Superfluidity of coherent light in self-focusing nonlinear waveguides

Ze Cheng(成泽)†

School of Physics, Huazhong University of Science and Technology, Wuhan 430074, *China*

(Received 2 December 2016; revised manuscript received 4 January 2017; published online 20 February 2017)

We establish the superfluidity theory of coherent light in waveguides made of nonlinear polar crystals. It is found that the pairing state of photons in a nonlinear polar crystal is the photonic superfluid state. The photon–photon interaction potential is an attractive effective interaction by exchange of virtual optical phonons. In the traveling-wave pairing state of photons, the photon number is conserved, which is similar to the Bose–Einstein condensation (BEC) state of photons. In analogy to the BCS-BEC crossover theory of superconductivity, we derive a set of coupled order parameter and number equations, which determine the solution of the traveling-wave superfluid state of photons. This solution gives the critical velocity of light in a self-focusing nonlinear waveguide. The most important property of the photonic superfluid state is that the system of photon pairs evolves without scattering attenuations.

Keywords: superfluidity, photon pairing, nonlinear waveguide

PACS: 67.25.dj, 64.60.–i, 67.10.Ba, 78.30.–j **DOI:** 10.1088/1674-1056/26/4/046701

1. Introduction

In recent decades there have been tremendous advances in the optical communications field. The present optical communications technology has improved low loss fibers, high quantum efficiency detectors, and stable lasers to the point where the performance of optical communications systems is approaching the intrinsic limits. The intrinsic limits are caused by the wave-particle duality and thermal scattering of light. The wave nature of light leads to the dispersion of laser pulses in fibers, and the particle nature of light gives rise to the quantum noise of laser light. Thermal scattering from fibers produces the energy loss and thermal noise of laser light. The desire to go beyond the intrinsic limits has greatly driven the development of optical communications theory. Hasegawa and Tappert have proposed the theory of optical solitons in nonlinear fibers.[1–3] The optical solitons occur owing to the cancelation of the group velocity dispersion by the Kerr nonlinearity and so propagate without distortion. The squeezed state of light in nonlinear fibers has been an active theoretical subject and has been observed in a long propagation distance.[4–7] The squeezed state has less quantum noise in one quadrature than a coherent state, and squeezing arises due to self-phase modulation. Originally we have presented the superfluid state of photons in nonlinear polar crystals.[8,9] In the superfluid state, the Rayleigh and Brillouin scatterings are overcome by the virtual Raman scattering, so that the photon system propagates without thermal scattering.

The present optical communications systems consist of transmitters, transmission media, and receivers. The transmitters all use laser pulses, the transmission media are highly flexible waveguides, i.e., optical fibers, and the receivers generally employ direct photo-detectors. Under ideal circumstances where the laser emits coherent states and the photo-detector has unit quantum efficiency and no dark current, the ultimate performance of the system is limited by the optical fiber. Thermal scattering exists normally in optical fibers. Thermal scattering originates from thermal fluctuations in the dielectric function of fibers. The scattering loss limits the propagation distance of laser light in fibers. The thermal amplitude noise degrades the signal. The thermal phase noise tends to erase the coherence of laser light and therefore is most dangerous. To perform quantum communications and measurement, we must know what the thermal scattering effects are and how they can be suppressed. The photonic superfluid theory proposed in Refs. [8] and [9] is very crude for the reason that the theory does not develop a set of calculable equations. The present paper develops a calculable theory of the superfluid state of laser light in nonlinear waveguides. Our theory describes the remarkable properties of photonic superfluid state in waveguides of a self-focusing nonlinearity. The thermal scattering effects are suppressed in the superfluid state, and the traveling-wave superfluid state is expected to have the squeezing property and the soliton effect.

The definition of superfluidity in standing-wave configurations is that the matter behaves like a fluid with zero viscosity below a critical temperature.[10] It should be noted that the standing-wave superfluid state is a thermodynamic equilibrium state.[11] The definition of superfluidity in traveling-wave configurations means the existence of a finite critical velocity v_c below which the motion of the fluid is dissipationless.[10] It should be noted that the traveling-wave superfluid state is also a thermodynamic equilibrium state,[11] but it is metastable.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10174024 and 10474025).

†Corresponding author. E-mail: zcheng@mail.hust.edu.cn

© 2017 Chinese Physical Society and IOP Publishing Ltd

According to the definition of superfluidity in standing-wave configurations, in the past decade we have presented the standing-wave superfluid state of photons in a Kerr nonlinear blackbody.[12,13] It must be stressed that the thermalization of photonic gas is achieved in a blackbody. According to the definition of superfluidity in traveling-wave configurations, in the present paper we shall investigate the traveling-wave superfluid state of photons in nonlinear waveguides. The photons in a nonlinear waveguide are thermalized because of thermal scattering of photons by acoustic and optical phonons, such as inelastic Brillouin and Raman scatterings. The thermal equilibrium of photons with a crystal lattice is reached as photons are scattered by phonons many times. The average time between two adjacent scatterings of a photon is $\tau_{\rm ph} \cong 20$ ps.[14] Therefore, a time of thermalization of photons in nonlinear waveguides is $t_{\rm ph} \cong 0.1$ ns. A finite lifetime of photons in an absorbing waveguide is $T_{\rm ph} \cong 100$ μs. A key requirement in obtaining photonic superfluid state is that $t_{\rm ph} \leq T_{\rm ph}$. The photonic superfluid described above is a pairing condensate of photons, which is somewhat similar to a Bardeen–Cooper–Schrieffer (BCS) condensate of fermions. According to the definition of superfluidity in traveling-wave configurations, Bolda, Chiao, and Zurek have investigated a self-defocusing refractive medium inside a Fabry–Pérot cavity with a cylindrical obstacle.[15] However, no clear evidence of the existence of a superfluid critical velocity was provided. Leboeuf and Moulieras have investigated an array of self-defocusing waveguides with a localized defect.[16] Controlling the speed of a light packet with respect to a defect, they have demonstrated the presence of superfluidity. It is well known that in the three-dimensional (3D) space the photons have a vanishing rest mass. Theoretical works have considered mass-generating processes, involving a two-dimensional (2D) photon gas in a planar Fabry–Pérot cavity[17,18] or in a curved-mirror optical microcavity.[14,19,20] The 2D photons may acquire a nonzero rest mass and a nonzero chemical potential. In the presence of thermalization processes that conserve photon number, Weitz and colleagues have observed the Bose–Einstein condensation (BEC) of massive photons in a dye-filled optical microcavity.[19] This Bose–Einstein condensate is a photonic superfluid. Kirton and Keeling have established a non-equilibrium model of photonic BEC in the dye-filled microcavity.[21]

Recently there has been a growing interest in investigation of superfluid properties of light particles, like polaritons and photons. Theoretically it has been shown that a laser beam which propagates through a cubic-quintic nonlinear optical material may reach a condensed state resembling a liquid drop.[22] Szymanska, Littlewood, and Simons have studied the equilibrium BEC of cavity polaritons in the presence of decoherence.[23] Their results indicate that decoherence drives the polariton condensate towards the laser regime. Alodjants, Chestnov, and Arakelian have studied the high-temperature BEC of atomic polaritons in the presence of optical collisions.[24] The process of optical collisions represents nonresonant interaction of a quantized light field with an atom in the presence of buffer gas particle. More recently, Sanvitto and Timofeev have edited a book entitled 'Exciton Polaritons in Microcavities: New Frontiers'. In this book the top physicists in the field describe the most interesting and important up-to-date achievements in the BEC of microcavity polaritons, and in particular Snoke has given the similarities between polariton condensation and lasing.[25] Carusotto and Ciuti have reviewed quantum fluids of light in nonlinear cavities.[26] In this review, they described superfluid flow, solitons, vortices, and some new quantum phases. Alodjants, Barinov, and Arakelian have studied critical properties of optical group velocity in the traveling-wave BEC state of atomic polaritons.[27]

Now it is time for us to compare the results obtained in the pairing state of massless photons with the current achievements obtained in the BEC of two-dimensional photons and polaritons posing small effective mass. First of all, there are the following similarities: (i) Pairing condensate and Bose–Einstein condensate are all superfluids. (ii) Pairing condensation of massless photons and BEC of 2D polaritons all take place in the momentum space. (iii) In the traveling-wave pairing state, the number of photons is conserved and thus the chemical potential of photons does not vanish, which is similar to the BEC state of photons. However, there are the following contrasts: 1) The pairing state is formed through the association of photons in pairs with opposite wave vectors and helicities, whereas the BEC state is produced when a large fraction of the bosons occupy the lowest quantum state. 2) The photons in the pairing state are all three-dimensional, whereas the photons and polaritons in the BEC state are all two-dimensional or one-dimensional. 3) Pairing state of massless photons appears in the momentum space but BEC of 2D photons takes place in the coordinate space. 4) The photons in the pairing state are all massless, whereas the photons and polaritons in the BEC state pose small effective mass. 5) The elementary excitations in the pairing state are dressed photons, whereas the elementary excitations in the BEC state are phonons. With the above advantages, we believe that the pairing state of photons will find its extensive applications in science and technology.

The remainder of this paper is organized as follows. Section 2 describes our physical model and derives the effective Hamiltonian of the photon system. The traveling-wave superfluid state is studied in Section 3. In Section 3, we illuminate the formation of photon pairs and dressed photons, derive a set of coupled order parameter and number equations, and analyze the dynamical stability of traveling-wave superfluid states. In

Section 4, we present numerical calculation and results. The comprehensive discussion is given in Section 5.

2. Attractive interaction Hamiltonian between photons

A laser light field is incident on a nonlinear polar crystal and induces a macroscopic em field inside the crystal. The present paper investigates a coupled system consisting of the em field and the crystal. By nonlinearity we mean that the crystal is first-order Raman active. For simplicity one assumes the crystal to be of the cubic symmetry. The ion lattice within a volume V has N primitive cells. Let $\boldsymbol{r}_n$ be a suitable reference point inside the n-th cell. The instantaneous displacement of the l ion in cell n from its equilibrium position is given by the vector $\boldsymbol{s}_{nl}$. The incident light field is a linearly polarized coherent light field of a single mode. The incident frequency ω_0 is assumed to be well below the electronic transition frequencies, so that the photon–electron interaction and the related photon absorption by electrons can be ignored. Resonant Raman scattering of photons by phonons occurs when incident or scattered photons are near resonance with fundamental electronic transitions.[28] With the assumption that the incident frequency is well below the electronic transition frequencies, resonant Raman scattering of photons can be ignored. Further we hypothesize that the incident frequency is well above the transverse-optical (TO) phonon frequencies of the crystal, so that the photon–TO–phonon interaction and the related multiphonon absorption of a photon can be omitted. This approximation allows us to make a phenomenological treatment to the interaction between the EM field and the crystal.

