s}})\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y} + o(\| \boldsymbol{y} \|^2). \qquad (11)
$$

Since

$$
\lim_{\|\boldsymbol{y}\| \to 0} \frac{o(\| \boldsymbol{y} \|^2)}{\| \boldsymbol{y} \|^2} = 0,
$$

it is shown that there exists a constant $\delta > 0$ such that if $\| \boldsymbol{y} \| \le \delta$, then

$$
o(\| \boldsymbol{y} \|^2) \le -\frac{1}{2}\lambda_{\max}(\boldsymbol{Q}^{\mathrm{s}})\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}.
$$

By the Rayleigh–Ritz theorem,

$$
\frac{2V_1(\boldsymbol{y})}{\lambda_{\max}(\boldsymbol{\Pi})\lambda_{\max}(\boldsymbol{P})} \le \boldsymbol{y}^{\mathrm{T}}\boldsymbol{y},
$$

where $\lambda_{\max}(\boldsymbol{\Pi}) > 0$ and $\lambda_{\max}(\boldsymbol{P}) > 0$. Then we obtain that if $\| \boldsymbol{y} \| \le \delta$, then

$$
\dot{V}_1(t) \le \frac{\lambda_{\max}(\boldsymbol{Q}^{\mathrm{s}})}{\lambda_{\max}(\boldsymbol{\Pi})\lambda_{\max}(\boldsymbol{P})} V_1(t).
$$

This tells us that the key to proving this theorem is to find the appropriate initial states $\boldsymbol{y}(0)$ such that $\| \boldsymbol{y} \| \le \delta$.

In fact, it can be shown that when

$$
\| \boldsymbol{y}(0) \| \le \frac{\delta}{2}\sqrt{\frac{\lambda_{\min}(\boldsymbol{\Pi})\lambda_{\min}(\boldsymbol{P})}{\lambda_{\max}(\boldsymbol{\Pi})\lambda_{\max}(\boldsymbol{P})}},
$$

where $\lambda_{\min}(\boldsymbol{\Pi}) > 0$ and $\lambda_{\min}(\boldsymbol{P}) > 0$, we have $\| \boldsymbol{y}(t) \| \le \delta$ on its domain. At this time, we can prove that both $\boldsymbol{y}(t)$ and $V_1(t)$ are defined on $[0, +\infty)$, and moreover, exponentially converge to 0 as $t \to +\infty$. Hence, this theorem is proved.

Theorem 1 and Theorem 2 give the criteria for the exponential synchronization of the LCODEs. For convenience, we propose the following corollaries.

**Corollary 1** Assume that the continuous map $\boldsymbol{f}$ satisfies the so-called weak-QUAD condition,[20] i.e., there exists a real positive definite diagonal matrix $\boldsymbol{P}$ and a real diagonal matrix $\boldsymbol{\Delta}$ such that $\boldsymbol{f} \in \mathscr{F}_{\mathrm{quad}}(\boldsymbol{\Delta}, \boldsymbol{P})$. If $\boldsymbol{E}_{m-1} \otimes \boldsymbol{P}\boldsymbol{\Delta} + \boldsymbol{C} \otimes \boldsymbol{P}\boldsymbol{\Gamma}$ is negative definite, then system (1) is globally exponentially synchronized.

**Proof** Corollary 1 can be directly obtained from Theorem 1 with $\boldsymbol{\Xi} = \boldsymbol{E}_{m-1}$.

**Corollary 2** Assume that the continuous map $\boldsymbol{f}$ satisfies the global Lipschitz condition, i.e., there exists a constant $\alpha > 0$ such that

$$
\| \boldsymbol{f}(t,\boldsymbol{u}) - \boldsymbol{f}(t,\boldsymbol{v}) \| \le \alpha \| \boldsymbol{u} - \boldsymbol{v} \|,
$$

for all $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{R}^n$ and all $t\geq 0$. If $\boldsymbol{E}_{m-1}\otimes\alpha\boldsymbol{E}_{\boldsymbol{n}}+\boldsymbol{C}\otimes\boldsymbol{\Gamma}$ is negative definite, then system (1) is globally exponentially synchronized.

