Experimentally testing Hardy's theorem on nonlocality with entangled mixed states*

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Hardy's theorem on nonlocality has been verified by a series of experiments with two-qubit entangled pure states. However, in this paper we demonstrate the experimental test of the theorem by using the two-photon entangled mixed states. We first investigate the generic logic in Hardy's proof of nonlocality, which can be applied for arbitrary two-qubit mixed polarization entangled states and can be reduced naturally to the well-known logic tested successfully by the previous pure state experiments. Then, the optimized violations of locality for various experimental parameters are delivered by the numerical method. Finally, the logic argued above for testing Hardy's theorem on nonlocality is demonstrated experimentally by using the mixed entangled-photon pairs generated via pumping two type-I BBO crystals. Our experimental results shows that Hardy's proof of nonlocality can also be verified with two-qubit polarization entangled mixed states, with a violation of about 3.4 standard deviations.

Keywords: Hardy's theorem, nonlocality, entangled mixed state, spontaneous parametric down conversion (SPDC)

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1. Introduction

It is well-known that quantum nonlocality[1] is one of the most important features in quantum theory. Testing experimentally its existence takes an important role, not only to verify the correctness of the standard quantum mechanics, but also for the implementation of quantum information processing. Historically, the existence of quantum nonlocality is verified by testing the violations of the Bell theorem.[2] This theorem takes two forms: one is the so-called Bell inequality tested already by many experiments (see, e.g., Ref. [3]), and the other one is Hardy's theorem,[4] i.e., Hardy's nonlocality proof (HNP). Based on the HNP, the two-body nonlocality can be verified, without any statistical inequality, by a single joint measurement of a two-body entangled state. Therefore, HNP is regarded as "the best version of Bell's theorem",[5] and thus the best way to test the existence of quantum nonlocality.

A series of experiments[6] to test Hardy's theorem have been done by using various photon pairs entangled in either polarizations[7] or orbital angular momenta[8] or energy-times.[9] All these results verified the HNP and thus violated the prediction of any local theory. However, these experiments are based on the logic of HNP for pure states, i.e., for which there exist certain probabilities (the maximal one is about 9% for one ladder,[10] and could be reached to 50% for infinity ladders[11,12]) that the local theory is violated without inequality. For a realistic experiment, the generated quantum entangled state is always a mixed one, due to various unpredictable noises and imperfections. Therefore, testing the HNP with the mixed entangled state is really an important and open question. In 2006, the possibility of testing HNP with mixed state was discussed theoretically,[13,14] but the relevant demonstration with the experimental mixed state has not been reported yet, to the best of our knowledge.

In this paper, we demonstrate the first experimental test of HNP with the entangled mixed states, i.e., the mixed polarization entangled photon pairs generated by the spontaneous parametric down conversion (SPDC) processes.[15] Our work begins with a generic logic of the HNP test by using arbitrary mixed quantum state. Such a condition can be reduced naturally to the usual one using the pure state.[4] Then we provide a defined criterion for the test of HNP specifically with the Werner-like mixed states.[16] Finally, we experimentally test the generic logic for the HNP by using the mixed entangled photon pairs generated by the SPDC processes. It is shown that the experimental results agree well with the deductions of the logic.

2. A generic logic for testing Hardy's theorem with entangled mixed states

We firstly investigate the logical process in HNP testing for an arbitrary two-qubit state. For simplicity, the logic is explained specifically with the mixed polarization entangled

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photon pairs (i.e., the signal photon s and the idler photon i) but can be applied to any mixed entangled state of the other objections.

Let us consider the joint probability $P(\alpha_s, \beta_i)$ of the photon s polarized in the α direction and the photon i simultaneously polarized in the β direction. Following White *et al.*,[7] if

$$P(\alpha_s, -\alpha_i) = 0, \tag{1a}$$

$$P(\beta_s, -\alpha_i^{\perp}) = 0, \tag{1b}$$

$$P(\alpha_s^{\perp}, -\beta_i) = 0, \tag{1c}$$

with $\alpha_s^{\perp}$ ($\alpha_i^{\perp}$) being the perpendicular direction of α_s (α_i), then any local realistic theory yields

$$P(\beta_s, -\beta_i) = 0. \tag{2}$$

Comparably, the nonlocal quantum mechanics can deliver the contradictory result, i.e.,

$$P(\beta_s, -\beta_i) \neq 0. \tag{3}$$

This implies that, experimentally, there should be a nonzero probability of the photon s polarized in the β direction and the photon i polarized in the $-\beta$ direction, if the conditions (1a)–(1c) are satisfied initially. The above contradiction can be easily delivered as follows. If the probability of $P(\beta_s, -\beta_i)$ is not equal to zero, then together with Eq. (1b) and Eq. (1c), the local realistic model will give a conclusion that the photon s (i) will definitely polarize at α ($-\alpha$) direction, i.e., $P(\alpha_s, -\alpha_i) = 1$. But, this is contradictory obviously with the initial condition Eq. (1a).

