Comparative analysis of entanglement measures based on monogamy inequality

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We evaluate the monogamy inequality for symmetric, non-symmetric pure states of importance in terms of squared concurrence, squared entanglement of formation, squared negativity of partial transpose and compare the corresponding tangles. We show that though concurrence and concurrence tangle are zero for two special classes of mixed entangled states, both negativity tangle and entanglement of formation (EOF) tangle turn out to be non-zero. A comparison of different tangles is carried out in each case and it is shown that while the concurrence tangle captures the genuine multiqubit entanglement in N-qubit pure states with N distinct spinors (containing GHZ and superposition of W-, obverse W states) either negativity tangle or EOF tangle is to be used as a better measure of entanglement in the W-class of states with two distinct spinors and in the special classes of mixed multiqubit states.

Keywords: monogamy of quantum entanglement, concurrence tangle, negativity tangle, Majorana representation of symmetric pure states

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1. Introduction

Characterizing and quantifying entanglement of multipartite states are fundamental issues in quantum information theory. One of the most important properties of multipartite quantum systems, the limited shareability of quantum entanglement/correlation, has evoked great interest in recent years.[1–38] While the entanglement measure called concurrence-tangle was obtained via the monogamy relation using squared concurrence in N-qubit pure states,[1,7] the measure called negativity-tangle was obtained through the formulation of monogamy inequality using squared negativity of partial transpose in Ref. [10] for N-qubit states. Though entanglement of formation does not obey the monogamy relation, recently it was shown[37] that squared entanglement of formation satisfies the monogamy inequality and hence the tangle in terms of entanglement of formation is shown to quantify genuine multiqubit entanglement in N-qubit pure states.[37] We observe here that, by definition, the multiqubit entanglement measures such as concurrence tangle, negativity tangle, and entanglement of formation (EOF) tangle formulated using monogamy inequality, quantify the three-party or residual entanglement[1] which is the entanglement not accounted for by the two-qubit entanglements in a multiqubit state.[38] In fact, the generalized monogamy inequalities and the associated partition-dependent residual entanglement quantifiers accounting for arbitrary partitions of a multiqubit system have also been proposed[21] and were used in studying entanglement dynamics of N-qubit GHZ, W states interacting with N independent reservoirs. It is important to acknowledge the fact that a partition dependent residual entanglement gives a better characterization of multiqubit entanglement as it accounts for the entanglement between any two blocks which is over and above the sum of all two-qubit entanglements across the bipartition between blocks. While we intend to use partition dependent residual entanglement quantifiers in future works, we have restricted ourselves to a comparative analysis of the measures for usual residual entanglement in this article.

The motivation for the present work lies in the fact that there are different residual entanglement quantifiers such as concurrence tangle,[1,7] negativity tangle[10] (tangle in terms of squared negativity of partial transpose[39–41]), and EOF tangle[37] (tangle in terms of squared entanglement of formation[42,43]). The suitability of a measure to quantify residual or three-party entanglement in SLOCC inequivalent classes of pure states and special classes of mixed states is an issue that is worth examining. In fact, a comparitive analysis of different measures of residual entanglement in 3-qubit, 3-qudit states[20] and in N-qubit W-class of states[38] have been carried out respectively in Refs. [20] and [38]. We have also observed that for non-symmetric multiqubit states (states that do not remain invariant under interchange of qubits), the definitions of concurrence tangle[1,7] and EOF tangle[37] are to be slightly modified to account for the average entanglement in all tripartitions. While the definition of negativity tangle[10] is suitable for both symmetric and non-symmetric states, the need for an analogous definition for concurrence tangle, EOF

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tangle and comparison of these tangles for non-symmetric states are other motivations behind this work.

It can be recalled here that by using the Majorana representation[44–46] of pure symmetric N-qubit states, it was shown in Ref. [38] that the concurrence tangle vanishes for the W-class of states (N-qubit states with two distinct Majorana spinors) whereas the negativity tangle, being non-zero, properly quantifies the residual entanglement in this family. Here we evaluate the value of EOF tangle for the W-class of states to check whether it is non-vanishing and hence can quantify the three-party entanglement in this class. We also consider superposition of W and obverse W states which belong to the family of pure symmetric N-qubit states with N distinct spinors and evaluate their concurrence tangle, negativity tangle, and EOF tangle. This is done in order to examine whether measures of entanglement suitable to quantify the three-party entanglement in W-class of states are also suitable for states belonging to an SLOCC inequivalent class, the class of states with N-distinct spinors. We will show here that unlike the W-class where concurrence tangle fails to quantify the residual entanglement, all the three tangles are able to capture the three-party entanglement in the family of pure states with N distinct spinors. Quite similar to the W states, it is shown that their non-symmetric counterparts, the generalized W states, have vanishing concurrence tangle and both negativity tangle and EOF tangle serve as good measures of three-party entanglement for these states.

For mixed states, the evaluation of each tangle requires the knowledge of an optimal pure state decomposition[1,10,37] (that results in a minimum value of the tangle) for the state and hence all the three tangles cannot be readily evaluated for the mixed states. Efforts are underway to find optimal pure state decompositions of mixed states of interest and it is seen that there exists a special class of mixed entangled states having zero concurrence and concurrence tangle.[35,36] In Ref. [35], the authors showed that the concurrence tangle fails to identify genuine entanglement in a class of entangled three-qubit mixed states with no bipartite entanglement. Another class of N-qubit mixed entangled states having no bipartite entanglement, its genuine multiqubit entanglement not recognized by the concurrence tangle, has been identified in Ref. [36]. But the formulation of monogamy inequality in terms of squared entanglement of formation has led to the revelation that the genuine tripartite entanglement in the three-qubit mixed states in Ref. [35] can be effectively captured by the EOF tangle.[37] In this work, we examine whether the EOF tangle identifies the genuine multiqubit entanglement in the N qubit mixed entangled states without two-qubit entanglement and vanishing concurrence tangle.[36] We also evaluate the negativity tangle for these special states and examine whether, along with the EOF tangle, it serves the purpose of quantifying genuine multiqubit entanglement in mixed states.