Under the influence of the macroscopic EM field, the charge center of the electron shell of an ion shifts relative to that of the nucleus and hence an electric dipole moment is induced in the ion. For the isotropic medium, the interaction Hamiltonian involving the ionic deformations is quadratic in $\boldsymbol{D}$ and has the form[29]

$$H_{\mathrm{I}} = -\frac{1}{2\varepsilon_0^2}\sum_{n,l}\int \mathrm{d}t'\,\tilde{\alpha}_{nl}(t,t')\boldsymbol{D}(\boldsymbol{r}_n,t')\cdot\boldsymbol{D}(\boldsymbol{r}_n,t), \qquad (1)$$

where ε_0 is the permittivity of the vacuum and $\tilde{\alpha}_{nl}(t,t')$ is the effective polarizability of the nl-th ion. In Eq. (1) the local displacement field at an ion location is replaced by the macroscopic displacement field. Because the Hamiltonian of the EM field can be second quantized in light of the procedure of canonical coordinates,[30] the quantized Hamiltonian of the EM field reads as

$$H_{\mathrm{L}} = \sum_{\boldsymbol{k}\sigma}\hbar\omega_{\boldsymbol{k}}a^{\dagger}_{\boldsymbol{k}\sigma}a_{\boldsymbol{k}\sigma}, \qquad (2)$$

where $a^{\dagger}_{\boldsymbol{k}\sigma}$ and $a_{\boldsymbol{k}\sigma}$ are the creation and annihilation operators of circularly polarized photons with wave vector $\boldsymbol{k}$ and helicity $\sigma = \pm 1$, $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ is the frequency of photons, and the zero-point energy is removed. The acoustic vibration of an ion represents a pressure fluctuation at the ion site. If we regard the pressure at an ion site as an independent state variable of the crystal, the lattice vibrations refer to the optical vibrations. When the phonons of the j-th optical branch are created or annihilated by the operators $b^{\dagger}_{\boldsymbol{q}j}$ and $b_{\boldsymbol{q}j}$, in the harmonic approximation, the quantized Hamiltonian for the lattice vibrations has the form

$$H_{\mathrm{C}} = \sum_{\boldsymbol{q}j}\hbar\omega_j(\boldsymbol{q})b^{\dagger}_{\boldsymbol{q}j}b_{\boldsymbol{q}j}, \qquad (3)$$

where $\omega_j(\boldsymbol{q})$ is the frequency of the j-th optical branch and the zero-point energy is discarded also. The operators of photons and optical phonons obey the Bose commutation relations. The whole system Hamiltonian reads as

$$H = H_{\mathrm{L}} + H_{\mathrm{C}} + H_{\mathrm{I}}. \qquad (4)$$

Because the wave vector of light is smaller than the dimensions of the Brillouin zone, conservation of momentum requires that the phonons involved in optical processes be near the center of the Brillouin zone. The symmetries of the small-wave-vector lattice vibrations can then be described by irreducible representations of the crystal point group. The optical vibrations of polar crystals fall into two distinct categories, i.e., polar modes and nonpolar modes. Polar modes carry electric dipole moments and are infrared active, whereas nonpolar modes carry no electric dipole moments and are infrared inactive. In the cubic crystal, each group-theoretical threefold polar mode splits into a TO doublet and a longitudinal-optical singlet, while each nonpolar mode is threefold degenerate at zero wave vector.

For simplicity, the crystal studied is also taken to be centrosymmetric. Polar modes in centrosymmetric crystals have odd parity and are Raman-inactive.[31] In the cubic system, the common polar crystals that are both centrosymmetric and Raman-active have the fluorite structure. At this point, the crystal studied is determined as a certain crystal of the fluorite structure, such as CaF_2. In the fluorite-structure crystal, a primitive cell contains two anions and one cation giving a single nonpolar mode, which is Raman active. For the Raman-active mode, the two anions in the primitive cell move in antiphase, while the cation remains stationary. Since the following treatment has no relation to polar modes, the optical vibrations of the crystal are limited to the Raman-active mode. Now the index l is only used to distinguish the two anions in the basis. In Eq. (3) the Raman-mode frequency $\omega_j(\boldsymbol{q})$ at small wave vectors is replaced with the zero-wave-vector value ω_{R} and the branch index j is deleted. Furthermore, the incident intensity is assumed to be below the threshold of stimulated Brillouin scattering, so that the pressure fluctuations and Raman-active mode of the crystal are excited thermally.

The polarizability $\tilde{\alpha}_{nl}$ is a function of the displacement at the nl-th ion site. Near the equilibrium state of the ion system, the polarizability $\tilde{\alpha}_{nl}$ can be expanded as

$$\tilde{\alpha}_{nl}(t,t') = \tilde{\alpha}_l(t-t') + \delta(t-t')\nabla\alpha_l \cdot \boldsymbol{s}_{nl}(t). \tag{5}$$

Here $\tilde{\alpha}_l(t-t')$ is the polarizability of the l-th basis ion in the equilibrium state and its Fourier transformation $\alpha_l(\omega)$ reflects the material dispersion due to the ionic deformation. $\nabla\alpha_l$ is the differential polarizability vector of the l-th basis ion in the equilibrium state and represents the nonlinearity of the crystal. Thereby the interaction Hamiltonian $H_{\rm I}$ contains two terms. The first term involving $\tilde{\alpha}_l(t-t')$ can be incorporated in the em field energy by introducing a $\boldsymbol{k}$-dependent linear dielectric function $\bar{\varepsilon}(\boldsymbol{k})$. Now, in Eq. (2) $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|/\sqrt{\bar{\varepsilon}(\boldsymbol{k})}$. If $\boldsymbol{e}_{\boldsymbol{k}\sigma}$ is the polarization vector of circularly polarized photons, the second term in $H_{\rm I}$ is quantized as follows:

$$H_{\rm v} = \sum_{\boldsymbol{k}\sigma,\boldsymbol{q}\sigma'} M_{\boldsymbol{k}\sigma}(\boldsymbol{q}\sigma') a^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\sigma'} a_{\boldsymbol{k}\sigma}(b_{\boldsymbol{q}} + b^{\dagger}_{-\boldsymbol{q}}), \tag{6}$$

$$M_{\boldsymbol{k}\sigma}(\boldsymbol{q}\sigma') = -\left(\frac{N\hbar\omega_{\boldsymbol{k}+\boldsymbol{q}}\omega_{\boldsymbol{k}}}{2\omega_R}\right)^{1/2} \times \left[\frac{\hbar}{2V\varepsilon_0\bar{\varepsilon}(\boldsymbol{k})}\right] \mathscr{P}(\boldsymbol{q})(\boldsymbol{e}^{*}_{\boldsymbol{k}+\boldsymbol{q},\sigma'} \cdot \boldsymbol{e}_{\boldsymbol{k}\sigma}),$$

$$\mathscr{P}(\boldsymbol{q}) = \sum_l \frac{1}{\sqrt{m_l}}[\nabla\alpha_l \cdot \boldsymbol{e}_l(\boldsymbol{q})]. \tag{7}$$

Here m_l is the mass of the l-th basis atom and $\boldsymbol{e}_l(\boldsymbol{q})$ is the eigenvector of the optical branch. $\mathscr{P}(\boldsymbol{q})$ is the Raman coefficient that is characteristic of a crystal. One easily finds that $\boldsymbol{e}^{*}_{\boldsymbol{k}'\sigma'} \cdot \boldsymbol{e}_{\boldsymbol{k}\sigma} = (1+\sigma'\sigma\cos\theta)/2$, where θ is the angle between $\boldsymbol{k}'$ and $\boldsymbol{k}$. The Hamiltonian of the coupled system becomes

$$H = H_{\rm L} + H_{\rm C} + H_{\rm v}. \tag{8}$$

In the first order the Hamiltonian $H_{\rm v}$ as given by Eq. (6) leads to Raman scattering of photons by phonons. It is such scattering that gives rise to thermal equilibrium between the radiation and the crystal. To determine the second-order effect of $H_{\rm v}$ on the system of photons, it is desirable to make the unitary transformation $H_{\rm T} = {\rm e}^{-{\rm i}S} H {\rm e}^{{\rm i}S}$ in which S is Hermitian.[32] The exponential functions in the expression can be expanded. The unitary transformation can eliminate $H_{\rm v}$ from H in the first order provided S is given by ${\rm i}[H_{\rm L}+H_{\rm C}, S] = -H_{\rm v}$. In this way S is obtained as

$$S = \sum_{\boldsymbol{k}\sigma,\boldsymbol{q}\sigma'} {\rm i}M_{\boldsymbol{k}\sigma}(\boldsymbol{q}\sigma') \times \left[\frac{a^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\sigma'} a_{\boldsymbol{k}\sigma} b_{\boldsymbol{q}}}{\hbar(\omega_{\boldsymbol{k}+\boldsymbol{q}} - \omega_{\boldsymbol{k}} - \omega_{\rm R})} + \frac{a^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\sigma'} a_{\boldsymbol{k}\sigma} b^{\dagger}_{-\boldsymbol{q}}}{\hbar(\omega_{\boldsymbol{k}+\boldsymbol{q}} - \omega_{\mathbf{k}} + \omega_{\rm R})}\right]. \tag{9}$$

To the second order in S, the transformed Hamiltonian is

$$H_{\rm T} = H_{\rm L} + H_{\rm C} + \frac{1}{2}{\rm i}[H_{\rm v}, S], \tag{10}$$

where $H_{\rm T}$ contains an additional term.

Averaging the additional term with the equilibrium phonon density operator

$$\rho = \frac{\exp(-H_{\rm C}/k_{\rm B}T)}{{\rm Tr}\exp(-H_{\rm C}/k_{\rm B}T)}, \tag{11}$$

where $k_{\rm B}$ is Boltzmann's constant, we find the interaction Hamiltonian among photons:

$$H'_{\rm v} = \frac{1}{2}{\rm i}\langle[H_{\rm v}, S]\rangle = \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}\sigma} \sum_{\sigma_1\sigma'\sigma_2} \frac{M_{\boldsymbol{k}\sigma}(-\boldsymbol{q},\sigma')M_{\boldsymbol{k}'\sigma_1}(\boldsymbol{q}\sigma_2)\omega_{\rm R}}{\hbar[(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}-\boldsymbol{q}})^2 - \omega_{\rm R}^2]} \times a^{\dagger}_{\boldsymbol{k}-\boldsymbol{q},\sigma'} a^{\dagger}_{\boldsymbol{k}'+\boldsymbol{q},\sigma_2} a_{\boldsymbol{k}'\sigma_1} a_{\boldsymbol{k}\sigma}. \tag{12}$$

The interaction matrix elements given by Eq. (12) can be either attractive or repulsive. If the states $\boldsymbol{k}$ and $\boldsymbol{k}-\boldsymbol{q}$ are separated by an energy smaller than $\hbar\omega_{\rm R}$, an attraction is present. The system will have to adjust itself to the presence of this attraction. A physical model for this attraction is as follows. Photons are correlated when the coupling between photons and thermal optical phonons is ignored. Since the energy of the system of thermal optical phonons is constant at a certain temperature, the energy of the system of correlated photons must decrease. The effective photon–photon attraction results from this decrease in the system's energy. In this case, the conservation of energy requires there to be sufficient virtual optical phonons in the crystal to account for the energy deficit. Usually evolution of one of the interacting subsystems could be adiabatically eliminated if characteristic timescales are completely different. The timescales of thermalization for photons and phonons are completely different. The timescale of thermalization for phonons is essentially shorter than that for photons.