To perform the testing experimentally, a system wherein the conditions in Eqs. (1a)–(1c) are required to be satisfied perfectly, and then whether the value of the joint probability $P(\beta_s, -\beta_i)$ being zero (predicted by the local theory) or nonzero (predicted by nonlocal quantum mechanics) would be tested. However, in a realistic experiment, the conditions (1a)–(1c) could not be fulfilled exactly, as any measurement always has a certain uncertainty. Therefore, we alternatively require the quantity

$$H_1 = P(\alpha_s, -\alpha_i) + P(\beta_s, -\alpha_i^{\perp}) + P(\alpha_s^{\perp}, -\beta_i), \tag{4}$$

is minimum by properly adjusting the measurements settings $\{\alpha, \beta\}$ that varied from 0 to π independently. Once such a setting is found, then the values of β_s and $-\beta_i$ are defined. As a consequence, Hardy's theorem could be tested by checking if the following inequality

$$H = P(\beta_s, -\beta_i) - H_1 > 0, \tag{5}$$

is satisfied (predicted by the nonlocal quantum mechanics) or not (supported by the local theory). Correspondingly, the measured value of the $P(\beta_s, -\beta_i)$ refers to Hardy's fraction for the testing of the nonlocality. Certainly, for an ideal experiment the value of H_1 should be 0 and the measured joint probability $P(\beta_s, -\beta_i)$ should approach to 1, which means that the nonlocality is verified deterministically. However, this is practically impossible, even for the testing using the pure nonmaximal entangled states. In fact, with the pure entangled state the theoretical minimal value of H_1 could reach 0, but the maximal value of the $P(\beta_s, -\beta_i)$ is just about 9%.[4] Formally, we can introduce a quantity

$$S \equiv \frac{P(\beta_s, -\beta_i)}{H_1}, \tag{6}$$

to describe the robustness of the experiment testing the HNP. Obviously, the test is perfect if $S \to +\infty$. On the contrary, $S < 1$ means that the test fails.

In the following we prove that the above logic, which was suitable for testing any of the quantum states (including the mixed entangled state), can be reduced naturally to that in the original HNP with pure entangled state.[4] For example, with the following pure entangled state

$$|\varphi\rangle = (|H\rangle_s|H\rangle_i + r\mathrm{e}^{\mathrm{i}\phi}|V\rangle_s|V\rangle_i)/\sqrt{1+r^2}, \tag{7}$$

the measurement settings required to satisfy Eq. (4) can be found

$$\begin{aligned} P(\alpha_s, -\alpha_i) &= |\langle\alpha_s|\langle -\alpha_i|\varphi\rangle|^2 \\ &= \frac{(\cos\alpha_s\cos\alpha_i - r\sin\alpha_s\sin\alpha_i)^2}{(1+r^2)}, \end{aligned} \tag{8}$$

by minimizing the values of H_1. In the above discussion, $|H\rangle$ ($|V\rangle$) represents the horizontal (vertical) polarization state of the photon, r the proportion between $|H\rangle_s|H\rangle_i$ and $|V\rangle_s|V\rangle_i$ components and is adjustable ($0 < r < 1$) freely by rotating the polarization direction of the pump beam for the SPDC. Also, ϕ represents the phase difference and is treated as zero without loss of the generality. Given the H_1 can be minimized to zero for the present pure entangled state (i.e., Eq. (7)), the joint probability $P(\beta_s, -\beta_i)$ is determined accordingly and thus

$$H = P(\beta_s, -\beta_i) - H_1 > 0$$

can be achieved.

Figure 1 shows the numerical results (blue circle) of the joint probabilities $P(\beta_s, -\beta_i)$, to verify the HNP for different values of r, based on the above generic logic for any mixed entangled state, and that based on a pure entangled state (red solid line.[4]). It is seen that they are matched perfectly, which proved that the generic logic demonstrated above could be naturally reduced to the previous one used successfully for the pure entangled state.