**Proof** Since $\boldsymbol{f}$ satisfies the global Lipschitz condition, we can obtain

$$(\boldsymbol{u}-\boldsymbol{v})^{\mathrm{T}}[(\boldsymbol{f}(t,\boldsymbol{u})-\boldsymbol{f}(t,\boldsymbol{v}))-\alpha(\boldsymbol{u}-\boldsymbol{v})]\leq 0.$$

Thus, $\boldsymbol{f}\in\mathscr{F}_{\mathrm{quad}}(\alpha\boldsymbol{E}_n,\boldsymbol{E}_n)$. By corollary 1, corollary 2 is proved.

**Corollary 3** Assume that the continuous map $\boldsymbol{f}$ satisfies the following condition:

$$(\boldsymbol{u}-\boldsymbol{v})^{\mathrm{T}}(\boldsymbol{f}(t,\boldsymbol{u})-\boldsymbol{f}(t,\boldsymbol{v}))\leq 0,$$

for all $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{R}^n$ and all $t\geq 0$. Then the following results hold:

(a) If $\boldsymbol{C}\otimes\boldsymbol{\Gamma}$ is negative definite, then system (1) is globally exponentially synchronized.

(b) If $\boldsymbol{f}(t,\boldsymbol{v})$ is differential with respect to $\boldsymbol{v}$ on its domain, $\det(\boldsymbol{C})\neq 0$, and $\boldsymbol{\Gamma}$ is symmetric and positive definite, then system (1) is locally exponentially synchronized.

**Proof** It is easy to see that $\boldsymbol{f}\in\mathscr{F}_{\mathrm{quad}}(\boldsymbol{O}_{n\times n},\boldsymbol{E}_n)$. Thus, by corollary 1, (a) holds.

Next, we need to prove (b). It can be shown that $\boldsymbol{f}\in\mathscr{F}_{\mathrm{pard}}(\boldsymbol{O}_{n\times n},\boldsymbol{E}_n)$. Thus, by Theorem 2, we only need to find a symmetric and positive definite matrix $\boldsymbol{\Pi}\in\boldsymbol{R}^{(m-1)\times(m-1)}$ such that $\boldsymbol{\Pi C}\otimes\boldsymbol{\Gamma}$ is negative definite.

Let $\lambda_1=0,\lambda_2,\ldots,\lambda_m$ denote all the eigenvalues of $\boldsymbol{A}$. By Lemma 2, all the eigenvalues of $\boldsymbol{C}$ are $\lambda_2,\ldots,\lambda_m$. Because $\det(\boldsymbol{C})\neq 0$, all the eigenvalues of $C$ are non-zero. Again by Lemma 1, $\mathrm{Re}(\lambda_j)<0$ for all $j=2,\ldots,m$. This implies that the real parts of the eigenvalues of $-\boldsymbol{C}$ are all positive, namely, $-\boldsymbol{C}$ is a positive stable matrix. Thus, there exists a symmetric and positive definite matrix $\boldsymbol{W}\in\boldsymbol{R}^{(m-1)\times(m-1)}$ such that $(-\boldsymbol{C})^{\mathrm{T}}\boldsymbol{W}+\boldsymbol{W}(-\boldsymbol{C})$ is symmetric and positive definite,[29] i.e., $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{W}+\boldsymbol{WC}$ is symmetric and negative definite. Moreover, $(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{W}+\boldsymbol{WC})\otimes\boldsymbol{\Gamma}$ is symmetric and negative definite. When $\boldsymbol{\Pi}=\boldsymbol{W}$,

$$\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{\Pi C}\otimes\boldsymbol{\Gamma})y=\frac{1}{2}y^{\mathrm{T}}[(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{W}+\boldsymbol{WC})\otimes\Gamma]\boldsymbol{y},$$

so $\boldsymbol{\Pi C}\otimes\boldsymbol{\Gamma}$ is negative definite. Hence, (b) is proved.