In deriving the Hamiltonian (12), we also obtain the change in the one-photon energy due to the interaction of photons with the crystal. This change reflects the material dispersion due to the lattice vibrations. For this the linear dielectric function $\bar{\varepsilon}(\boldsymbol{k})$ is renormalized as the linear dielectric function $\varepsilon(\boldsymbol{k})$. $\varepsilon(\boldsymbol{k})$ is real and spherically symmetric. $\varepsilon(|\boldsymbol{k}|)$ is an ascending function of $|\boldsymbol{k}|$ and $\varepsilon(\infty) = \varepsilon_{\rm h}$, where $\varepsilon_{\rm h}$ is a high-frequency dielectric constant. It is assumed that the crystal is a nondispersion medium having a linear index of refraction $n_0 = \sqrt{\varepsilon}$. Then the photon frequency in the Hamiltonian (2) is rewritten as $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|/n_0$. The effective Hamiltonian of the photon system is $H_{\rm p} = H_{\rm L} + H'_{\rm v}$. For future study we make a remark. Since the incident light field is linearly polarized, the numbers of clockwise and counterclockwise circularly polarized photons are equal for each wave vector in the isotropic medium.

In the optical Kerr effect, the medium possesses an intensity dependent refractive index: $n = n_0 + n_2 I$, where n_2 is the second-order nonlinear refractive index, and I is the

intensity of the light traveling through the medium. A self-focusing Kerr nonlinearity corresponds to $n_2 > 0$, while a self-defocusing Kerr nonlinearity corresponds to $n_2 < 0$. A self-focusing Kerr nonlinearity requires that the sign of the interaction matrix elements given by Eq. (12) should be negative. Namely, an attractive photon–photon interaction is consistent with the positive nonlinearity. It is well known that in the nonlinear dispersive medium, soliton formation is due to the exact balance between the nonlinearity and the group velocity dispersion. When the signs of nonlinearity and group velocity dispersion are opposite, the interaction between nonlinearity and group velocity dispersion may lead to the generation of bright solitons.[33] In particular, bright solitons occur in a self-focusing nonlinear medium with a negative group velocity dispersion.

3. Traveling-wave superfluid state

3.1. Formation of photon pairs in nonlinear waveguides

We are going to investigate the traveling-wave superfluid state in the following. As shown in Fig. 1, the traveling-wave configuration corresponds to a cylindrical dielectric waveguide whose core is occupied by the first crystal and whose cladding is occupied by the second crystal. The physical quantities in the first and second crystals are marked with subscripts 1 and 2. The two high-frequency dielectric constants must satisfy the inequality $\varepsilon_{\rm h1} > \varepsilon_{\rm h2}$. The laser light field is normally incident on the end face $z = 0$ of the core at the instant $t = 0$. In Section 2 we assumed that the incident light field was a monochromatic field. Now the incident light field is required to be a quasimonochromatic field of a central frequency ω_0. The incident light field induces a macroscopic em field in the waveguide, which propagates with a group velocity $v_{\rm g}$. At time t we investigate a length of waveguide in the interval $z = 0$ to z, where $z = v_{\rm g}t$. The inequality $\varepsilon_{\rm h1} > \varepsilon_{\rm h2}$ limits the plane-wave modes appearing in the Hamiltonian (2) to the guided modes. Each wave vector of the guided modes is separated into $\boldsymbol{k} = \boldsymbol{K} + \boldsymbol{p}$, where $\boldsymbol{K}$ and $\boldsymbol{p}$ are the components parallel to and transverse to the z axis. The axial wave vector $\boldsymbol{K}$ is real everywhere. $\boldsymbol{p}$ is real in the core, but imaginary in the cladding. For a fixed K, the values of allowed p in the core are in the range $0 \leq p < K\tan\theta_{\rm c}$, where $\theta_{\rm c}$ is the complement of the critical angle of total internal reflection defined by $\sqrt{\varepsilon_{\rm h1}}\cos\theta_{\rm c} = \sqrt{\varepsilon_{\rm h2}}$. While propagating in the core, a guided wave undergoes a series of total internal reflections at the core-cladding interface. Consequently, the energy carried by the guided modes is confined to the vicinity of the core and diffraction effects are eliminated. The axial wave number K is a function of the incident frequency ω. The function $K(\omega)$ must be specified by the Maxwell equations together with the boundary conditions at the core-cladding interface [34]. Next we introduce the waveguide frequency $F = \omega_0\rho(\varepsilon_{\rm h1} - \varepsilon_{\rm h2})^{1/2}/c$, where ρ is the core radius, and the profile height parameter $\delta = (\varepsilon_{\rm h1} - \varepsilon_{\rm h2})/2\varepsilon_{\rm h1}$. If $0 < F < 2.4048$, the waveguide supports only the fundamental mode. Under the weak-guidance approximation $\delta \ll 1$, the propagation constant $K_0 = K(\omega_0)$ of the fundamental mode is the largest value of K_0 determined by the eigenvalue equation

$$U\frac{\mathrm{J}_1(U)}{\mathrm{J}_0(U)} = W\frac{\mathrm{K}_1(W)}{\mathrm{K}_0(W)}, \tag{13}$$

where $U = \rho(\varepsilon_{\rm h1}\omega_0^2/c^2 - K_0^2)^{1/2}$ and $F^2 = U^2 + W^2$. $\mathrm{J}_0(x)$ and $\mathrm{J}_1(x)$ are the Bessel functions of the first kind, while $\mathrm{K}_0(x)$ and $\mathrm{K}_1(x)$ are the modified Bessel functions of the second kind. By the definition $v_{\rm g}^{-1} = (\partial \mathrm{K}/\partial\omega)_{\omega=\omega_0}$, the group velocity $v_{\rm g}$ of the fundamental mode is obtained as

$$v_{\rm g} = \frac{c}{\sqrt{\varepsilon_{\rm h1}}}\left\{1 + \delta\frac{U^2}{F^2}\left[1 - 2\frac{\mathrm{K}_0^2(W)}{\mathrm{K}_1^2(W)}\right]\right\}. \tag{14}$$

Strictly speaking, the above definition of group velocity $v_{\rm g}$ is valid for a non-absorbing dielectric only.[35] But when a dielectric is absorbing, the axial wave number K is now complex, so that the concept of group velocity breaks down. In this case, the true energy velocity $v_{\rm e}$ for an em wave can be defined as in Ref. [35]. Generally speaking, the photons in a waveguide are three-dimensional and massless. The reasons for this are that the wave vector in a waveguide is a three-dimensional wave vector $\boldsymbol{k} = \boldsymbol{K} + \boldsymbol{p}$ and that the photonic energy in a waveguide is $\hbar\omega_{\boldsymbol{k}} = \hbar c|\boldsymbol{k}|/n_0$. Under the approximation $|\boldsymbol{K}| \ll |\boldsymbol{p}|$, the photons in a waveguide are effectively one-dimensional photons, which pose a small effective mass determined by the optical field properties in a transverse plane. However, the approximation $|\boldsymbol{K}| \ll |\boldsymbol{p}|$ holds only for a hollow metallic waveguide.[24] For a dielectric waveguide, the approximation $|\boldsymbol{K}| > |\boldsymbol{p}|$ holds.

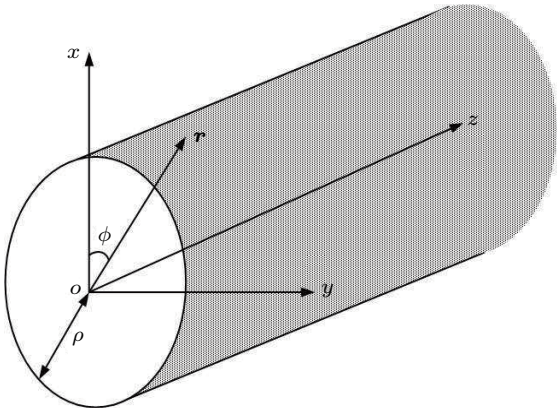


Fig. 1. Section of a cylindrical dielectric waveguide. The z axis coincides with the waveguide axis of symmetry and ρ is the core radius.

The fundamental mode photons of wave vectors $\boldsymbol{K} + \boldsymbol{p}$ and $\boldsymbol{K} - \boldsymbol{p}$ always occur simultaneously and have an equal number. By exchange of virtual optical phonons, the fundamental mode photons can experience an attractive effective interaction. Such an interaction leads to the traveling-wave superfluid state, in which a propagating photon pair is

the combination of $(\boldsymbol{K}+\boldsymbol{p},\sigma)$ with $(\boldsymbol{K}-\boldsymbol{p},-\sigma)$ and thus has the composite momentum $2\hbar\boldsymbol{K}$. The propagating photon pair is created by the operator $a^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}a^{\dagger}_{\boldsymbol{K}-\boldsymbol{p},-\sigma}$. In general, the creation operator of a propagating photon pair is $a^{\dagger}_{\boldsymbol{K}+\boldsymbol{k},\sigma}a^{\dagger}_{\boldsymbol{K}-\boldsymbol{k},-\sigma}$, where $\boldsymbol{k}$ is an arbitrary wave vector. If $\boldsymbol{k}$ has a nonzero axial component, however, the pairing probability of these two photons is almost zero. The two photons with different axial wave vectors propagate independently. Where $a^{\dagger}_{\boldsymbol{K}+\boldsymbol{k},\sigma}a^{\dagger}_{\boldsymbol{K}-\boldsymbol{k},-\sigma}$ is concerned, we therefore let $\boldsymbol{k}=\boldsymbol{p}$. In Eq. (12) we set $\boldsymbol{k}=\boldsymbol{K}+\boldsymbol{p}$, $\boldsymbol{k}'=\boldsymbol{K}-\boldsymbol{p}$, $\sigma_1=-\sigma$, and $\sigma_2=-\sigma'$. Then one changes index $\boldsymbol{q}$ of the summation into $\boldsymbol{p}'=\boldsymbol{p}-\boldsymbol{q}$. The pair Hamiltonian of the photon system in the core region reads, consequently,

$$H_{\rm p}=\frac{1}{2}\sum_{\boldsymbol{K},\boldsymbol{p}\sigma}\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}(a^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}a_{\boldsymbol{K}+\boldsymbol{p},\sigma}+a^{\dagger}_{\boldsymbol{K}-\boldsymbol{p},-\sigma}a_{\boldsymbol{K}-\boldsymbol{p},-\sigma})+\sum_{\substack{\boldsymbol{K},\boldsymbol{p}\sigma\\ \boldsymbol{p}'\sigma'}}V_{\boldsymbol{K}+\boldsymbol{p},\sigma;\boldsymbol{K}+\boldsymbol{p}',\sigma'}a^{\dagger}_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}a^{\dagger}_{\boldsymbol{K}-\boldsymbol{p}',-\sigma'}a_{\boldsymbol{K}-\boldsymbol{p},-\sigma}a_{\boldsymbol{K}+\boldsymbol{p},\sigma}, \quad (15)$$

$$V_{\boldsymbol{K}+\boldsymbol{p},\sigma;\boldsymbol{K}+\boldsymbol{p}',\sigma'}=\frac{|M_{\boldsymbol{K}+\boldsymbol{p},\sigma}(\boldsymbol{p}'-\boldsymbol{p},\sigma')|^2\omega_R}{\hbar[(\omega_{\boldsymbol{K}+\boldsymbol{p}}-\omega_{\boldsymbol{K}+\boldsymbol{p}'})^2-\omega_R^2]}. \quad (16)$$