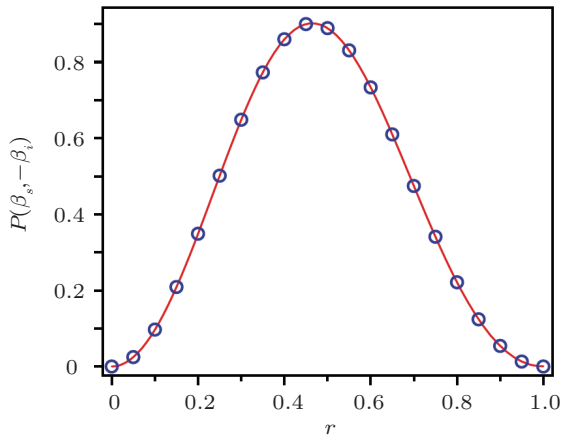


Fig. 1. (color online) The joint probability $P(\beta_s,-\beta_i)$ for verifying the HNP for different r, based on the generic logic with realistic experiments (blue circles) and Hardy's original one[4] with pure entangled state (red solid line).

As a specific example, we consider a Werner-like three-parameter (i.e., p, r, ϕ) mixed entangled state

$$\hat{\rho} = p|\varphi\rangle\langle\varphi| + \frac{(1-p)}{4}I, \tag{9}$$

to investigate the above generic logic. Equation (9) describes a nonmaximally entangled pure state $|\varphi\rangle$, defined by Eq. (7), was mixed by the white noise with $0 \leq p < 1$. Here, I represents the identity operator. This mixed entangled state could be generated by an approach[17] which has been used to prepare the Werner state. Generally, the degree of purity of the state with density operator ρ could be described by $\mathrm{Tr}(\rho^2)$, (i.e., $\mathrm{Tr}(\rho^2) = 1$ corresponds to the pure state and $\mathrm{Tr}(\rho^2) < 1$ corresponds to the mixed state).

Without loss of the generality, we chose $r = 0.47$, for which the Hardy's fraction reaches the maximal value for the testing with pure entangled state $|\varphi\rangle$.

For such a mixed entangled state, all the joint probabilities in Eq. (4) can be calculated. For example, $P(\alpha_s,-\alpha_i) = \langle\psi_M|\hat{\rho}|\psi_M\rangle$, with

$$\begin{aligned}|\psi_M\rangle = &\cos(\alpha_s)\cos(-\alpha_i)|H\rangle_s|H\rangle_i \\ &+ \cos(\alpha_s)\sin(-\alpha_i)|H\rangle_s|V\rangle_i \\ &+ \sin(\alpha_s)\cos(-\alpha_i)|V\rangle_s|H\rangle_i \\ &+ \sin(\alpha_s)\sin(-\alpha_i)|V\rangle_s|V\rangle_i,\end{aligned} \tag{10}$$

being the generic form of the measurement setting. Thus, H_1 can be calculated and minimized to determine the measurement setting $(\beta_s,-\beta_i)$. Consequently, $P(\beta_s,-\beta_i)$ can be calculated and then equation (5) is tested.

Figure 2 is our numerical results based on the above logic. It is seen from Fig. 2 that, once the criterion

$$p > 0.848,\ \mathrm{Tr}(\rho^2) > 0.789, \tag{11}$$

is satisfied, the mixed entangled state equation (9) could be used to test Hardy's theorem. It is emphasized that, the criterion demonstrated in the present work, compared with the theoretical conditions proposed in Ref. [13], for successfully testing HNP with mixed entangled state has been significantly relaxed and thus could be more easily implemented for a realistic experiment. The derivation for this argument can be found in Appendix A in detail.

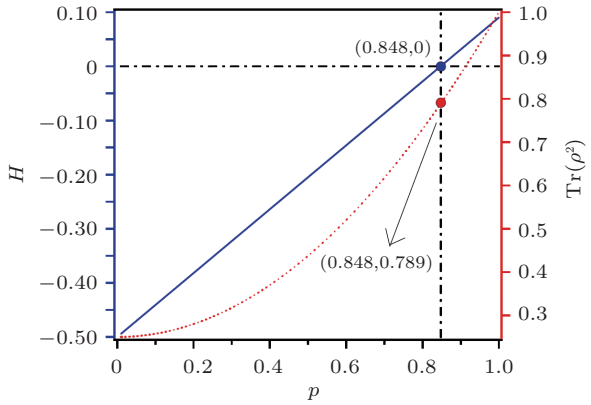


Fig. 2. (color online) Testing Hardy's theorem for mixed entangled state Eq. (9) for different p. The right axis corresponds to $\mathrm{Tr}(\rho^2)$ with the red dotted line. The left axis corresponds to H with the blue solid line. The horizontal and vertical dash-dotted lines represent the critical values: $H = 0$ and $p = 0.848$ respectively, for a successful testing.