The article is divided into five sections. In Section 2, we evaluate and compare the tangles corresponding to pure symmetric N-qubit states belonging to two inequivalent SLOCC classes. In Section 3, on defining concurrence tangle, EOF tangle to suit their use for non-symmetric states, we evaluate and compare all the three tangles for generalized W states. Section 4 gives the evaluation and comparison of the tangles in the two special classes of mixed entangled states. Section 5 provides a concise summary of the results.

2. Monogamy of pure symmetric states

2.1. N qubit states belonging to W-class: symmetric states with 2 distinct spinors

The W-class or the Dicke class of states are symmetric (invariant under interchange of qubits) N-qubit states with 2 distinct spinors.[44–46] An arbitrary N-qubit pure symmetric state belonging to the W class is written[38] using Majorana representation[44–46] as

$$\begin{aligned}&|\Psi_{N-1,1}\rangle\\ &\equiv \cos\frac{\theta}{2}|000\cdots0\rangle\\ &\quad+\sin\frac{\theta}{2}\left(\frac{|100\cdots0\rangle+|010\cdots0\rangle+\cdots+|00\cdots01\rangle}{\sqrt{N}}\right),\end{aligned}\tag{1}$$

with a single parameter θ, $0<\theta\leq\pi$ describing the state.

It has been shown in Ref. [38] that the monogamy inequality with squared concurrence holds good with equality for the states $|\Psi_{N-1,1}\rangle$, in other words, the N-concurrence tangle given by

$$\tau_{\rm w}=C^2_{A_1:A_2A_3\ldots A_N}-(N-1)C^2_{A_1A_2}=0$$

vanishes for the family of states $|\Psi_{N-1,1}\rangle$. The fact that N-concurrence tangle underestimates the genuine multiqubit entanglement in N-qubit pure states has been illustrated in Ref. [38] through the evaluation of non-zero negativity-tangle

$$\Pi_{\rm w}=(N-1)\left(\frac{(1-\cos\theta)^2}{N^2}-N^2_{A_1A_k}\right)$$

for the W-class of states $|\Psi_{N-1,1}\rangle$, with $N_{A_1A_k}$ being the negativity of partial transpose between the qubits A_1 and A_k. Continuing on this work,[38] we will show here that tangle in terms of squared EOF is non-zero for the family of states $|\Psi_{N-1,1}\rangle$.

For N-qubit pure symmetric states, tangle in terms of squared EOF is given by

$$\begin{aligned}\xi&=E^2_{A_1:A_2A_3\ldots A_N}-(E^2_{A_1A_2}+E^2_{A_1A_3}+E^2_{A_1A_4}+\cdots+E^2_{A_1A_N})\\&=E^2_{A_1:A_2A_3\ldots A_N}-(N-1)E^2_{A_1A_j},\end{aligned}\tag{2}$$

where the entanglement of formation in the bipartitions[42,43] $A_1:A_j$, $j=2,3,4,\ldots,N$ and in the bipartition $A_1:A_2A_3\ldots A_N$ are respectively given by

$$E_{A_1A_j}=h\left(\frac{1+\sqrt{1-C^2_{A_1A_j}}}{2}\right),$$

$$E_{A_1:A_2A_3\ldots A_N} = h\left(\frac{1+\sqrt{1-C^2_{A_1:A_2A_3\ldots A_N}}}{2}\right), \quad (3)$$

with $h(x) = -x\log_2 x - (1-x)\log_2(1-x)$ being the binary entropy. The squared concurrence between any two qubits and that between a single qubit and the remaining $N-1$ qubits of the state $|\Psi_{N-1,1}\rangle$ are shown to be

$$C^2_{A_1A_j} = \frac{4}{N^2}\sin^4\left(\frac{\theta}{2}\right), \quad C^2_{A_1:A_2A_3\ldots A_N} = \frac{N-1}{N^2}(1-\cos\theta)^2. \quad (4)$$

On explicit evaluation of $E_{A_1:A_2A_3\ldots A_N}$, $E_{A_1A_j}$ using the concurrences in Eq. (4), the EOF tangle $\xi_{\rm w}$ of the state $|\Psi_{N-1,1}\rangle$ is obtained as a function of the parameters θ and N, the number of qubits. Figure 1 shows the variations of negativity tangle $\Pi_{\rm w}$ and EOF tangle $\xi_{\rm w}$ with respect to θ and N.

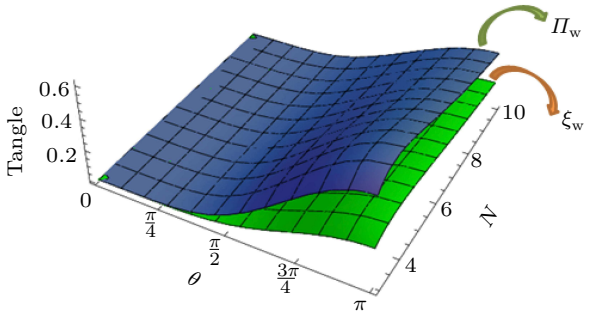


Fig. 1. (color online) Negativity tangle $\Pi_{\rm w}$ and EOF tangle $\xi_{\rm w}$ versus θ and N for the state $|\Psi_{N-1,1}\rangle$ belonging to the W-class.

Though the concurrence tangle is zero for the W-class of states, from Fig. 1, it is readily seen that both EOF tangle and negativity tangle are non-zero, indicating genuine multiqubit entanglement in this class of states. It can also be seen that the negativity tangle is greater than the EOF tangle.

When $\theta = \pi$ in $|\Psi_{N-1,1}\rangle$, one obtains the N-qubit W-state

$$|W_N\rangle \equiv \frac{1}{\sqrt{N}}\left(|100\cdots0\rangle + |010\cdots0\rangle + \cdots + |00\cdots01\rangle\right).$$

By substituting $\theta = \pi$ in Eq. (4), one has

$$C^2_{A_1A_j} = \frac{4}{N^2}, \quad C^2_{A_1:A_2A_3\ldots A_N} = \frac{4(N-1)}{N^2}, \quad (5)$$

leading to the vanishing concurrence tangle $\tau_{\rm w} = C^2_{A_1:A_2A_3\ldots A_N} - (N-1)C^2_{A_1A_2}$ for N qubit W states $|W_N\rangle$. In Ref. [38], the negativity-tangle $\Pi_{\rm w}$ for N-qubit W states is shown to be

$$\Pi_{\rm w} = \frac{N-1}{N^2}\left(4 - \left[\sqrt{(N-2)^2+4} - (N-2)\right]^2\right). \quad (6)$$

Using the concurrences in Eq. (5), we calculate the EOF tangle for N-qubit W-states and tabulate the values of these tangles for different numbers N of qubits. It can be readily seen that both negativity tangle and EOF tangle can be used as multiqubit entanglement measures for N-qubit W states.