Here the pair potential is real and satisfies the symmetric properties $V_{\boldsymbol{K}+\boldsymbol{p},\sigma;\boldsymbol{K}+\boldsymbol{p}',\sigma'}=V_{\boldsymbol{K}-\boldsymbol{p},-\sigma;\boldsymbol{K}-\boldsymbol{p}',-\sigma'}$ and $V_{\boldsymbol{K}+\boldsymbol{p},\sigma;\boldsymbol{K}+\boldsymbol{p}',\sigma'}=V_{\boldsymbol{K}+\boldsymbol{p}',\sigma';\boldsymbol{K}+\boldsymbol{p},\sigma}$. We also need the Hamiltonian of the em field in the core region, which from Eq. (2) takes the form

$$H_{\rm L}=\frac{1}{2}\sum_{\boldsymbol{K},\boldsymbol{p}\sigma}\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}(a^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}a_{\boldsymbol{K}+\boldsymbol{p},\sigma}+a^{\dagger}_{\boldsymbol{K}-\boldsymbol{p},-\sigma}a_{\boldsymbol{K}-\boldsymbol{p},-\sigma}). \quad (17)$$

The double sums over $\boldsymbol{K}$ and $\boldsymbol{p}$ in Eqs. (15) and (17) represent a sum over wave vectors $\boldsymbol{k}=\boldsymbol{K}+\boldsymbol{p}$ in three dimensions. The direction Ω of $\boldsymbol{k}=\boldsymbol{K}+\boldsymbol{p}$ is specified by a polar angle θ and an azimuthal angle ϕ. The double sums are calculated in the following way. For a fixed $\boldsymbol{K}$ we first find the sum over $\boldsymbol{p}$ in the region $0\leq p<K\tan\theta_c$ and $0\leq\phi\leq 2\pi$. Then we find the sum over the axial wave vectors $\boldsymbol{K}(\omega)$ in a small interval near $\boldsymbol{K}_0$. The sum over $\boldsymbol{K}(\omega)$ reflects the fact that the incident light field is a quasimonochromatic field of a central frequency ω_0. Since $\boldsymbol{p}'$ is a 2D wave vector, the sum over $\boldsymbol{p}'$ in Eq. (15) is inconsistent with the three-dimensional photon system. The reason for this is that in deriving Eq. (15) we replace a small three-dimensional wave vector $\boldsymbol{q}$ with $\boldsymbol{p}'$. Therefore $\boldsymbol{p}'$ is replaced self-consistently by $\boldsymbol{q}$ in the end.

A laser beam is regarded as a gas of coherent photons. One may consider that in the absence of an attractive photon–photon interaction, the boundary conditions of a waveguide can induce a photon pair of $(\boldsymbol{K}+\boldsymbol{p},\sigma)$ and $(\boldsymbol{K}-\boldsymbol{p},-\sigma)$. However, such a photon pair is a free photon pair and hence a laser beam propagating in a waveguide is a gas of free photon pairs. An attractive photon–photon interaction can lead to a bound photon pair of $(\boldsymbol{K}+\boldsymbol{p},\sigma)$ and $(\boldsymbol{K}-\boldsymbol{p},-\sigma)$. In this case, a laser beam propagating in a waveguide is a liquid of bound photon pairs. A liquid of bound photon pairs is in a condensation state, which is the traveling-wave superfluid state of photons. The physical behavior of the traveling-wave superfluid state resembles a coherent liquid droplet. One could expect a phase transition from a gas cloud to a coherent liquid droplet. In the following, we shall investigate the properties of the traveling-wave superfluid state in detail.

3.2. Production of dressed photons in nonlinear waveguides

Single, unpaired bare photons in the photon system are transformed into a new kind of quasiparticle: the dressed photon. A dressed photon is the photon clothed with a cloud of virtual transverse-optical phonons. Diagonalization of the pair Hamiltonian in Eq. (15) can be performed by the Bogoliubov transformation:

$$\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}=Ua_{\boldsymbol{K}+\boldsymbol{p},\sigma}U^{\dagger}=a_{\boldsymbol{K}+\boldsymbol{p},\sigma}\cosh\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}-a^{\dagger}_{\boldsymbol{K}-\boldsymbol{p},-\sigma}\sinh\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}, \quad (18)$$

$$\alpha^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}=Ua^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}U^{\dagger}=a^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}\cosh\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}-a_{\boldsymbol{K}-\boldsymbol{p},-\sigma}\sinh\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}, \quad (19)$$

where the parameter $\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}$ is assumed to be real and has the symmetry: $\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}=\varphi_{\boldsymbol{K}-\boldsymbol{p},-\sigma}$. $\alpha^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}$ and $\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}$ are the creation and annihilation operators, respectively, of quasiparticles in the photon system; they also obey the Bose commutation relations. The transition from the operators of bare photons to those of quasiparticles can be effected by a unitary transformation:

$$U=\exp\Big[\frac{1}{2}\sum_{\boldsymbol{K},\boldsymbol{p}\sigma}\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}(a^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}a^{\dagger}_{\boldsymbol{K}-\boldsymbol{p},-\sigma}-a_{\boldsymbol{K}-\boldsymbol{p},-\sigma}a_{\boldsymbol{K}+\boldsymbol{p},\sigma})\Big]. \quad (20)$$

The unitary transformation U given by Eq. (20) is called a squeeze operator. It is well known that the unitary transformation does not change the energy spectrum of the photon system. The normalized state vector of photon pairs in the photon system may be constructed as $|G\rangle=U|0\rangle$, where $|0\rangle$ is the vacuum state of photons, such that $\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}|G\rangle=0$. Because the incident light field

is fully coherent, dressed photons are in a coherent state of many modes. Concomitantly, we introduce the displacement operator

$$D(\eta)=\exp\left[\sum_{\boldsymbol{K},\boldsymbol{p}\sigma}\left(\eta_{\boldsymbol{K}+\boldsymbol{p},\sigma}\alpha^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}-\eta^{*}_{\boldsymbol{K}+\boldsymbol{p},\sigma}\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}\right)\right], \tag{21}$$

where the parameter $\eta_{\boldsymbol{K}+\boldsymbol{p},\sigma}$ is complex and spherically symmetric. The state vector of the photon system may be written as $|\eta\rangle=D(\eta)|G\rangle$, such that $\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}|\eta\rangle=\eta_{\boldsymbol{K}+\boldsymbol{p},\sigma}|\eta\rangle$. Note that the state vector $|\eta\rangle$ represents the pairing state of the photon system. The squeeze operator U given by Eq. (20) embodies the pairing correlation. $|\eta_{\boldsymbol{K}+\boldsymbol{p},\sigma}|^2$ characterizes the mean dressed-photon number of a mode field and can be described by a distribution function $f_{\boldsymbol{K}+\boldsymbol{p},\sigma}(u)=|\eta_{\boldsymbol{K}+\boldsymbol{p},\sigma}|^2$, where the argument u is the average energy density of the em field.

The transformation in Eqs. (18) and (19) can easily be inverted

$$a_{\boldsymbol{K}+\boldsymbol{p},\sigma}=\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}\cosh\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}+\alpha^{\dagger}_{\boldsymbol{K}-\boldsymbol{p},-\sigma}\sinh\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma},\quad a^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}=\alpha^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}\cosh\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}+\alpha_{\boldsymbol{K}-\boldsymbol{p},-\sigma}\sinh\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}. \tag{22}$$

It is convenient to define the number operators $N_{\boldsymbol{K}+\boldsymbol{p},\sigma}=\alpha^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}$ for quasiparticles. We substitute Eq. (22) into Eq. (15). After some arrangement and the omission of fourth-order nondiagonal terms, the pair Hamiltonian in Eq. (15) becomes

$$\begin{aligned}H_{\rm p}=E_{\rm p}+\frac{1}{2}\sum_{\boldsymbol{K},\boldsymbol{p}\sigma}&\Bigg[\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}\cosh2\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}+\sinh2\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}\sum_{\boldsymbol{p}'\sigma'}V_{\boldsymbol{K}+\boldsymbol{p},\sigma;\boldsymbol{K}+\boldsymbol{p}',\sigma'}\sinh2\varphi_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}\\&\times\left(1+\frac{1}{2}N_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}+\frac{1}{2}N_{\boldsymbol{K}-\boldsymbol{p}',-\sigma'}\right)\Bigg](N_{\boldsymbol{K}+\boldsymbol{p},\sigma}+N_{\boldsymbol{K}-\boldsymbol{p},-\sigma})\\&+\frac{1}{2}\sum_{\boldsymbol{K},\boldsymbol{p}\sigma}\Bigg[\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}\sinh2\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}+\cosh2\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}\sum_{\boldsymbol{p}'\sigma'}V_{\boldsymbol{K}+\boldsymbol{p},\sigma;\boldsymbol{K}+\boldsymbol{p}',\sigma'}\\&\times\sinh2\varphi_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}(1+N_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}+N_{\boldsymbol{K}-\boldsymbol{p}',-\sigma'})\Bigg](\alpha^{\dagger}_{\boldsymbol{K}+\boldsymbol{p},\sigma}\alpha^{\dagger}_{\boldsymbol{K}-\boldsymbol{p},-\sigma}+\alpha_{\boldsymbol{K}-\boldsymbol{p},-\sigma}\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}),\end{aligned} \tag{23}$$

where $E_{\rm p}$ is the energy of the system of photon pairs, as given by

$$E_{\rm p}=\sum_{\boldsymbol{K},\boldsymbol{p}\sigma}\left[\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}\sinh^2\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}+\frac{1}{4}\sinh2\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}\sum_{\boldsymbol{p}'\sigma'}V_{\boldsymbol{K}+\boldsymbol{p},\sigma;\boldsymbol{K}+\boldsymbol{p}',\sigma'}\sinh2\varphi_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}\right]. \tag{24}$$

It is necessary to define excitation energy $E_{\boldsymbol{K}+\boldsymbol{p}}(u)$ of quasiparticles for any em energy density u. To this end, let $\langle H_{\rm p}\rangle$ and $\langle N_{\boldsymbol{K}+\boldsymbol{p},\sigma}\rangle$ denote, respectively, thermal averages of the Hamiltonian $H_{\rm p}$ in Eq. (23) and the number operator $N_{\boldsymbol{K}+\boldsymbol{p},\sigma}$ for energy density u. The basis states used in constructing this average are the state vectors of the photon system, which are given by $|\eta\rangle=D(\eta)|G\rangle$, where $|G\rangle=U|0\rangle$. The photon system in the pairing state consists of dressed photons and photon pairs. The subsystem of photon pairs is in the squeezed state $|G\rangle=U|0\rangle$. The squeezed state $|G\rangle$ of photon pairs acts as the vacuum state of dressed photons, so that the subsystem of dressed photons is in the coherent state $|\eta\rangle=D(\eta)|G\rangle$. Namely, the pairing state of the photon system is a coherent squeezed state. Our definition is

$$E_{\boldsymbol{K}+\boldsymbol{p}}(u)=\partial\langle H_{\rm p}\rangle/\partial\langle N_{\boldsymbol{K}+\boldsymbol{p},\sigma}\rangle.$$

Within the framework of a mean-field theory,[32] we can approximate the averages of products of number operators as

$$\langle N_{\boldsymbol{K}+\boldsymbol{p},\sigma}N_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}\rangle\approx\langle N_{\boldsymbol{K}+\boldsymbol{p},\sigma}\rangle\langle N_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}\rangle. \tag{25}$$