The above three corollaries all involve the negative definiteness of the matrices in the form of $\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}+\boldsymbol{C}\otimes\boldsymbol{P\Gamma}$, where $\boldsymbol{P}$, $\boldsymbol{\Delta}$, $\boldsymbol{C}$, and $\boldsymbol{\Gamma}$ are as in corollary 1. The following proposition lists some conditions to make sure that such matrices are negative definite.

**Proposition 1** Suppose that $\boldsymbol{P}=\mathrm{diag}\{p_1,p_2,\ldots,p_n\}$ is a real positive definite diagonal matrix and $\boldsymbol{\Delta}=\mathrm{diag}\{\delta_1,\delta_2,\ldots,\delta_n\}$ is a real diagonal matrix. Choose $\boldsymbol{C}=\boldsymbol{L}_1\boldsymbol{A}_{12}+\boldsymbol{A}_{22}$. Let $\lambda_1^{\mathrm{s}},\lambda_2^{\mathrm{s}},\ldots,\lambda_{m-1}^{\mathrm{s}}$ denote all the eigenvalues of $\boldsymbol{C}^{\mathrm{s}}$ and let $\lambda_1=0,\lambda_2,\ldots,\lambda_m$ denote all the eigenvalues of $\boldsymbol{A}$. Then we have that

(a) if $\boldsymbol{\Gamma}=\mathrm{diag}\{\gamma_1,\gamma_2,\cdots,\gamma_n\}$ is a diagonal matrix, then the negative definiteness of $\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}+\boldsymbol{C}\otimes\boldsymbol{P\Gamma}$ is equivalent to

$$\max\{p_i\delta_i+p_i\gamma_i\lambda_j^{\mathrm{s}}:1\leq i\leq n,\ 1\leq j\leq m-1\}<0,$$

in particular, if $\boldsymbol{C}$ is normal, then the negative definiteness of $\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}+\boldsymbol{C}\otimes\boldsymbol{P\Gamma}$ is equivalent to

$$\max\{p_i\delta_i+p_i\gamma_i\mathrm{Re}(\lambda_j):1\leq i\leq n,\ 2\leq j\leq m\}<0;$$

(b) if $\boldsymbol{P}=\boldsymbol{E}_n$, $\boldsymbol{\Delta}=\alpha\boldsymbol{E}_n$, and $\boldsymbol{\Gamma}$ is a symmetric matrix with all the eigenvalues $\lambda_1^*,\lambda_2^*,\ldots,\lambda_n^*$, where $\alpha\geq 0$, then the negative definiteness of $\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}+\boldsymbol{C}\otimes\boldsymbol{P\Gamma}$ is equivalent to

$$\max\{\lambda_i^*\lambda_j^{\mathrm{s}}:1\leq i\leq n,\ 1\leq j\leq m-1\}<-\alpha,$$

in particular, $\boldsymbol{C}$ is normal, then the negative definiteness of $\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}+\boldsymbol{C}\otimes\boldsymbol{P\Gamma}$ is equivalent to

$$\max\{\lambda_i^*\mathrm{Re}(\lambda_j):1\leq i\leq n,\ 2\leq j\leq m\}<-\alpha.$$

**Proof** Suppose $\boldsymbol{G}=\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}+\boldsymbol{C}\otimes\boldsymbol{P\Gamma}$ and $\boldsymbol{y}\in\boldsymbol{R}^{(m-1)n}$. Then $\boldsymbol{G}$ is negative definite if and only if $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Gy}<0$ for all the non-zero vectors $\boldsymbol{y}$.