Table 1. Concurrence tangle, negativity tangle, and EOF tangle of $|W_N\rangle$ for different N.

N	3	4	5	10	20	30
Concurrence tangle	0	0	0	0	0	0
EOF tangle	0.2381	0.2809	0.2707	0.1602	0.0698	0.0399
Negativity tangle	0.5493	0.6213	0.5813	0.3545	0.1894	0.1287

2.2. Superposition of W and obverse W states: symmetric states with N distinct spinors

We recall here that the so-called obverse W states are also states with 2-distinct spinors similar to the W states.[46] The N-qubit obverse W state is given by

$$|\bar{W}_N\rangle = \frac{1}{\sqrt{N}}\left(|011\cdots1\rangle + |101\cdots1\rangle + \cdots + |11\cdots10\rangle\right). \quad (7)$$

Though both W and obverse W states belong to the SLOCC family of 2-distinct spinors, their equal superposition given by

$$|W\bar{W}\rangle_N = \frac{1}{\sqrt{2}}\left(|W_N\rangle + |\bar{W}_N\rangle\right) \quad (8)$$

belongs to the family of N distinct spinors.[46] It is to be recalled here that N-qubit GHZ states $|{\rm GHZ}\rangle_N$ are symmetric states with N-distinct spinors[45,46] and hence both $|{\rm GHZ}\rangle_N$ and $|W\bar{W}\rangle_N$ belong to the same SLOCC class, the class of N distinct spinors. It has been shown in Ref. [46] that, inspite of belonging to the same SLOCC class of 3 distinct spinors, the entanglement properties of $|W\bar{W}\rangle_3$ are quite different from those of 3 qubit GHZ states.[46,47] Here, we examine the monogamous nature of the states $|W\bar{W}\rangle_N$ and show that except for $N=3$, the states $|W\bar{W}\rangle_N$ possess the maximal monogamous behavior of $|{\rm GHZ}\rangle_N$ states. In the following, we evaluate the tangles with respect to squared concurrence, negativity, and squared entanglement of formation for the states $|W\bar{W}\rangle_N$.

On evaluating the two-qubit and single-qubit density matrices of the 3-qubit state $|W\bar{W}\rangle_3$, we obtain

$$\begin{aligned}
C_{A_1A_2} &= C_{A_1A_3} = \frac{1}{3} \Longrightarrow E_{A_1A_2} = E_{A_1A_3} = 0.1873,\\
C_{A_1:A_2A_3} &= \frac{\sqrt{5}}{3} \Longrightarrow E_{A_1:A_2A_3} = 0.65,\\
\tau_{W\bar{W}_3} &= C^2_{A_1:A_2A_3} - \left(C^2_{A_1A_2} + C^2_{A_1A_3}\right) = \frac{1}{3},\\
\xi_{W\bar{W}_3} &= E^2_{A_1:A_2A_3} - \left(E^2_{A_1A_2} + E^2_{A_1A_3}\right) \approx 0.3524. \quad (9)
\end{aligned}$$

The evaluation of negativity tangle of $|W\bar{W}\rangle_3$ leads us to $\Pi_{W\bar{W}_3} = \tau_{W\bar{W}_3}$ as

$$\begin{aligned}
N_{A_1A_2} &= N_{A_1A_3} = \frac{1}{3}, \quad N_{A_1:A_2A_3} = \frac{\sqrt{5}}{3},\\
\Pi_{W\bar{W}_3} &= N^2_{A_1:A_2A_3} - \left(N^2_{A_1A_2} + N^2_{A_1A_3}\right) = \frac{1}{3}. \quad (10)
\end{aligned}$$

For the 4-qubit state $|W\bar{W}\rangle_4$, it can be seen that

$$C_{A_1A_j} = E_{A_1A_j} = N_{A_1A_j} = 0, \quad j = 2,3,4,$$

$$C_{A_1:A_2A_3A_4} = E_{A_1:A_2A_3A_4} = N_{A_1:A_2A_3A_4} = 1,$$
$$\tau_{W\bar{W}_4} = \xi_{W\bar{W}_4} = \Pi_{W\bar{W}_4} = 1. \quad (11)$$

For N-qubit states $|W\bar{W}\rangle_N$, $N > 4$, while the single qubit density matrix turns out be the maximally mixed state $I_2/2$, the two-qubit density matrix is given by

$$\rho_{A_1A_j} = \frac{1}{2N}\begin{pmatrix} N-2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & N-2 \end{pmatrix}, \quad j = 2, 3, \ldots, N. \quad (12)$$

The concurrence $C_{A_1A_j}$ as well as negativity $N_{A_1A_j}$ of the state $\rho_{A_1A_j}$ are readily seen to be zero. The entanglement between the first qubit (or any other) and the remaining $N-1$ qubits being $C_{A_1:A_2A_3\cdots A_N} = 2\sqrt{\det\rho_{A_1}}$, we obtain $C_{A_1:A_2A_3\cdots A_N} = 1$ as $\det\rho_{A_1} = 1/4$, with ρ_{A_1} being a maximally mixed state. We therefore have all the three tangles equal and equal to 1 for $|W\bar{W}\rangle_N$, $N > 4$. On explicit evaluation, we obtain $\tau_{W\bar{W}_4} = \xi_{W\bar{W}_4} = \Pi_{W\bar{W}_4} = 1$ and can conclude that

$$\tau_{W\bar{W}_N} = \xi_{W\bar{W}_N} = \Pi_{W\bar{W}_N} = 1 \ \text{ for } \ N \geq 4. \quad (13)$$

In fact, the residual entanglement of the state $|W\bar{W}\rangle_N$ with $N \geq 4$ matches exactly with that of the N-qubit GHZ state for which[38]

$$\tau_{\mathrm{GHZ}_N} = \xi_{\mathrm{GHZ}_N} = \Pi_{\mathrm{GHZ}_N} = 1 \ \text{ for } \ N \geq 3. \quad (14)$$

It can thus be concluded that, except for $N = 3$, the two states $|W\bar{W}\rangle_N$ and $|\mathrm{GHZ}\rangle_N$ belonging to the SLOCC family of N distinct spinors have the same monogamous behavior and maximal residual entanglement.