We then find that

$$\begin{aligned}E_{\boldsymbol{K}+\boldsymbol{p}}(u)=\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}&\cosh2\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}+\sinh2\varphi_{\boldsymbol{K}+\boldsymbol{p},\sigma}\sum_{\boldsymbol{p}'\sigma'}V_{\boldsymbol{K}+\boldsymbol{p},\sigma;\boldsymbol{K}+\boldsymbol{p}',\sigma'}\sinh2\varphi_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}\\&\times(1+\langle N_{\boldsymbol{K}+\boldsymbol{p}',\sigma'}\rangle+\langle N_{\boldsymbol{K}-\boldsymbol{p}',-\sigma'}\rangle).\end{aligned} \tag{26}$$

Because $\alpha_{\boldsymbol{K}+\boldsymbol{p},\sigma}|\eta\rangle=\eta_{\boldsymbol{K}+\boldsymbol{p},\sigma}|\eta\rangle$, the thermal average of the number operator $N_{\boldsymbol{K}+\boldsymbol{p},\sigma}$ is given by

$$\langle N_{\boldsymbol{K}+\boldsymbol{p},\sigma}\rangle=\langle\eta|N_{\boldsymbol{K}+\boldsymbol{p},\sigma}|\eta\rangle=|\eta_{\boldsymbol{K}+\boldsymbol{p},\sigma}|^2.$$

Concomitantly, we set

$$\langle N_{\boldsymbol{K}+\boldsymbol{p},\sigma}\rangle=f_{\boldsymbol{K}+\boldsymbol{p},\sigma}(u)=\frac{1}{\exp[(E_{\boldsymbol{K}+\boldsymbol{p}}-\mu)/k_{\rm B}T]-1}, \tag{27}$$

where $E_{\boldsymbol{K}+\boldsymbol{p}}$ is the energy of a dressed photon. In the traveling-wave pairing state, the number of dressed photons is conserved and thus the dressed photons possess a chemical potential μ.

In the second-order nondiagonal terms of Eq. (23), one can replace the number operators with their thermal averages. The second-order nondiagonal terms should vanish on condition that

$$\hbar\omega_{K+p}\sinh 2\varphi_{K+p,\sigma}+\cosh 2\varphi_{K+p,\sigma}\sum_{p'\sigma'}V_{K+p,\sigma;K+p',\sigma'}\sinh 2\varphi_{K+p',\sigma'}(1+\langle N_{K+p',\sigma'}\rangle+\langle N_{K-p',-\sigma'}\rangle)=0. \tag{28}$$

One must solve Eqs. (26), (27), and (28) simultaneously to determine the parameter $\varphi_{K+p,\sigma}$. The solution is facilitated if we introduce a density-dependent quantity $\Delta_{K+p}(u)$ by

$$\Delta_{K+p}(u)=-\sum_{p'\sigma'}V_{K+p,\sigma;K+p',\sigma'}\sinh 2\varphi_{K+p',\sigma'}[1+2f_{K+p',\sigma'}(u)]. \tag{29}$$

Equation (29) is inserted into Eq. (28), which immediately yields

$$\tanh 2\varphi_{K+p,\sigma}=\frac{\Delta_{K+p}(u)}{\hbar\omega_{K+p}}. \tag{30}$$

As the hyperbolic tangent function is real, it is required that $\Delta_{K+p}(u)\leq\hbar\omega_{K+p}$. The parameter $\varphi_{K+p,\sigma}$ can also be determined by the relations

$$\cosh 2\varphi_{K+p,\sigma}=\frac{\hbar\omega_{K+p}}{\sqrt{\hbar^2\omega_{K+p}^2-\Delta_{K+p}^2(u)}}, \tag{31}$$

$$\sinh 2\varphi_{K+p,\sigma}=\frac{\Delta_{K+p}(u)}{\sqrt{\hbar^2\omega_{K+p}^2-\Delta_{K+p}^2(u)}}. \tag{32}$$

Equations (29), (31), and (32) may now be substituted back into Eq. (26). As a result, the excitation energy of quasiparticles is derived as

$$E_{K+p}(u)=\left[(\hbar\omega_{K+p})^2-\Delta_{K+p}^2(u)\right]^{1/2}, \tag{33}$$

where the excitation spectrum $E_{K+p}(u)$ has no gap and $\Delta_{K+p}(u)$ is the order parameter for pairing of photons. $E_{K+p}(u)$ is dependent on energy density.

3.3. Order parameter and number equations

As we have seen, the pair Hamiltonian in Eq. (15) can be solved only when the pair potential $V_{K+p,\sigma;K+p',\sigma'}$ is negative. When equations (32) and (33) are substituted into Eq. (29), the order parameter is determined self-consistently by the following equation

$$\Delta_{K+p}(u)=-\sum_{q\sigma'}\frac{V_{K+p,\sigma;K+q,\sigma'}\Delta_{K+q}(u)}{E_{K+q}(u)}[1+2f_{K+q,\sigma'}(u)], \tag{34}$$

which is suitable for any energy density u. Under the mean-field approximation, the pair Hamiltonian of the photon system in Eq. (23) becomes

$$H_{\rm p}=E_{\rm p}(u)+\sum_{K,p\sigma}E_{K+p}(u)\alpha_{K+p,\sigma}^{\dagger}\alpha_{K+p,\sigma}. \tag{35}$$

The idea of a mean-field approximation was introduced by the Weiss theory of ferromagnetism to deal with phase transitions.[36] In the present article, the idea is that individual dressed photons move independently in a mean field caused by all the other photons, which includes parts of the photon–photon interaction.

At this point, the transformation in Eq. (22) is substituted into Eq. (17). After some arrangement, the Hamiltonian of the em field in Eq. (17) becomes

$$H_{\rm L}=\frac{1}{2}\sum_{K,p\sigma}\hbar\omega_{K+p}\Big[(N_{K+p,\sigma}+N_{K-p,-\sigma})\cosh 2\varphi_{K+p,\sigma}+(\alpha_{K+p,\sigma}^{\dagger}\alpha_{K-p,-\sigma}^{\dagger}+\alpha_{K-p,-\sigma}\alpha_{K+p,\sigma})\sinh 2\varphi_{K+p,\sigma}$$
$$+\cosh 2\varphi_{K+p,\sigma}-1\Big]. \tag{36}$$

The energy $E_{\rm L}$ of the EM field in the core region is the expectation value of H_L with respect to the state vector $|\eta\rangle$ of the photon system, namely,

$$E_{\rm L}=\langle\eta|H_L|\eta\rangle=\frac{1}{2}\sum_{K,p\sigma}\hbar\omega_{K+p}\Big[(\langle N_{K+p,\sigma}\rangle+\langle N_{K-p,-\sigma}\rangle)\cosh 2\varphi_{K+p,\sigma}+\cosh 2\varphi_{K+p,\sigma}-1\Big]. \tag{37}$$

The second-order nondiagonal terms contain the factor $\eta_{K+p,\sigma}^{*}\eta_{K-p,-\sigma}^{*}+\eta_{K-p,-\sigma}\eta_{K+p,\sigma}$. $\eta_{K+p,\sigma}$ has a definite phase factor depending on its suffix $(K+p,\sigma)$. We shall assume that there is no correlation between these phase factors when $(K+p,\sigma)\neq(K+p',\sigma')$. Strictly speaking, the phase of $\eta_{K+p,\sigma}$ is random to some degree, because of thermal scattering. This is the thermal phase noise in the system of dressed photons. Consequently, the second-order nondiagonal terms disappear under the random-phase approximation. Equations (27), (31), and (33) may now be substituted back into Eq. (37). The energy of the EM field in the core region is acquired as

$$E_{\rm L}=\frac{1}{2}\sum_{K,p\sigma}\hbar\omega_{K+p}\left[\frac{\hbar\omega_{K+p}}{E_{K+p}(u)}-1\right]+\sum_{K,p\sigma}\frac{(\hbar\omega_{K+p})^2}{E_{K+p}(u)}f_{K+p,\sigma}(u). \tag{38}$$

The first and second terms represent the contributions from paired photons and dressed photons, respectively. Because Eq. (38) contains the mean number $f_{\boldsymbol{K}+\boldsymbol{p},\sigma}(u)$ of dressed photons, equation (38) is called the number equation.

In the coherent state, namely the normal state with $\Delta_{\boldsymbol{K}+\boldsymbol{p}}=0$, the behavior of photons is governed by the Hamiltonian (8) of the coupled system. The normal photons suffer spontaneous Raman scattering by optical phonons as well as Rayleigh and spontaneous Brillouin scatterings. It should be noted that the normal state with $\Delta_{\boldsymbol{K}+\boldsymbol{p}}=0$ is a thermal equilibrium state and can be described by thermal equilibrium theory. However, the normal state with $\Delta_{\boldsymbol{K}+\boldsymbol{p}}=0$ cannot be described by the pairing theory of photons. In the superfluid state, photon pairs and single dressed photons are present together. A photon pair is comprised of two bare photons by exchange of virtual optical phonons and a dressed photon is the bare photon clothed with a cloud of virtual transverse-optical phonons. Since Rayleigh and spontaneous Brillouin scatterings are overcome by attractive photon–phonon–photon interaction, the resultant photon pairs incorporate the optical phonons and hence do not suffer any scatterings. Furthermore, the system of photon pairs is a condensate because a macroscopically large number of photon pairs occupy every quantum state of wave vector $2\boldsymbol{K}$ and zero spin, where $\boldsymbol{K}$ varies only in a small interval near $\boldsymbol{K}_0$. However, the fourth-order non-diagonal terms of the Hamiltonian in Eq. (35) represent the interaction between dressed photons. As this interaction originates from the scattering mechanisms, individual dressed photons experience scatterings like the normal photons. Because of continual scattering in the waveguide, dressed photons are distributed continuously in the whole wave-vector space. The wave vector $\boldsymbol{K}+\boldsymbol{p}$ is specified by the frequency ω and the traveling direction Ω of bare photons, where Ω corresponds to a polar angle θ and an azimuthal angle ϕ. It should be noted that the frequency of bare photons is $\omega=\omega_{\boldsymbol{K}+\boldsymbol{p}}$ whereas the frequency of dressed photons is $\tilde{\omega}=E_{\boldsymbol{K}+\boldsymbol{p}}/\hbar$. The traveling direction Ω of bare photons is that of dressed photons.

The frequency spectrum of dressed photons can be described by a Gaussian distribution with frequency width $2\omega_b$ centered at the incident frequency ω_0. ω_b is an increasing function of temperature, but always far smaller than ω_0. We write the distribution function of dressed photons in the separation form of variables

$$f_{\boldsymbol{K}+\boldsymbol{p},\sigma}(u)=N(u)g(\Omega)\exp\left[-\left(\frac{\omega-\omega_0}{\omega_b}\right)^2\right]. \tag{39}$$

Here $g(\Omega)$ represents the direction distribution of dressed photons and must satisfy the conical symmetry

$$g(\Omega)=\begin{cases}1, & 0\leq\theta<\theta_{\rm c},\ 0\leq\phi<2\pi,\\ 0, & \theta_{\rm c}\leq\theta\leq\pi,\ 0\leq\phi<2\pi.\end{cases} \tag{40}$$

The average energy density of the em field is $u=E_{\rm L}/V$, where $E_{\rm L}$ is given by Eq. (38) and $V=\pi\rho^2 z$ is the volume of the core region. Therefore, equation (38) determines the functional relation of $N(u)$ to u and is called the number equation. Equation (34) determines the functional relation of $\Delta_{\boldsymbol{K}+\boldsymbol{p}}(u)$ to u and is called the order parameter equation. Equations (34) and (38) must be solved simultaneously. This case is similar to the BCS–BEC crossover theory of superconductivity, in which the superconducting order parameter $\Delta(T)$ and the chemical potential $\mu(T)$ are uniquely determined by the order parameter and number equations.