Suppose $\boldsymbol{y}$ is non-zero. We have

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Gy}=\boldsymbol{y}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{s}}\boldsymbol{y}=\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}+\boldsymbol{C}^{\mathrm{s}}\otimes\boldsymbol{P\Gamma})\boldsymbol{y}.$$

Because $\boldsymbol{C}^{\mathrm{s}}$ is a real symmetric matrix, we have that there exists a real orthogonal matrix $\boldsymbol{U}_1\in\boldsymbol{R}^{(m-1)\times(m-1)}$ such that $\boldsymbol{U}_1\boldsymbol{C}^{\mathrm{s}}\boldsymbol{U}_1{}^{\mathrm{T}}=\boldsymbol{\Lambda}_1=\mathrm{diag}\{\lambda_1^{\mathrm{s}},\lambda_2^{\mathrm{s}},\ldots,\lambda_{m-1}^{\mathrm{s}}\}$, where $\lambda_j^{\mathrm{s}}\in R$.[28] Thus,

$$\begin{aligned}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Gy}&=\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{U}_1{}^{\mathrm{T}}\otimes\boldsymbol{E}_n)(\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}\\&\quad+\boldsymbol{U}_1\boldsymbol{C}^{\mathrm{s}}\boldsymbol{U}_1{}^{\mathrm{T}}\otimes\boldsymbol{P\Gamma})(\boldsymbol{U}_1\otimes\boldsymbol{E}_n)\boldsymbol{y}\\&=[(\boldsymbol{U}_1\otimes\boldsymbol{E}_n)y]^{\mathrm{T}}(\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}\\&\quad+\boldsymbol{\Lambda}_1\otimes\boldsymbol{P\Gamma})[(\boldsymbol{U}_1\otimes\boldsymbol{E}_n)\boldsymbol{y}].\end{aligned}$$

Obviously, $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Gy}<0$ for all the non-zero vectors $\boldsymbol{y}$ if and only if

$$\max\{p_i\delta_i+p_i\gamma_i\lambda_j^{\mathrm{s}}:1\leq i\leq n,\ 1\leq j\leq m-1\}<0.$$

Also, by Lemma 2(c), all the eigenvalues of $\boldsymbol{C}$ are $\lambda_2,\lambda_3,\ldots,\lambda_m$. If $C$ is normal, then there exists a unitary matrix $\boldsymbol{U}_2\in\mathbb{C}^{(m-1)\times(m-1)}$ such that $\boldsymbol{U}_2\boldsymbol{C}\boldsymbol{U}_2{}^{\mathrm{H}}=\boldsymbol{\Lambda}_2=\mathrm{diag}\{\lambda_2,\lambda_3,\ldots,\lambda_m\}$, where $\lambda_j\in\mathbb{C}$.[28] Moreover,

$$\begin{aligned}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Gy}&=\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{U}_2{}^{\mathrm{H}}\otimes\boldsymbol{E}_n)(\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}\\&\quad+\boldsymbol{U}_2\boldsymbol{C}\boldsymbol{U}_2{}^{\mathrm{T}}\otimes\boldsymbol{P\Gamma})(\boldsymbol{U}_2\otimes\boldsymbol{E}_n)\boldsymbol{y}\\&=[(\boldsymbol{U}_2\otimes\boldsymbol{E}_n)\boldsymbol{y}]^{\mathrm{T}}(\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}\\&\quad+\boldsymbol{\Lambda}_2\otimes\boldsymbol{P\Gamma})[(\boldsymbol{U}_2\otimes\boldsymbol{E}_n)\boldsymbol{y}].\end{aligned}$$

Suppose $\mathrm{Re}(\boldsymbol{\Lambda}_2)=\mathrm{diag}\{\mathrm{Re}(\lambda_2),\mathrm{Re}(\lambda_3),\ldots,\mathrm{Re}(\lambda_m)\}$. Then

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Gy}=[(\boldsymbol{U}_2\otimes\boldsymbol{E}_n)\boldsymbol{y}]^{\mathrm{T}}(\boldsymbol{E}_{m-1}\otimes\boldsymbol{P\Delta}$$

$$+\mathrm{Re}(\boldsymbol{\Lambda}_2)\otimes \boldsymbol{P\Gamma})[(\boldsymbol{U}_2\otimes \boldsymbol{E}_n)\boldsymbol{y}].$$

Of course, $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{y}<0$ for all the non-zero vectors $\boldsymbol{y}$ if and only if

$$\max\{p_i\delta_i+p_i\gamma_i\mathrm{Re}(\lambda_j):1\le i\le n,\ 2\le j\le m\}<0.$$

Hence, (a) is proved.