2.3. Generalized GHZ State

Consider the N-qubit generalized GHZ state[25] given by

$$|\mathrm{GHZ}\rangle_{\mathrm{gen}} = a|000\cdots0\rangle + b|111\cdots1\rangle,$$
$$a = r_1 e^{i\alpha},\ b = r_2 e^{i\beta};\ \ r_1^2 + r_2^2 = 1.$$

The two-qubit and single-qubit density matrices of $|\mathrm{GHZ}\rangle_{\mathrm{gen}}$ are readily seen to be

$$\rho_{A_1A_j} = \mathrm{diag}\,(r_1, 0, 0, r_2),\ j = 2, 3, \ldots N,$$
$$\rho_{A_j} = \mathrm{diag}\,(r_1, r_2), \quad (15)$$

implying $C_{A_1A_j} = 0$, $j = 2, 3, \ldots, N$, and $C_{A_1:A_2A_3\cdots A_N} = 2\sqrt{\det\rho_{A_j}} = 2r_1r_2$. The concurrence tangle for the generalized GHZ state thus turns out to be

$$\tau_{\mathrm{gghz}} = 4r_1^2r_2^2 = 4r_1^2(1 - r_1^2). \quad (16)$$

As $N_{A_1A_j} = 0$, $j = 2, 3, \ldots, N$ and $C_{A_1:A_2A_3\cdots A_N} = N_{A_1:A_2A_3\cdots A_N}$, the negativity tangle Π_{gghz} of the state $|\mathrm{GHZ}\rangle_{\mathrm{gen}}$ is the same as that of concurrence tangle τ_{gghz}. Owing to the vanishing $C_{A_1:A_j}$, we have $E_{A_1:A_j} = 0$, $j = 2, 3, \ldots, N$, but $E_{A_1:A_2A_3\cdots A_N}$ is non-zero and given by

$$E_{A_1:A_2A_3\cdots A_N} = h\left(\frac{1 + \sqrt{1 - C^2_{A_1:A_2A_3\cdots A_N}}}{2}\right). \quad (17)$$

As

$$\frac{1 + \sqrt{1 - C^2_{A_1:A_2A_3\cdots A_N}}}{2} = \frac{(r_1^2 + r_2^2) + \sqrt{(r_1^2 + r_2^2)^2 - 4r_1^2r_2^2}}{2}$$
$$= \frac{(r_1^2 + r_2^2) + (r_1^2 - r_2^2)}{2} = r_1^2,$$
$$\frac{1 - \sqrt{1 - C^2_{A_1:A_2A_3\cdots A_N}}}{2} = \frac{(r_1^2 + r_2^2) - \sqrt{(r_1^2 + r_2^2)^2 - 4r_1^2r_2^2}}{2}$$
$$= \frac{(r_1^2 + r_2^2) - (r_1^2 - r_2^2)}{2} = r_2^2, \quad (18)$$

hence the EOF tangle for $|\mathrm{GHZ}\rangle_{\mathrm{gen}}$ is obtained as

$$\xi_{\mathrm{gghz}} = E^2_{A_1:A_2A_3\cdots A_N}, \quad (19)$$

with $E_{A_1:A_2A_3\cdots A_N} = h(r_1^2)$, and $h(x)$ being the binary entropy.

A comparison of all the three tangles for the generalized GHZ state is shown in Fig. 2.

It can be readily seen from Fig. 2 that the maximal monogamous behavior with $\tau_{\mathrm{gghz}} = \xi_{\mathrm{gghz}} = \Pi_{\mathrm{gghz}} = 1$ is realized when $r_1 = r_2 = \frac{1}{\sqrt{2}}$, i.e., for the GHZ state $|\mathrm{GHZ}\rangle_N$.

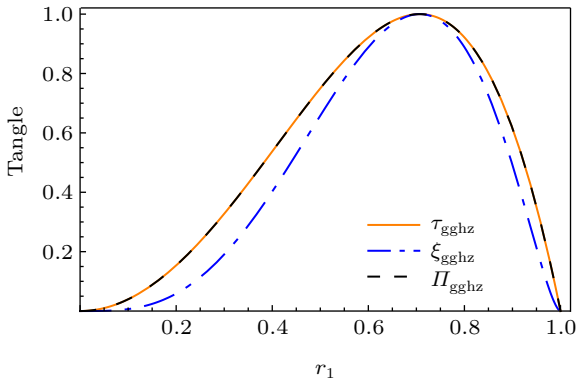


Fig. 2. (color online) Concurrence tangle τ_{gghz}, EOF tangle ξ_{gghz}, and negativity tangle Π_{gghz} versus r_1 for the generalized GHZ-state.

3. Non-symmetric pure states

Before examining the monogamy inequality and entanglement in non-symmetric pure states, we observe that only the tangle in terms of squared negativity of partial transpose defined in Ref. [10] is suitable for quantifying entanglement in non-symmetric states and both concurrence tangle[1,7] and EOF tangle[37] are to be redefined so that different partitions of the non-symmetric state are taken into account.