Because the intensity I of incident light must be equal to the intensity of the light propagating in a waveguide, the chemical potential μ is determined by the number equation (38). We consider the limit of low temperatures where

$$(E_{\boldsymbol{K}+\boldsymbol{p}}-\mu)/k_{\rm B}T\gg 1.$$

In the limit of low temperatures, equation (27) is rewritten as

$$f_{\boldsymbol{K}+\boldsymbol{p},\sigma}(u)=\exp(\mu/k_{\rm B}T)\exp(-E_{\boldsymbol{K}+\boldsymbol{p}}/k_{\rm B}T). \tag{41}$$

In Eq. (41), for the first factor we let

$$\exp(\mu/k_{\rm B}T)=N(u)g(\Omega). \tag{42}$$

The order parameter $\Delta_{\boldsymbol{K}+\boldsymbol{p}}(u)$ is a function of wave vector $\boldsymbol{K}+\boldsymbol{p}$, energy density u, and temperature T. Below Eq. (30), we have established the following relation

$$\Delta_{\boldsymbol{K}+\boldsymbol{p}}(u)\leq\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}.$$

If we ignore the dependence of the order parameter on energy density u, at zero temperature ($T=0$ K) the order parameter attains its maximal value, namely $\Delta_{\boldsymbol{K}+\boldsymbol{p}}=\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}$. It should be noted that because the order parameter $\Delta_{\boldsymbol{K}+\boldsymbol{p}}$ represents the degree for pairing of photons, $\Delta_{\boldsymbol{K}+\boldsymbol{p}}=\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}}$ at zero temperature corresponds to the limit of strong pairing. In the limit of low temperatures, in an ad hoc way we can write the order parameter as

$$\Delta^2_{\boldsymbol{K}+\boldsymbol{p}}=(\hbar\omega_{\boldsymbol{K}+\boldsymbol{p}})^2-(k_{\rm B}T)^2\left(\frac{\omega-\omega_0}{\omega_b}\right)^4.$$

Concomitantly, from Eq. (33) one obtains the energy of dressed photons as

$$E_{\boldsymbol{K}+\boldsymbol{p}}=k_B T\left(\frac{\omega-\omega_0}{\omega_b}\right)^2.$$

Once the last expression is substituted into the second factor of equation (41), we find that

$$\exp(-E_{\boldsymbol{K}+\boldsymbol{p}}/k_{\rm B}T)=\exp\left[-\left(\frac{\omega-\omega_0}{\omega_b}\right)^2\right]. \tag{43}$$

As a result, equation (41) is equivalent to Eq. (39). Generally speaking, the order parameter equation (34) and the

number equation (38) are nonlinear and are hardly solved. The reason for this is that the quantity E_{K+p} in the factor $\exp(-E_{K+p}/k_{\rm B}T)$ is related to the order parameter Δ_{K+p} through Eq. (33) and hence E_{K+p} is a quantity to be determined by the order parameter equation (34). Equation (43) means that the factor $\exp(-E_{K+p}/k_{\rm B}T)$ to be determined is replaced by the frequency distribution of incident light, which is known. In this case, the order parameter equation (34) and the number equation (38) are linearized. Therefore, the distribution function given by Eq. (39) is not a form of hypothesis. We conclude that the distribution function given by Eq. (39) is the low-temperature and linearized form of the well-known Bose–Einstein distribution.

4. Calculation of properties of the superfluid state

Since the em energy density u is not a good physical quantity for the waveguide, we introduce the light intensity I in the core region by the relation $I = uv_g$. Below some light intensity equation (34) has the superfluid-state solution, namely, $\Delta_{K+p}(I) \neq 0$ for all $K+p$. In the following we introduce the frequency $\tilde{\omega}_{K+p}(I)$ of dressed photons by the relation $E_{K+p}(I) = \hbar\tilde{\omega}_{K+p}(I)$. Because the frequency $\tilde{\omega}_{K+p}(I)$ of dressed photons must be directly proportional to the wave number $|K+p|$, according to Eq. (33) this criterion demands that $\Delta_{K+p}(I) \propto |K+p|$. Correspondingly, we let $\Delta_{K+p}(I) = \hbar\omega_{K+p}\Delta(I)$, where $\omega_{K+p} = c|K+p|/n_0$ and $\Delta(I)$ is the reduced order parameter that depends on light intensity but not on wave vector. Consequently, the frequency of dressed photons is acquired as

$$\tilde{\omega}_{K+p}(I) = v(I)|K+p|,$$

where the velocity of dressed photons has been introduced by

$$v(I) = v_{\rm c}\sqrt{1-\Delta^2(I)}, \tag{44}$$

where $v_{\rm c} = c/n_0$ is the critical velocity. The reduced order parameter $\Delta(I)$ is nonzero below some light intensity and is meaningful only if $\Delta(I) < 1$. However, if $\Delta(I) \geq 1$, the photon system goes into a coherent state. Concomitantly, the excitation energy of a dressed photon is given by

$$E_{K+p}(I) = \hbar\omega_{K+p}\frac{v(I)}{v_{\rm c}}. \tag{45}$$

In order to derive the equation for $v(I)$ from Eq. (34), we need to obtain a simple expression of the pair potential $V_{K+p,\sigma;K+p',\sigma'}$ from Eq. (16). In Eq. (16), we let $\omega_{K+p} = \omega_{K+p'}$, so that

$$V_{K+p,\sigma;K+p',\sigma'} = -\frac{|M_{K+p,\sigma}(p'-p,\sigma')|^2}{\hbar\omega_{\rm R}}, \tag{46}$$

if expressions $|\omega_{K+p}-\omega_0| < \omega_{\rm R}$ and $|\omega_{K+p'}-\omega_0| < \omega_{\rm R}$, and $V_{K+p,\sigma;K+p',\sigma'} = 0$ otherwise. The coupling coefficient in Eq. (46) is simplified from Eq. (7) as

$$M_{K+p,\sigma}(p'-p,\sigma') = -\left(\frac{N\hbar\omega_{K+p}\omega_{K+p'}}{2\omega_{\rm R}}\right)^{1/2}\left(\frac{\hbar\mathscr{P}(\mathbf{0})}{2V\varepsilon_0\varepsilon}\right). \tag{47}$$

This yields

$$V_{K+p,\sigma;K+p',\sigma'} = -V_0\hbar\omega_{K+p}\hbar\omega_{K+p'}, \tag{48}$$

if expressions $|\omega_{K+p}-\omega_0| < \omega_{\rm R}$ and $|\omega_{K+p'}-\omega_0| < \omega_{\rm R}$, and $V_{K+p,\sigma;K+p',\sigma'} = 0$ otherwise. Here

$$V_0 = \frac{1}{2}N[\mathscr{P}(\mathbf{0})/2V\varepsilon_0\varepsilon\omega_{\rm R}]^2.$$

The equation for $v(I)$ is then

$$v(I) = 2v_{\rm c}V_0{\sum_{q}}'\hbar\omega_{K+q}\left[1+2f_{K+q,\sigma}(I)\right], \tag{49}$$

where the prefactor 2 arises from the summation over helicities and the prime on the summation means that $|\omega_{K+q}-\omega_0| < \omega_{\rm R}$.

In calculating Eq.(49), we let $K+q = k$ and convert the sum over q into the sum over k, so that equation (49) is rewritten as

$$v(I) = 2v_{\rm c}V_0{\sum_{k}}'\hbar\omega_k\left[1+2f_{k\sigma}(I)\right]. \tag{50}$$

The distribution function $f_{k\sigma}(I)$ of dressed photons is given by Eq. (39). We substitute Eq. (39) into Eq. (50). In the usual way, by altering the summation to an integration over bare frequencies $\omega = v_{\rm c}|k|$ we obtain

$$v(I) = 4\hbar V_0\frac{V}{(2\pi)^2v_{\rm c}^2}\int_{\omega_0-\omega_{\rm R}}^{\omega_0+\omega_{\rm R}}d\omega\,\omega^3\left\{1+N(I)\exp\left[-\left(\frac{\omega-\omega_0}{\omega_{\rm b}}\right)^2\right](1-\cos\theta_{\rm c})\right\}, \tag{51}$$

where Eq. (40) is used. The above integral is easily calculated and the integration yields the following result

$$v(I) = 4\gamma v_{\rm c}\frac{\omega_0}{\omega_{\rm R}^4}\left[2\omega_{\rm R}(\omega_0^2+\omega_{\rm R}^2)+N(I)(1-\cos\theta_{\rm c})\omega_b^3Z\left(\frac{\omega_{\rm R}}{\omega_{\rm b}}\right)\right], \tag{52}$$

$$Z(x) = \sqrt{\pi}\left[\frac{3}{2}+\left(\frac{\omega_0}{\omega_{\rm b}}\right)^2\right]{\rm erf}(x) - 3x\exp\left(-x^2\right), \tag{53}$$

where $\mathrm{erf}(x)$ is the error function defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t\, \mathrm{e}^{-t^2}. \tag{54}$$

γ is a dimensionless constant that is characteristic of a self-focusing nonlinear crystal. In Ref. [12], we have established that the constant γ signifies the coupling strength between a dressed photon and its virtual nonpolar phonons and is directly proportional to the Kerr nonlinear coefficient of a crystal. As shown in Ref. [12], γ is given by

$$\gamma = \frac{\hbar}{2c^3 \Omega n_0} \left[\frac{\mathscr{P}(\mathbf{0})\omega_{\mathrm{R}}}{4\pi\varepsilon_0} \right]^2, \tag{55}$$

where Ω is the volume of a primitive cell in the crystal. In Ref. [12], we have pointed out that the interaction strength γ is meaningful only if $0 < \gamma < 1$.

It remains for us to calculate the parameter $N(I)$ of dressed photons. Our starting point is Eq. (38) for E_{L}. The average energy density of the em field is $u = E_{\mathrm{L}}/V$. The light intensity in the waveguide is $I = uv_{\mathrm{g}}$ and thereby the em energy is given by

$$E_{\mathrm{L}} = IV/v_{\mathrm{g}}. \tag{56}$$

We substitute Eqs. (39), (45), and (56) into Eq. (38). In calculating Eq. (38), we let $\boldsymbol{K} + \boldsymbol{q} = \boldsymbol{k}$ and convert the sum over $\boldsymbol{q}$ into the sum over $\boldsymbol{k}$. In the usual way, by altering the summation to an integration over bare frequencies $\omega = c|\boldsymbol{k}|/n_0$ we obtain

$$v(I)IV/v_{\mathrm{g}} = \frac{\hbar[v_{\mathrm{c}} - v(I)]V}{2\pi^2 v_{\mathrm{c}}^3} \int_{\omega_0-\omega_{\mathrm{b}}}^{\omega_0+\omega_{\mathrm{b}}} \omega^3\, \mathrm{d}\omega + (1-\cos\theta_{\mathrm{c}})N(I)\frac{\hbar V}{2\pi^2 v_{\mathrm{c}}^2} \int_{\omega_0-\omega_{\mathrm{R}}}^{\omega_0+\omega_{\mathrm{R}}} \mathrm{d}\omega\omega^3 \exp\left[-\left(\frac{\omega-\omega_0}{\omega_{\mathrm{b}}}\right)^2 \right], \tag{57}$$

where equation (40) is used. The last integrals are easily calculated and the integration produces the following result:

$$v(I)I/v_{\mathrm{g}} = \frac{\hbar[v_{\mathrm{c}} - v(I)]}{\pi^2 v_{\mathrm{c}}^3} \omega_0\omega_{\mathrm{b}}(\omega_0^2 + \omega_{\mathrm{b}}^2) + (1-\cos\theta_{\mathrm{c}})N(I)\frac{\hbar}{2\pi^2 v_{\mathrm{c}}^2}\omega_0\omega_{\mathrm{b}}^3 Z\left(\frac{\omega_{\mathrm{R}}}{\omega_{\mathrm{b}}}\right), \tag{58}$$

where $Z(x)$ is given by Eq. (53).