(b) can be proved in the same way as that of (a).

## 4. Applications and simulations

In this section, we apply the theoretical results of this paper to the exponential synchronization analysis of linearly coupled Hopfield neural networks and illustrate the effectiveness of the theoretical results through simulations.

In system (1), for each $i$, let $\boldsymbol{f}(t,\boldsymbol{x}^i)=-\boldsymbol{D}\boldsymbol{x}^i+\boldsymbol{T}\boldsymbol{g}(\boldsymbol{x}^i)+\boldsymbol{I}(t)$, where $\boldsymbol{D}=\mathrm{diag}\{D_1,D_2,\ldots,D_n\}$ is a real diagonal matrix, $\boldsymbol{T}\in\boldsymbol{R}^{n\times n}$, $\boldsymbol{g}:\boldsymbol{R}^n\to\boldsymbol{R}^n$ is a continuous map, and $\boldsymbol{I}(t)=[I_1(t),I_2(t),\ldots,I_n(t)]^{\mathrm{T}}\in\boldsymbol{R}^n$. Then system (1) can be written as

$$\begin{aligned}\dot{\boldsymbol{x}}^i &= -\boldsymbol{D}\boldsymbol{x}^i+\boldsymbol{T}\boldsymbol{g}(\boldsymbol{x}^i)+\boldsymbol{I}(t)\\ &\quad+\sum_{j=1,j\ne i}^{m}a_{ij}\boldsymbol{\Gamma}(\boldsymbol{x}^j-\boldsymbol{x}^i),\ \ i=1,2,\ldots,m.\end{aligned}\tag{12}$$

Let $\boldsymbol{g}$ be chosen as $g(\boldsymbol{x}^i)=[g_1(x_1^i),g_2(x_2^i),\ldots,g_n(x_n^i)]^{\mathrm{T}}$ with each $g_j$ continuous on $R$. Then as we know, system (12) describes the linearly coupled Hopfield neural networks, where $\boldsymbol{A}=(a_{ij})\in\boldsymbol{R}^{n\times n}$ with $a_{ii}=-\sum_{j=1,j\ne i}^{m}a_{ij}$ for all $i=1,2,\ldots,m$ is the coupling matrix and the differential equation $\dot{\boldsymbol{x}}^i=-\boldsymbol{D}\boldsymbol{x}^i+\boldsymbol{T}\boldsymbol{g}(\boldsymbol{x}^i)+\boldsymbol{I}(t)$ describes the dynamics of the $i$-th Hopfield neural network with no coupling.

For simplicity, we assume that $\boldsymbol{\Gamma}=\mathrm{diag}\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$, $g(x^i)=[g_0(x_1^i),g_0(x_2^i),\ldots,g_0(x_n^i)]^{\mathrm{T}}$ with $g_0:R\to R$ satisfying that $g_0$ is a continuous function and there exists a constant $\alpha>0$ such that

$$|g_0(u)-g_0(v)|\le\alpha|u-v|,$$

for all $u,v\in\mathbb{R}$. Then it is not difficult to verify that

$$f\in\mathscr{F}_{\mathrm{quad}}(-D+\alpha\parallel T\parallel E_n,E_n).$$

Let $\boldsymbol{C}=\boldsymbol{L}_1\boldsymbol{A}_{12}+\boldsymbol{A}_{22}$ and let $\lambda_1^{\mathrm{s}},\lambda_2^{\mathrm{s}},\ldots,\lambda_{m-1}^{\mathrm{s}}$ denote all the eigenvalues of $\boldsymbol{C}^{\mathrm{s}}$. Then by Corollary 1, Corollary 3, and Proposition 1, we have the following conclusions:

(B1) if

$$\max\{-D_i+\gamma_i\lambda_j^{\mathrm{s}}:1\le i\le n,\ 1\le j\le m-1\}<-\alpha\parallel\boldsymbol{T}\parallel,$$

then system (7) is globally exponentially synchronized;

(B2) if $g_0$ is differential on $R$, $\det(\boldsymbol{C})\ne0$, $\gamma_i>0$ for all $i=1,2,\ldots,n$, and

$$\min_{1\le i\le n}D_i\ge\alpha\parallel\boldsymbol{T}\parallel,$$

then system (7) is locally exponentially synchronized. In this case, we can see that system (7) is also globally exponentially synchronized because of the particularity of the map $g$.