The negativity tangle for N-qubit pure states is given by[10,38]

$$\Pi = \frac{1}{N}(\Pi_1 + \Pi_2 + \ldots + \Pi_N), \quad (20)$$

where

$$\begin{aligned}\Pi_1 &= N^2_{A_1:A_2A_3\ldots A_N}-\left(N^2_{A_1A_2}+N^2_{A_1A_3}+\cdots+N^2_{A_1A_N}\right),\\ \Pi_2 &= N^2_{A_2:A_1A_3\ldots A_N}-\left(N^2_{A_2A_1}+N^2_{A_2A_3}+\cdots+N^2_{A_2A_N}\right),\\ &\vdots\\ \Pi_N &= N^2_{A_N:A_1A_2\ldots A_{N-1}}-\left(N^2_{A_NA_1}+N^2_{A_NA_2}+\cdots+N^2_{A_NA_{N-1}}\right). \quad (21)\end{aligned}$$

It is observed here that though $\Pi_1=\Pi_2=\cdots=\Pi_N$ leading to $\Pi=\Pi_k$, $k=1,2,\ldots,N$ for symmetric states, one has to take into account all possible bipartitions as given in Eq. (20) for non-symmetric states. Analogous to the definition of negativity tangle Π in Eq. (20), we redefine the concurrence tangle and EOF tangle as

$$\tau = \frac{1}{N}(\tau_1+\tau_2+\ldots+\tau_N), \quad (22)$$

where

$$\begin{aligned}\tau_1 &= C^2_{A_1:A_2A_3\ldots A_N}-\left(C^2_{A_1A_2}+C^2_{A_1A_3}+\ldots+C^2_{A_1A_N}\right),\\ \tau_2 &= C^2_{A_2:A_1A_3\ldots A_N}-\left(C^2_{A_2A_1}+N^2_{A_2A_3}+\ldots+C^2_{A_2A_N}\right),\\ &\vdots\\ \tau_N &= C^2_{A_N:A_1A_2\ldots A_{N-1}}-\left(C^2_{A_NA_1}+C^2_{A_NA_2}+\ldots+C^2_{A_NA_{N-1}}\right); \quad (23)\end{aligned}$$

and

$$\xi = \frac{1}{N}(\xi_1+\xi_2+\ldots+\xi_N), \quad (24)$$

where

$$\begin{aligned}\xi_1 &= E^2_{A_1:A_2A_3\ldots A_N}-\left(E^2_{A_1A_2}+E^2_{A_1A_3}+\ldots+E^2_{A_1A_N}\right),\\ \xi_2 &= E^2_{A_2:A_1A_3\ldots A_N}-\left(E^2_{A_2A_1}+E^2_{A_2A_3}+\ldots+E^2_{A_2A_N}\right),\\ &\vdots\\ \xi_N &= E^2_{A_N:A_1A_2\ldots A_{N-1}}-\left(E^2_{A_NA_1}+E^2_{A_NA_2}+\ldots+E^2_{A_NA_{N-1}}\right), \quad (25)\end{aligned}$$

so that they properly quantify the entanglement in non-symmetric states.

In the following, we make use of Eqs. (20)–(25) to evaluate the three different measures of three-party entanglement in the generalized W state.

3.1. Generalized W state

The 3-qubit generalized W state[25] is of the form

$$\begin{aligned}&|W\rangle_{\rm gen} = a|100\rangle+b|010\rangle+c|001\rangle,\\ &a=r_1\,{\rm e}^{{\rm i}\alpha},\ \ b=r_2\,{\rm e}^{{\rm i}\beta},\ \ c=r_3\,{\rm e}^{{\rm i}\gamma};\ \ r_1^2+r_2^2+r_3^2=1, \quad (26)\end{aligned}$$

and is readily seen to be non-symmetric under interchange of qubits.

The two-qubit and single-qubit density matrices of the state $|W\rangle_{\rm gen}$ are given by

$$\rho_{A_1A_2}=\begin{pmatrix} {r_1}^2 & 0 & 0 & 0\\ 0 & {r_2}^2 & {\rm e}^{{\rm i}(\beta-\gamma)}r_2r_3 & 0\\ 0 & {\rm e}^{-{\rm i}(\beta-\gamma)}r_2r_3 & {r_3}^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\rho_{A_1}=\begin{pmatrix} {r_1}^2+{r_2}^2 & 0\\ 0 & {r_3}^2\end{pmatrix},$$

$$\rho_{A_1A_3}=\begin{pmatrix} {r_2}^2 & 0 & 0 & 0\\ 0 & {r_1}^2 & {\rm e}^{{\rm i}(\alpha-\gamma)}r_1r_3 & 0\\ 0 & {\rm e}^{-{\rm i}(\alpha-\gamma)}r_1r_3 & {r_3}^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\rho_{A_2}=\begin{pmatrix} {r_1}^2+{r_3}^2 & 0\\ 0 & {r_2}^2\end{pmatrix},$$

$$\rho_{A_2A_3}=\begin{pmatrix} {r_3}^2 & 0 & 0 & 0\\ 0 & {r_1}^2 & {\rm e}^{{\rm i}(\alpha-\beta)}r_1r_2 & 0\\ 0 & {\rm e}^{-{\rm i}(\alpha-\beta)}r_1r_2 & {r_2}^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\rho_{A_3}=\begin{pmatrix} {r_2}^2+{r_3}^2 & 0\\ 0 & {r_1}^2\end{pmatrix}. \quad (27)$$

The evaluation of concurrences in different bipartitions of the state $|W\rangle_{\rm gen}$ leads us to

$$\begin{aligned}C_{A_1A_2} &= 2r_2r_3,\ \ C_{A_1A_3}=2r_1r_3,\ \ C_{A_2A_3}=2r_1r_2,\\ C_{A_1:A_2A_3} &= 2r_3(\sqrt{{r_1}^2+{r_2}^2}),\ \ C_{A_2:A_3A_1}=2r_2(\sqrt{{r_1}^2+{r_3}^2}),\\ C_{A_3:A_1A_2} &= 2r_1(\sqrt{{r_2}^2+{r_3}^2}).\end{aligned}$$

Now it can be readily seen that

$$\begin{aligned}\tau_1 &= C^2_{A_1:A_2A_3}-C^2_{A_1A_2}-C^2_{A_1A_3}\\ &= 4r_3^2(r_1^2+r_2^2)-4r_2^2r_3^2-4r_1^2r_3^2=0,\\ \tau_2 &= C^2_{A_2:A_3A_1}-C^2_{A_1A_2}-C^2_{A_2A_3}\\ &= 4r_2^2(r_1^2+r_3^2)-4r_2^2r_3^2-4r_1^2r_2^2=0,\\ \tau_3 &= C^2_{A_3:A_1A_2}-C^2_{A_1A_3}-C^2_{A_2A_3}\\ &= 4r_1^2(r_2^2+r_3^2)-4r_1^2r_3^2-4r_1^2r_2^2=0,\end{aligned}$$

implying that the concurrence tangle $\tau_{\rm genW}=(\tau_1+\tau_2+\tau_3)/3=0$ is quite similar to the case of their symmetric counterparts $|W_3\rangle$.