One must solve Eqs. (52) and (58) simultaneously to determine the velocity $v(I)$ of dressed photons. The solution is easily obtained as

$$v(I) = v_{\mathrm{c}} \frac{I_{\mathrm{a}}}{I + I_{\mathrm{b}}(\gamma)}, \tag{59}$$

where I_{a} and $I_{\mathrm{b}}(\gamma)$ are two characteristic intensities of laser light in the waveguide and are given by

$$I_{\mathrm{a}} = \frac{\hbar\omega_0 v_{\mathrm{g}}}{\pi^2 v_{\mathrm{c}}^3}[\omega_{\mathrm{b}}(\omega_0^2 + \omega_{\mathrm{b}}^2) - \omega_{\mathrm{R}}(\omega_0^2 + \omega_{\mathrm{R}}^2)], \tag{60}$$

$$I_{\mathrm{b}}(\gamma) = \frac{\hbar v_{\mathrm{g}}}{\pi^2 v_{\mathrm{c}}^3}\left[\omega_0\omega_{\mathrm{b}}(\omega_0^2 + \omega_{\mathrm{b}}^2) - \frac{\omega_{\mathrm{R}}^4}{8\gamma}\right]. \tag{61}$$

The velocity $v(I)$ determined by Eq. (59) is a monotonically decreasing function of intensity I. The properties of the superfluid state are closely related to the values of γ. There is a critical coupling constant γ_{c}, which is given by

$$\gamma_{\mathrm{c}} = \frac{\omega_{\mathrm{R}}^3}{8\omega_0(\omega_0^2 + \omega_{\mathrm{R}}^2)}. \tag{62}$$

If the parameter γ satisfies the inequality $0 < \gamma \le \gamma_{\mathrm{c}}$, then it follows that $I_{\mathrm{b}}(\gamma) \le I_{\mathrm{a}}$. In this case, there is a critical light intensity I_{c}, at which $v(I_{\mathrm{c}}) = v_{\mathrm{c}}$. As $I < I_{\mathrm{c}}$, $v(I) > v_{\mathrm{c}}$, so that $v(I)$ has no meaning. Then, the photon system is in the coherent state, in which photons are unpaired and dressed photons do not exist. As $I > I_{\mathrm{c}}$, $v(I) < v_{\mathrm{c}}$, so that $v(I)$ is meaningful. Hence, the photon system is in the superfluid state, in which photons are paired, and unpaired photons are transformed into dressed photons. Thus, we substitute $v(I_{\mathrm{c}}) = v_{\mathrm{c}}$ into Eq. (59) and obtain a simple expression of I_{c}:

$$I_{\mathrm{c}}(\gamma) = I_{\mathrm{a}} - I_{\mathrm{b}}(\gamma). \tag{63}$$

If the parameter γ satisfies the inequality $\gamma_{\mathrm{c}} < \gamma < 1$, then it follows that $I_{\mathrm{b}}(\gamma) > I_a$. In this case, $v(I) < v_{\mathrm{c}}$ for all values of I. That is to say, no matter how large the incident light intensity is, the photon system is always in the superfluid state. According to Eq. (44), at the critical intensity I_{c} the order parameter is $\Delta(I_{\mathrm{c}}) = 0$. The order parameter $\Delta(I)$ is a monotonically increasing function of intensity I.

The key parameter in this theory is γ defined by Eq. (55). γ is a dimensionless coupling constant of a 3D photon with Raman phonons. It would be valuable to have a concrete value of γ. In the present paper, the crystal studied is determined as a certain crystal of the fluorite structure, such as CaF_2. Crystals with the fluorite structure are Kerr nonlinear crystals. Kerr nonlinear crystals must be centrosymmetric and can possess a nonvanishing third-order susceptibility. More importantly the third-order response leads to the intensity-dependent refractive index, which is the basis of most nonlinear optical switching devices. In Kerr nonlinear crystals, the index n of refraction can be written as $n = n_0 + n_2 I$, where I is the intensity of the light propagating in the crystal, n_0 is the linear index of refraction, and n_2 is the nonlinear index of refraction. By use of the nonlinear refractive index n_2, the dimensionless interaction constant can be acquired as[37]

$$\gamma = \frac{\hbar\omega_{\mathrm{R}}^4}{4\pi^2 v_{\mathrm{c}}^2}\left(\frac{m_{\mathrm{e}}c^2}{\hbar\omega_{\mathrm{R}}}\right)^2 n_2, \tag{64}$$

where m_e is the rest mass of electron. We take CaF_2 crystals, for example.[38] CaF_2 crystals have a wide band gap $E_g = 12.2$ eV. In CaF_2, the frequencies of the TO and the Raman-active mode are $\hbar\omega_T = 33.10$ meV and $\hbar\omega_R = 40.02$ meV, respectively. In other words, the frequency of the Raman-active mode of CaF_2 crystals is $\omega_R = 6.216 \times 10^{13}$ s^{-1}. If we use $\omega_0 = 20\omega_R$, the prerequisite for superfluidity that $\hbar\omega_T \ll \hbar\omega_0 \ll E_g$ is satisfied. CaF_2 crystal is a self-focusing Kerr nonlinear crystal ($n_2 > 0$). The linear and nonlinear refractive indices of CaF_2 crystals are as follows[39] $n_0 = 1.4285$, and $n_2 = 1.261 \times 10^{-20}$ m$^2 \cdot$ W^{-1}. From Eq. (64) the dimensionless interaction constant is calculated as $\gamma = 1.7815 \times 10^{-3}$. From Eq. (62) the critical coupling constant is calculated as $\gamma_c = 1.5586 \times 10^{-5}$. One sees that $\gamma > \gamma_c$. If the core of the waveguide is made of CaF_2 crystals, the photon system in the core is always in the superfluid state. According to Eq. (58) in Ref. [12], the transition temperature T_c from the superfluid state to the normal state is a monotonically decreasing function of γ. At $\gamma = 1.7815 \times 10^{-3}$ we find that $T_c > 1000$ K. Experimental verification of predicated effects with CaF_2 crystals can be conveniently carried out at room temperature.

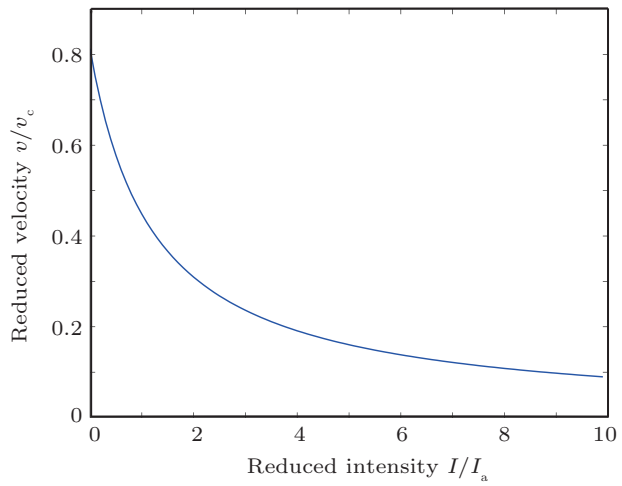


Fig. 2. (color online) According to Eq. (59), variation of reduced velocity v/v_c with reduced intensity I/I_a.

As seen in Section 3, the core of the waveguide has the high-frequency dielectric constant ε_{h1} and its cladding has the high-frequency dielectric constant ε_{h2}. The two high-frequency dielectric constants must satisfy the inequality $\varepsilon_{h1} > \varepsilon_{h2}$. In CaF_2 crystals, $\varepsilon_{h1} = 2.047$. For the second crystal in the cladding, we only require it to be a dielectric with high-frequency dielectric constant $\varepsilon_{h2} = 2.014$. The core radius of the waveguide has a typical value $\rho = 2.10$ μm. One acquires the waveguide frequency $F = 1.576$ and the profile height parameter $\delta = 0.008$, so that the single-mode and weak-guidance conditions are satisfied. We solve numerically Eqs. (13) and (14) and gain the group velocity $v_g = 2.097 \times 10^8$ m$\cdot$s^{-1}. For the frequency halfwidth ω_b of dressed photons, we take $\omega_b = 5\omega_R$. The calculation of Eqs. (60) and (61) gives the results: $I_a = 1.2479 \times 10^8$ W$\cdot$m^{-2} and $I_b(\gamma) = 1.5356 \times 10^8$ W$\cdot$m^{-2}. According to Eq. (59), the variation of reduced velocity $v(I)/v_c$ with reduced intensity I/I_a is shown in Fig. 2. The velocity of dressed photons decreases continuously from $v_c(I_a/I_b)$ as the light intensity increases from zero. According to Eq. (44) we have $v(I)/v_c = \sqrt{1-\Delta^2(I)}$. The variation of order parameter $\Delta(I)$ with reduced intensity I/I_a is shown in Fig. 3. The order parameter ascends continuously from $\sqrt{1-(I_a/I_b)^2}$ to unity as the light intensity increases from zero. The reduced velocity and the order parameter are complementary. In the traveling-wave configuration, the threshold intensity of stimulated Brillouin scattering of light has an order of magnitude $I_t \approx 10^{11}$ W$\cdot$m^{-2}. Therefore when $0 < I < I_t$, the photon system in the waveguide is in the traveling-wave superfluid state, whereas if $I \geq I_t$, the photon system in the waveguide is in the traveling-wave coherent state. According to Ref. [37], the transition from the superfluid state to the coherent state may be a first-order phase transition.

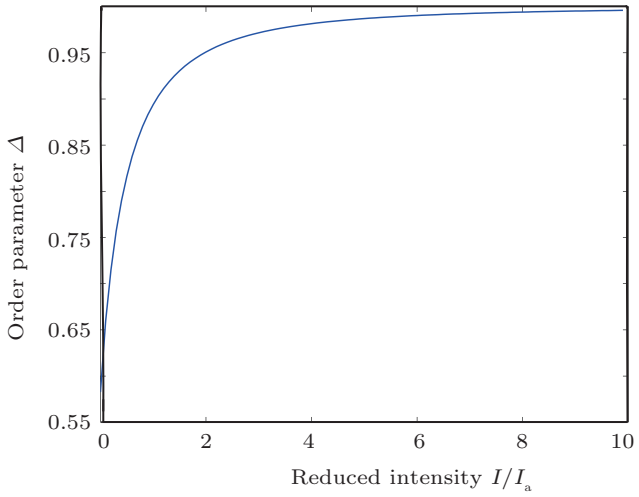


Fig. 3. (color online) According to Eq. (44), variation of order parameter Δ with reduced intensity I/I_a.