Next, we illustrate the effectiveness of our results through simulations.

In simulations, suppose that in system (7), $n=m=3$, $\boldsymbol{D}=\boldsymbol{\Gamma}=\boldsymbol{E}_3$, $\boldsymbol{I}(t)\equiv\boldsymbol{O}_{n\times1}$, and

$$\boldsymbol{T}=\begin{bmatrix}1.25&-3.2&-3.2\\-3.2&1.1&-4.4\\-3.2&4.4&1\end{bmatrix}.$$

Then conclusions (B1) and (B2) mean that if $\max\{\lambda_1^{\mathrm{s}},\lambda_2^{\mathrm{s}}\}<1-7.0099\alpha$, or if $\alpha\le1/7.0099$, $g_0$ is differential on $R$, and $\det(\boldsymbol{C})\ne0$, then system (7) is globally exponentially synchronized.

Let $z=\parallel x^2-x^1\parallel+\parallel x^3-x^1\parallel$ and $w=\ln z$. If for any initial states $x^1(0),x^2(0),x^3(0)$, the curve $z=z(t)$ exponentially converges to 0, i.e., the curve $w=w(t)$ is located below some straight line related with the initial states, and all the straight lines related with the initial states are parallel to each other, then system (7) is exponentially synchronized. Hence, we can roughly determine the exponential synchronization of system (7) by observing the trajectories of $z$ and $w$ with respect to time $t$ in the following simulations.

**Simulation 1** In system (7), let $g_0(u)=(|u+1|-|u-1|)/2$ with $\alpha=1$ and let the coupling matrix be

$$\boldsymbol{A}=\begin{bmatrix}-4&1&3\\5.5&-6.5&1\\0.5&4&-4.5\end{bmatrix}.$$

Then it is easy to obtain that

$$\boldsymbol{C}=\begin{bmatrix}-7.5&-2\\3&-7.5\end{bmatrix},$$

and $\max\{\lambda_1^{\mathrm{s}},\lambda_2^{\mathrm{s}}\}=-7<1-7.0099$. Figure 1(a) shows that for the randomly chosen initial states, all the curves $w=w(t)$ are respectively located below the straight lines represented as the dotted lines, namely, $w(t)\le w(0)-1.9802t$; figure 1(b) shows that for the randomly chosen initial states, all the curves $z=z(t)$ converge to 0 when time $t$ is long enough. Thus, figure 1 reflects the exponential synchronization of system (7).

**Simulation 2** In system (12) as in Simulation 1, we only change the coupling matrix $\boldsymbol{A}$ into

$$\boldsymbol{A}=\begin{bmatrix}-0.1&0.1&0\\0&-4.1&4.1\\0&0&0\end{bmatrix}.$$

Then we have

$$\boldsymbol{C}=\begin{bmatrix}-4.2&4.1\\-0.1&0\end{bmatrix},$$

and $\max\{\lambda_1^{\mathrm{s}},\lambda_2^{\mathrm{s}}\}=0.8000>1-7.0099$. In Fig. 2, we can see that system (7) is not synchronized. This tells us that when $\max\{\lambda_1^{\mathrm{s}},\lambda_2^{\mathrm{s}}\}>1-7.0099$, system (7) might not be exponentially synchronized.