The two-qubit and single qubit-density matrices in Eq. (27) allow us to evaluate the negativity of partial transpose in different bipartitions of the state $|W\rangle_{\rm gen}$ and we have

$$\begin{aligned}N_{A_1A_2} &= -{r_1}^2+\sqrt{{r_1}^4+4{r_2}^2{r_3}^2},\\ N_{A_1:A_2A_3} &= 2r_3(\sqrt{{r_1}^2+{r_2}^2}),\\ N_{A_1A_3} &= -{r_2}^2+\sqrt{{r_2}^4+4{r_1}^2{r_3}^2},\\ N_{A_2:A_3A_1} &= 2r_2(\sqrt{{r_1}^2+{r_3}^2}),\\ N_{A_2A_3} &= -{r_3}^2+\sqrt{{r_3}^4+4{r_1}^2{r_2}^2},\\ N_{A_3:A_1A_2} &= 2r_1(\sqrt{{r_2}^2+{r_3}^2}). \quad (28)\end{aligned}$$

The negativity tangles $\Pi_1\neq\Pi_2\neq\Pi_3$ for each focus qubit can now be readily evaluated and we obtain negativity tangle $\Pi_{\rm genW}=\frac{1}{3}(\Pi_1+\Pi_2+\Pi_3)$ for the state $|W\rangle_{\rm gen}$ as a function of the parameters r_1 and r_2 by using $r_3=\sqrt{1-r_1^2-r_2^2}$.

Though the concurrence tangle vanishes for $|W\rangle_{\rm gen}$, the EOF tangle for each focus qubit is seen to be non-zero. In fact, $\xi_1 \neq \xi_2 \neq \xi_3$ and the EOF tangle $\xi_{\rm genW} = \frac{1}{3}(\xi_1+\xi_2+\xi_3)$ depends on the parameters r_1, r_2, r_3. Figure 3 gives the comparison of negativity tangle with EOF tangle for $|W\rangle_{\rm gen}$.

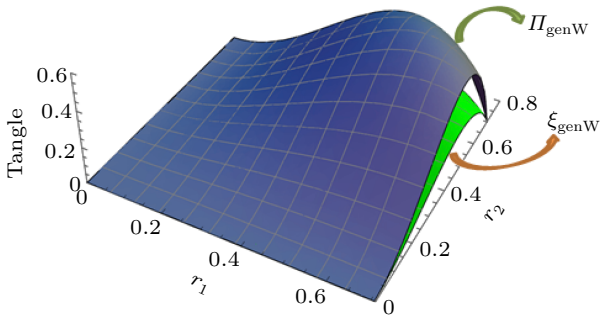


Fig. 3. (color online) EOF tangle $\xi_{\rm genW}$ and negativity tangle $\Pi_{\rm genW}$ with respect to the parameters r_1 and r_2 for the generalized W-state. Observe that $\Pi_{\rm genW} > \xi_{\rm genW}$ is quite similar to the W states.

4. Monogamy of mixed states

In this section, we are going to evaluate the different tangles for two special families of mixed states. Here, we recall that

$$\begin{aligned}\tau_{\rm mixed} &= \min\sum_i p_i\tau(|\psi_i\rangle),\quad \xi_{\rm mixed} = \min\sum_i p_i\xi(|\psi_i\rangle),\\ \Pi_{\rm mixed} &= \min\sum_i p_i\Pi(|\psi_i\rangle),\end{aligned} \tag{29}$$

where $\tau(|\psi_i\rangle)$, $\xi(|\psi_i\rangle)$, $\Pi(|\psi_i\rangle)$ are the tangles of the pure states $|\psi_i\rangle$ appearing in an arbitrary pure state decomposition $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ ($0 \le p_i \le 1, \sum_i p_i = 1$ of the mixed state ρ). The minimum in Eq. (29) is to be taken over all possible pure state decompositions of ρ. If

$$\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i| \tag{30}$$

is the optimal pure state decomposition that gives the minimum tangle for ρ, then one can readily have

$$\begin{aligned}\tau_{\rm mixed} &= \sum_i p_i\tau(|\phi_i\rangle),\quad \xi_{\rm mixed} = \sum_i p_i\xi(|\phi_i\rangle),\\ \Pi_{\rm mixed} &= \sum_i p_i\Pi(|\phi_i\rangle)\end{aligned} \tag{31}$$

as the concurrence-tangle, EOF tangle, negativity tangle, respectively of the mixed state ρ. It is thus evident that the evaluation of the tangles for a mixed state requires the knowledge of its optimal pure state decomposition.

4.1. N-qubit mixed state without two-qubit entanglement

Consider an N-qubit mixed entangled state[36] given by

$$\rho_N = \alpha|1_N\rangle\langle 1_N| + (1-\alpha)|W_N\rangle\langle W_N|,\quad \alpha = \frac{1}{N+1}, \tag{32}$$

with $|1_N\rangle$ and $|W_N\rangle$ being N-qubit spin-down and W-states, respectively.

This is an entangled state but without two-qubit entanglement, and its genuine multiqubit entanglement not detected by concurrence tangle as $\tau(\rho_N)$ vanishes for any $N \ge 3$.[36] Here, we evaluate the EOF tangle and negativity tangle for ρ_N and check whether they can quantify the entanglement in ρ_N. We also compare negativity tangle $\Pi(\rho_N)$ with EOF tangle $\xi(\rho_N)$ for all N.