5. Discussion

In this paper, we point out that the photon system with attractive interactions in a waveguide can be in a pairing state. The important properties of the pairing state of such a photon system are expounded in the following. Firstly, the interaction between photons is the phonon-mediated interaction with negative coefficient. Secondly, bare photons with opposite transverse wave vectors and helicities are bound into pairs and the photon-pair system is a condensate or a superfluid. Thirdly, the quasiparticle excitations in the pairing condensation are dressed photons rather than phonons and the dressed-photon system is a normal fluid. Fourthly, the pairing condensate is metastable. If the incident light intensity surpasses the threshold intensity of stimulated Brillouin scattering of light, the pairing condensate will collapse. Fifthly, in the transition from the coherent to the pairing state, the photon system may undergo a first-order phase transition. In Ref. [12], we

have defined a critical temperature T_c for the standing-wave superfluid state of photons. However, the present theory has a fault that we cannot define a critical temperature T_c for the traveling-wave superfluid state of photons. This fault will be repaired in a future work. The predicted properties of the pairing state of the photon system in a waveguide can be verified in the present-day physics laboratories.

There are two questions which should be addressed in the following. Firstly, in the BEC of cold atoms, the condensates of attractive Bose atoms easily become unstable and collapse unless Bose atoms are rotated or trapped by confinement potential.[40–42] The attractive force between Bose atoms can even trigger Bose nova phenomena. Furthermore, introducing confinement potential to attractive Bose atoms can help a condensation (fragmented one) only if the attractive interaction energy is smaller than the trapping zero point kinetic energy. This leads to an upper critical value of the interaction strength between Bose atoms of having a (fragmented) condensate in traps. However, the motion of attractive photons in a waveguide is equivalent to the rotation of attractive Bose atoms in a torus. Therefore, photonic superfluid state in self-focusing nonlinear waveguides is very stable. In fact, there is an upper limit of the interaction strength between attractive photons. In Ref. [12], we have pointed out that the interaction strength γ is meaningful only if $0 < \gamma < 1$. In the parameter regime $0 < \gamma < 1$, the attractive photon system is in the superfluid state. In Ref. [12], the parameter regime $\gamma \geq 1$ was considered to be meaningless. Now we consider that when $\gamma \geq 1$ the system of attractive photons is in the normal state. The interaction strength $\gamma = 1$ corresponds to the transition point of superfluidity. Secondly, the present paper considers a situation where photons are injected to the nonlinear waveguide by the coherent laser whereas the relaxation process is not explicitly taken into account. In what follows, we are going to show the sufficient reasoning for the justification of the treatment. The incident photons emitted by a coherent laser are far away from thermal equilibrium. As we have pointed out in Section 1, a time of thermalization of photons in nonlinear waveguides is $t_{\rm ph} \cong 0.1$ ns. As shown in Fig. 4, near the incident end of an optical fiber there is a relaxation region, whose length is $l_{\rm r} \cong 3.0$ cm. In the relaxation region, the photon system is in a non-equilibrium state. Our thermal equilibrium theory is not applied for the relaxation region. The remaining region of an optical fiber is the equilibrium region, in which the photon system under consideration is in a thermal equilibrium state. Therefore, our theory is applied for the equilibrium region. When $l \lesssim l_{\rm r}$, our thermal equilibrium theory fails. However, in the present long-haul optical communications systems, the length of an optical fiber is in the range from $l = 100$ m to 100 km. Because of $l \gg l_{\rm r}$, we can ignore the relaxation region of an optical fiber. This is the reason why at the very start of the present paper we present our theory based on the thermal equilibrium treatment.

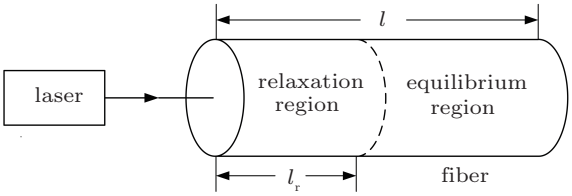


Fig. 4. The relaxation process of coherent light in an optical fiber: the relaxation region and the equilibrium region.

Now we summarize the macroscopic properties of the photonic superfluid state. (i) There is a critical coupling constant γ_c defined by Eq. (62) and a threshold intensity I_t of stimulated Brillouin scattering of light. If the coupling constant γ satisfies the inequality $0 < \gamma \leq \gamma_c$, there is a critical light intensity I_c defined by Eq. (63). When $0 < I < I_c$, the photon system in a waveguide is in the coherent state. If $I_c \leq I < I_t$, the photon system is in the superfluid state, whereas if $I \geq I_t$, the photon system goes into the coherent state. (ii) If the coupling constant γ satisfies the inequality $\gamma_c < \gamma < 1$, the photon system in a waveguide is always in the superfluid state in the range $0 < I < I_t$, whereas it goes into the coherent state in the range $I \geq I_t$. (iii) The system of photon pairs evolves without scattering attenuations, so that the traveling-wave superfluid state has a persistent light intensity. (iv) The traveling-wave superfluid state is expected to have the squeezing property. (v) The crystal in the core of a waveguide is a self-focusing nonlinear medium. (vi) In the waveguide the em wave may propagate as temporal solitons without dispersion. The present optical fibers are almost exclusively made of polar materials with a self-focusing nonlinearity. The scattering losses such as Rayleigh scattering exist inevitably in these fibers. For example, a well-fabricated silica fiber has a loss of 0.15 dB/km at 1.55 μm,[43] which almost corresponds to its intrinsic scattering-loss value. If we want to suppress the scattering losses of optical fibers it is necessary to develop the lossless waveguides, which are also made of polar materials with a self-focusing nonlinearity. Finally, will the traveling-wave superfluid state be the state of the photon system in a nonlinear waveguide? With the advances of science and technology of materials, we believe that scientists can fabricate an appropriate nonlinear waveguide, in which the traveling-wave superfluid state will appear.

To sum up, coherent light in self-focusing nonlinear waveguides can go into a superfluid state. In a superfluid state, photons sense an attractive effective interaction by exchange of virtual optical phonons and thus the attractive interaction

binds coherent photons in pairs. In the traveling-wave superfluid state, a propagating photon pair is the combination of the two photons with opposite transverse wave vectors and helicities. The photon system may undergo a first-order phase transition from the coherent to the superfluid state. The system of photon pairs evolves without scattering attenuations. The traveling-wave superfluid state is expected to have the squeezing property and to support quantum solitons without dispersion. With the above advantages of the traveling-wave superfluid state, scientists must fabricate an appropriate lossless waveguide in the near future.

References

[1] Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 142
[2] Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 171
[3] Hasegawa A 1990 *Optical Solitons in Fibers*, 2nd edn. (Berlin: Springer-Verlag)
[4] Carter S J, Drummond P D, Reid M D and Shelby R M 1987 *Phys. Rev. Lett.* **58** 1841
[5] Drummond P D, Carter S J and Shelby R M 1989 *Opt. Lett.* **14** 373
[6] Carter S J and Drummond P D 1991 *Phys. Rev. Lett.* **67** 3757
[7] Rosenbluh M and Shelby R M 1991 *Phys. Rev. Lett.* **66** 153
[8] Cheng Z 1991 *Phys. Rev. Lett.* **67** 2788
[9] Cheng Z 1995 *Phys. Rev. A* **51** 675
[10] Landau L D and Lifshitz E M 1980 *Statistical Physics*, 3rd edn. (Oxford: Reed Educational) Part 2 p. 88
[11] Leggett A J 2006 *Quantum Liquids: Bose Condensation and Cooper Pairing in Condensed-Matter Systems* (Oxford: Oxford University Press) p. 31
[12] Cheng Z 2002 *J. Opt. Soc. Am. B* **19** 1692
[13] Yin M and Cheng Z 2008 *Phys. Rev. A* **78** 063829
[14] Klaers J, Vewinger F and Weitz M 2010 *Nat. Phys.* **6** 512
[15] Bolda E L, Chiao R Y and Zurek W H 2001 *Phys. Rev. Lett.* **86** 416
[16] Leboeuf P and Moulieras S 2010 *Phys. Rev. Lett.* **105** 163904
[17] Chiao R Y and Boyce J 1999 *Phys. Rev. A* **60** 4114
[18] Chiao R Y 2000 *Opt. Commun.* **179** 157
[19] Klaers J, Schmitt J, Vewinger F and Weitz M 2010 *Nature* **468** 545
[20] Klaers J, Schmitt J, Damm T, Vewinger F and Weitz M 2012 *Phys. Rev. Lett.* **108** 160403
[21] Kirton P and Keeling J 2013 *Phys. Rev. Lett.* **111** 100404
[22] Michinel H, Campo-Táboas J, García-Fernández R, Salgueiro J R and Quiroga-Teixeiro M L 2002 *Phys. Rev. E* **65** 066604
[23] Szymanska M H, Littlewood P B and Simons B D 2003 *Phys. Rev. A* **68** 013818
[24] Alodjants A P, Chestnov I Yu and Arakelian S M 2011 *Phys. Rev. A* **83** 053802
[25] Snoke D 2012 in *Exciton Polaritons in Microcavities: New Frontiers*, eds. Sanvitto D and Timofeev V (Berlin: Springer-Verlag) p. 307
[26] Carusotto I and Ciuti C 2013 *Rev. Mod. Phys.* **85** 299, and references therein
[27] Alodjants A P, Barinov I O and Arakelian S M 2010 *J. Phys. B: At. Mol. Opt. Phys.* **43** 095502
[28] Martin R M and Damen T C 1971 *Phys. Rev. Lett.* **26** 86
[29] Schubert M and Wilhelmi B 1986 *Nonlinear Optics and Quantum Electronics* (New York: Wiley) p. 553
[30] Hillery M and Mlodinow L D 1985 *Phys. Rev. A* **31** 797
[31] Hayes W and Loudon R 1978 *Scattering of Light by Crystals* (New York: Wiley)
[32] Callaway J 1991 *Quantum Theory of the Solid State*, 2nd edn. (New York: Academic Press) pp. 720
[33] Kivshar Y S and Agrawal G P 2003 *Optical Solitons: From Fibres to Photonic Crystals* (San Diego: Academic Press)
[34] Snyder A W and Love J D 1983 *Optical Waveguide Theory* (London: Chapman and Hall)
[35] Loudon R 1970 *J. Phys. A: Gen. Phys.* **3** 233
[36] Reichl L E 1980 *A Modern Course in Statistical Physics* (Austin: University of Texas Press)
[37] Cheng Z 2012 *Phys. Rev. E* **86** 051102
[38] Hodby J W 1974 in *Crystals with the Fluorite Structure: Electronic, Vibrational, and Defect Properties*, ed. Hayes W (Oxford: Clarendon) p. 1
[39] Adair R, Chase L L and Payne S A 1989 *Phys. Rev. B* **39** 3337
[40] Wilkin N K, Gunn J M F and Smith R A 1998 *Phys. Rev. Lett.* **80** 2265
[41] Zhu Q Z and Wu B 2015 *Chin. Phys. B* **24** 050507
[42] Roberto O 2012 *Chin. Phys. B* **21** 070306
[43] Tanaka G, Watanabe M and Yano K 1988 *Fiber and Integrated Optics* **7** 47