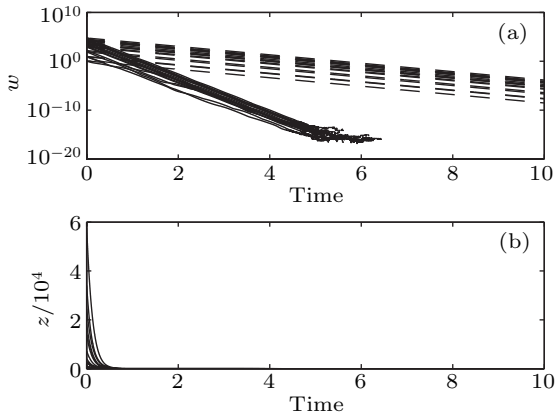



**Fig. 1.** The trajectories of (a) $w$ and (b) $z$ with respect to time $t$ obtained by choosing randomly initial states. In panel (a), the trajectories of $w$ with respect to time $t$ obtained by choosing randomly initial states disappear when time $t$ is long enough because the numerical values of $w(t)$ approximatively reach $-\infty$; the dotted lines are given by $w = w(0) - 1.9802t$.

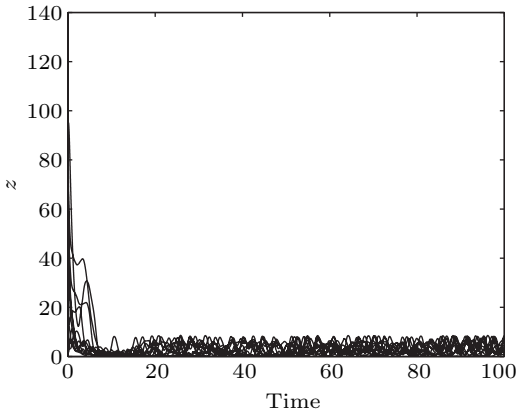



**Fig. 2.** The trajectories of $z$ with respect to time $t$ obtained by choosing randomly initial states.

**Simulation 3** In system (7), let $g_0(u) = 1/7.0099 \times \ln(1+u^2)$ with $\alpha = 1/7.0099$. Then $\alpha \leq 1/7.0099$ and $g_0$ is differential on $\mathbb{R}$. Thus, $\det(\boldsymbol{C}) \neq 0$ implies the exponential synchronization of system (7).

Let the coupling matrix

$$\boldsymbol{A} = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0 & -4.1 & 4.1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, it follows $\det(\boldsymbol{C}) \neq 0$. Figure 3(a) shows that for the randomly chosen initial states, all the curves $w = w(t)$ are respectively located below the straight lines represented as the dotted lines, namely, $w(t) \leq w(0) - 0.0370t$; figure 3(b) shows that for the randomly chosen initial states, all the curves $z = z(t)$ converge to 0 when time $t$ is long enough. Thus, figure 3 reflects the exponential synchronization of system (7). By comparing with simulation 2, it is shown that in system (7) with the coupling matrix $A$ invariant, we can achieve the exponential synchronization of the system through the change of the map $g$.

From the above analysis and simulations, it is concluded that the results of this paper can be applied to the exponential synchronization analysis of linearly coupled Hopfield neural networks.

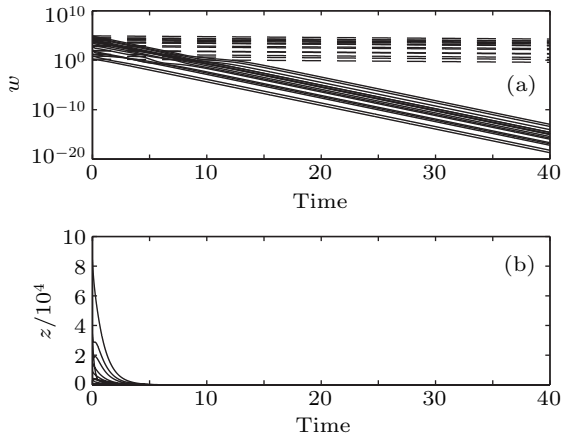



**Fig. 3.** The trajectories of (a) $w$ and (b) $z$ with respect to time $t$ obtained by choosing randomly initial states. In panel (a), the dotted lines are given by $w = w(0) - 0.0370t$.