The state ρ_N is readily seen to be symmetric and its two qubit density matrices $\rho_{A_1A_k}$, $k = 2, 4, \ldots, N$ are given by

$$\rho_{A_1A_k} = \frac{1}{N+1}\begin{pmatrix} N-2 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{33}$$

This gives $C_{A_1A_k} = E_{A_1A_k} = N_{A_1A_k} = 0$, $k = 2, 3, 4, \ldots, N$, indicating no two-qubit entanglement in the state. As the pure state decomposition of the state ρ_N in Eq. (32) is shown to be optimal in Ref. [35], one can readily evaluate the tangles for ρ_N using

$$\begin{aligned}\tau(\rho_N) &= \alpha\,\tau(|1_N\rangle) + (1-\alpha)\,\tau(|W_N\rangle),\\ \xi(\rho_N) &= \alpha\,\xi(|1_N\rangle) + (1-\alpha)\,\xi(|W_N\rangle),\\ \Pi(\rho_N) &= \alpha\,\Pi(|1_N\rangle) + (1-\alpha)\,\Pi(|W_N\rangle),\end{aligned} \tag{34}$$

with $|1_N\rangle$ being a separable state, $\tau(|1_N\rangle) = \xi(|1_N\rangle) = \Pi(|1_N\rangle) = 0$. By using the tangles for the N-qubit W state obtained in Subsection 2.1, one has

$$\begin{aligned}&\tau(\rho_N) = (1-\alpha)\,\tau(|W_N\rangle),\quad \xi(\rho_N) = (1-\alpha)\,\xi(|W_N\rangle),\\ &\Pi(\rho_N) = (1-\alpha)\,\Pi(|W_N\rangle).\end{aligned} \tag{35}$$

As $\tau(|W_N\rangle) = 0$, the concurrence tangle of ρ_N is zero for all N, but the EOF tangle $\xi(\rho_N)$ is nonzero and detects the genuine multiqubit entanglement in the state ρ_N as observed in Ref. [36]. We calculate the negativity tangle $\Pi(\rho_N)$ and compared it with the other two tangles in Fig. 4.

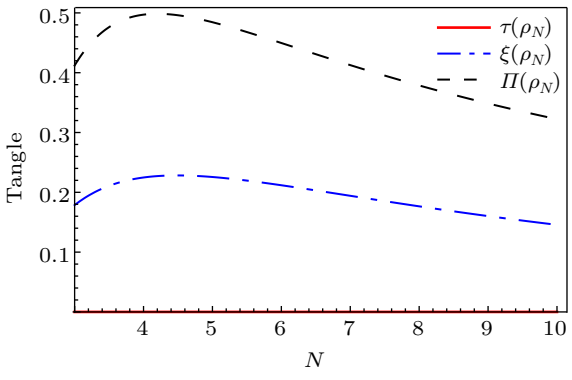


Fig. 4. (color online) The comparison of the tangles $\tau(\rho_N)$, $\xi(\rho_N)$, $\Pi(\rho_N)$ for the mixed symmetric state ρ_N. Notice that $\Pi(\rho_N) > \xi(\rho_N)$ for all N.

It can be readily seen from Fig. 4 that negativity tangle $\Pi(\rho_N)$ also helps to quantify the genuine multiqubit entanglement in the mixed state ρ_N. In fact, both negativity tangle $\Pi(\rho_N)$ and EOF tangle $\xi(\rho_N)$ serve as good measures of residual entanglement in the state ρ_N not having two-qubit entanglement and with vanishing concurrence tangle.

4.2. Mixed state containing GHZ and W states

Consider the mixed 3-qubit state ρ_{ABC}[35] composed of GHZ and W states

$$\rho_{ABC} = p|\mathrm{GHZ}_3\rangle\langle\mathrm{GHZ}_3| + (1-p)|W_3\rangle\langle W_3|, \quad 0 \le p \le 1. \tag{36}$$

It has been shown in Ref. [35] that when $p \in (0.292, 0.627)$, the state ρ_{ABC} corresponds to an entangled state without two-qubit concurrence and with vanishing concurrence tangle.

The optimal pure state decomposition for the state ρ_{ABC} when $p \le 0.627$ is given by[35,37]

$$\rho_{ABC} = \frac{\alpha}{3}\sum_{j=0}^{2}|\psi^j(p_0)\rangle\langle\psi^j(p_0)| + (1-\alpha)|W_3\rangle\langle W_3|,$$

where the pure state component $|\psi^j(p_0)\rangle = \sqrt{p_0}|\mathrm{GHZ}_3\rangle - \mathrm{e}^{(2\pi\mathrm{i}/3)j}\sqrt{1-p_0}|W_3\rangle$ and the parameter $\alpha = p/p_0$ varies from 0 to 1 as $0 < p < p_0$, $p_0 \approx 0.627$.

The two-qubit and single-qubit density matrices of the symmetric pure states $|\psi^j(p_0)\rangle$, $j = 0, 1, 2$ are respectively given by

$$\rho^j_{AB} = \rho^j_{AC} = \rho^j_{BC} = \begin{pmatrix} \frac{2+p_0}{6} & G & G & G^* \\ G^* & \frac{1-p_0}{3} & \frac{1-p_0}{3} & 0 \\ G^* & \frac{1-p_0}{3} & \frac{1-p_0}{3} & 0 \\ G & 0 & 0 & \frac{p_0}{2} \end{pmatrix}, \tag{37}$$

$$\rho^j_A = \rho^j_B = \rho^j_C = \begin{pmatrix} \frac{4-p_0}{6} & G \\ G^* & \frac{2+p_0}{6} \end{pmatrix}, \tag{38}$$

where $G = -\mathrm{e}^{-(2\pi\mathrm{i}/3)j}\sqrt{(1-p_0)p_0}/\sqrt{6}$. The concurrences C^j_{AB}, $C^j_{A:BC} = 2\sqrt{\det\rho^j_A}$ are obtained as functions of p_0 and it is found that $C^j_{AB} = C_{AB}$, $C^j_{A:BC} = C_{A:BC}$ independent of j. By substituting $p_0 = 0.627$, we obtain

$$C_{AB} \approx 0.644, \quad C_{A:BC} \approx 0.91, \\ \tau(|\psi_j\rangle) = C^2_{A:BC} - 2C^2_{AB} \approx 0, \quad j = 0, 1, 2. \tag{39}$$

As $\tau(|\psi_j\rangle) \approx 0$, $j = 0, 1, 2$ and the tangle $\tau(|W_3\rangle)$ for the W state is zero, the concurrence tangle of the state ρ_{ABC} given by

$$\tau(\rho_{ABC}) = \frac{\alpha}{3}\sum_{j=0}^{2}\tau(|\psi_j\rangle) + (1-\alpha)\tau(|W_3\rangle) \tag{40}$$

turns out to be zero.