## 5. Conclusion

This paper presents a new method of studying synchronization of LCODEs by using the matrix $C$ represented by the coupling matrix $A$ to simplify the LCODEs. Two theorems and their corollaries are presented for the exponential synchronization of the LCODEs. Among them, we propose two new conditions $\mathscr{F}_{\text{pard}}$ and $\mathscr{F}_{\text{quad}}$ for local and global exponential synchronizations of LCODEs, respectively. Compared with the obtained results in the references, the two new conditions are weaker than the QUAD condition that the intrinsic nonlinear dynamics satisfy. The results are suitable for any linearly coupling configuration. This is the requirement that all the existing results for the exponential synchronization of LCODEs can almost never meet.

## References

[1] Dhamala M, Jirsa V K and Ding M 2004 *Phys. Rev. Lett* **92** 537
[2] Lakshmanan S, Ju H P, Rihan F A and Rakkiyappan R 2014 *Chin. Phys. B* **23** 218
[3] Prakash M, Balasubramaniam P and Lakshmanan S 2016 *Neural Networks* **83** 86
[4] Shi X and Zhang J D 2016 *Chin. Phys. B* **25** 060502
[5] Bignami F and Agliari A 2010 *Stud. Nonlinear Dyn. Econ.* **14** 1
[6] Mcnally J C, Crowe M A, Roche J F and Beltman M E 2014 *Theriogenology* **82** 1263
[7] Wu D Y, Zhao Y P, Zheng M H, Zhou J and Liu Z H 2016 *Chin. Phys. B* **25** 028701
[8] Xu D G, Xu X Y, Yang C H and Gui W H 2017 *Physica A* **466** 544
[9] Yang J S, Strukov D B and Stewart D R 2013 *Nat. Nanotechnol.* **8** 13
[10] Pham V T, Buscarino A, Fortuna L and Frasca M 2012 *Int. J. Bifur. Chaos* **22** 2543
[11] Perez-Manuzuri V, Perez-Villar V and Chua L O 1993 *IEEE Trans. Circuits Syst. I* **40** 174
[12] Vanwiggeeren G D and Roy P 1988 *Science* **279** 1198
[13] Lu W L and Chen T P 2004 *IEEE Trans. Circuits Syst. I* **51** 2491

[14] Hoppensteadt F C and Lzhikevich E M 2000 *IEEE Trans. Neural Netw.* **11** 734
[15] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821
[16] Tang L K, Lu J A and Chen G R 2012 *Chaos* **22** 023121
[17] Tang L K, Lu J A and Wu X Q 2014 *Int. J. Bifur. Chaos* **24** 1450011
[18] Wu C W and Chua L O 1995 *IEEE Trans. Circuits Syst. I* **42** 430
[19] Yu W W, Chen G R, Lv J H and Kurths J 2013 *SIAM J. Optimiz.* **52** 1395
[20] Lu W L and Chen T P 2006 *Phys. D* **213** 214
[21] Liu X W and Chen T P 2008 *Phys. D* **237** 630
[22] Liu B, Lu W L and Chen T P 2012 *Neural Networks* **25** 5
[23] Yang X and Cao J 2013 *IEEE Trans. Circuits Syst. I* **60** 2431
[24] Xu Y, Lu R Q, Shi P, Li H Y and Xie S L 2016 *IEEE Trans. Cybern.* **60** 2635122
[25] Xu Y, Lu R Q, Shi P, Li H Y and Xie S L 2016 *IEEE Trans. Neural Netw. Learn. Syst.* **60** 2636325
[26] Yang K L 2016 *Acta Phys. Sin.* **65** 100501 (in Chinese)
[27] Zhu L H 2016 *Chin. Phys. Lett.* **33** 50501
[28] Horn R A and Johnson C R 2001 *Martix Analysis* (New York: Springer)
[29] Bhaya A, Kaszkurewicz E and Santos R 2003 *Linear Algebra Appl.* **374** 159