The negativity N^j_{AB} of the partial transpose of the states $|\psi^j(p_0)\rangle$, $j = 0, 1, 2$ is also found to be independent of j and $N^j_{AB} = N_{AB}$ is obtained as a function of p_0. Substituting $p_0 = 0.627$, we readily obtain $N_{AB} \approx 0.402$. As $N_{A:BC} = C_{A:BC} \approx 0.91$, the negativity tangle of the symmetric states $|\psi^j(p_0)\rangle$ turns out to be

$$\Pi(|\psi_j\rangle) = N^2_{A:BC} - 2N^2_{AB} = 0.505. \tag{41}$$

The negativity of the state ρ_{ABC} is thus given by

$$\begin{aligned}\Pi(\rho_{ABC}) &= \frac{\alpha}{3}\sum_{j=0}^{2}\Pi(|\psi_j\rangle) + (1-\alpha)\Pi(|W_3\rangle) \\ &= 0.505\alpha + (1-\alpha)\left(\frac{4}{9}(\sqrt{5}-1)\right) \\ &\approx 0.505 - 0.07p. \end{aligned} \tag{42}$$

An evaluation of the EOF tangle for the state $|\psi^j(p_0)\rangle$ using the concurrences C_{AB}, $C_{A:BC}$ gives us $\xi(|\psi_j\rangle) \approx 0.217$. Using the EOf tangle $\xi(|W_3\rangle) = 0.238$ (see Table 1), we have

$$\begin{aligned}\xi(\rho_{ABC}) &= \frac{\alpha}{3}\sum_{j=0}^{2}\xi(|\psi_j\rangle) + (1-\alpha)\xi(|W_3\rangle) \\ &\approx 0.217\alpha + (1-\alpha)0.238 \\ &\equiv 0.238 - 0.033p. \end{aligned} \tag{43}$$

In Fig. 5, we plot the tangles $\tau(\rho_{ABC})$, $\xi(\rho_{ABC})$, $\Pi(\rho_{ABC})$. It is seen that although the concurrence-tangle $\tau(\rho_{ABC})$ is zero, the non-zero values of EOF tangle $\xi(\rho_{ABC})$ and negativity tangle $\Pi(\rho_{ABC})$ are helpful in quantifying the genuine three-qubit entanglement in the state ρ_{ABC}.

It is to be noted that, in Ref. [37], EOF tangle ξ_1 with the first qubit A as the focus qubit has been evaluated and it was indicated that the EOF tangle is able to quantify the tripartite entanglement in ρ_{ABC}. Here, we have shown that due to the permutation symmetry of ρ_{ABC}, $\xi_1 = \xi_2 = \xi_3$ and $\xi = (\xi_1 + \xi_2 + \xi_3)/3 = \xi_1$ is the actual EOF tangle of the state ρ_{ABC}.

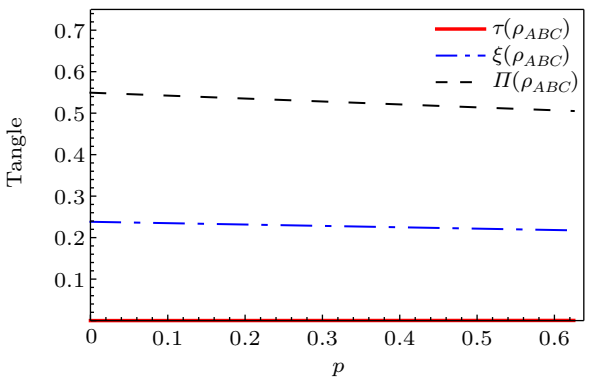


Fig. 5. (color online) The nonzero $\xi(\rho_{ABC})$ and $\Pi(\rho_{ABC})$ detect the genuine three-qubit entanglement in the mixed state ρ_{ABC} when $p < 0.627$.

5. Conclusion

A comparison of different three-party entanglement measures formulated using monogamy inequality of pure symmetric/non-symmetric and mixed multiqubit states has been carried out. Two inequivalent SLOCC families of pure symmetric states are considered and it is shown that though the concurrence tangle is zero, both negativity tangle and tangle in terms of squared entanglement of formation can be used to quantify the three party entanglement in the W-class of states with 2 distinct spinors. For the equal superposition of N-qubit W, obverse W states characterized by N distinct spinors, all the three tangles are shown to be equal and capable of quantifying the maximal entanglement when $N \geq 4$. By redefining the concurrence tangle, tangle in terms of squared entanglement of formation to account for all bipartitions of a non-symmetric state, we have evaluated these tangles for the 3-qubit generalized W state. For two special classes of mixed entangled states with zero concurrence and zero concurrence tangle, we have evaluated the negativity tangle as well as the tangle in terms of squared entanglement of formation. A comparison of tangles reveals the fact that the negativity tangle and the tangle in terms of squared entanglement of formation are non-zero for these mixed states, indicating genuine multiqubit entanglement in them. It is observed that except for superposition of 3-qubit W, obverse W states, the negativity tangle is greater than the tangles in terms of squared concurrence and squared entanglement of formation. Through our evaluation, comparison of tangles in different classes of pure, mixed states, it is illustrated that whereas the concurrence tangle is a good measure of multipartite entanglement in pure symmetric states with N-distinct spinors, one needs to use the negativity tangle and the tangle in terms of squared entanglement of formation for pure symmetric states with 2 distinct spinors and other mixed multiqubit states with vanishing concurrence tangle. It would be of interest to formulate different tangles using the generalized monogamy relations[21] and evaluate their values to obtain suitable entanglement quantifiers for multiqubit states of importance.

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