Figure 3 shows how the robustness of testing, using the mixed state Eq. (9) with $r = 0.47$, depends on the p-parameter within the range of $p = 0.95$ to $p = 1$. The two S-parameters marked on the line using red solid diamond and black solid circular will be introduced in the next section. Phenomenally, when $p \to 1$, (i.e., corresponds to a pure entangled state) the best testing (i.e., $S \to \infty$) of Hardy's theorem could be approached.

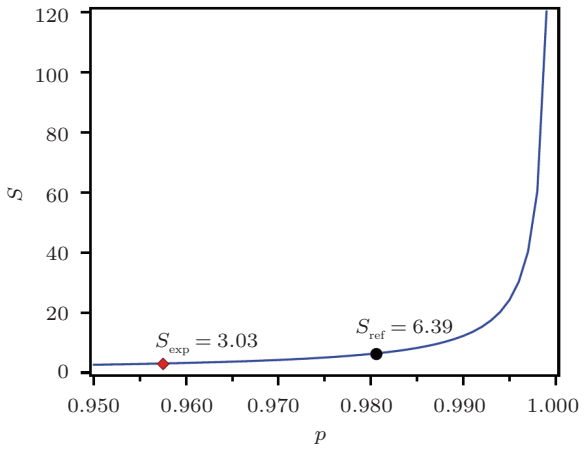


Fig. 3. (color online) Robustness of Hardy's theorem testing with the mixed entangled state Eq. (9) versus the p parameter. The blue solid line is the numerical results. The black solid circular, i.e., $S = 6.39$, represents the result from the pure state logic reported in Ref. [7]. The red solid diamond, i.e., $S = 3.03$, marks the result in our mixed-state experiments, which will be introduced in the next section.

Certainly, the mixed state defined in Eq. (9) with $r \neq 0.47$ can also be used to test Hardy's theorem. Figure 4 gives the required minimal value of p for a successful testing of

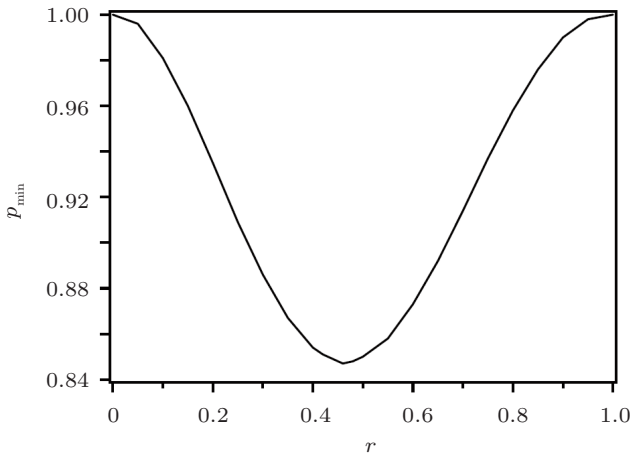


Fig. 4. The required minimal value of p for a successful testing of Hardy's theorem versus the value of r. The minimal value of p in this curve appeared at $r = 0.47$ with $p = 0.848$.

Hardy's theorem (i.e., satisfied the condition in Eq. (5)) versus the parameter r. One can see that, for a given r there always exists a minimal value of p for a successful testing. Of course, for $r = 0.47$, (i.e., the best value for testing Hardy's theorem with the pure entangled state) the range of the available p parameter is the largest, i.e., all the values satisfying the condition $p > 0.848$ can be utilized.

3. Experimentally testing Hardy's theorem using mixed entangled states

In this section, we tested Hardy's theorem using the above logic with a true experimental quantum mixed state. The mixed entangled state is the mixed polarization entangled photon pairs, generated by the standard SPDC process[18] via pumping two type-I BBO crystals. Here, we are only focusing on the density matrix of the generated quantum state in the experiments. For a realistic experimental system with various noises, the generated entangled state is not always a strictly pure one, but a mixed one.

The density matrix $\hat{\rho}_t$ of the quantum state generated in our experiment was reconstructed by using the Maximum Likelihood Estimation methods.[19] The result is

$$\hat{\rho}_t = \begin{pmatrix} 0.7673 & -0.1326+0.0044\mathrm{i} & 0.0005-0.0171\mathrm{i} & 0.3788-0.0451\mathrm{i} \\ -0.1326-0.0044\mathrm{i} & 0.0236 & -0.0003+0.0022\mathrm{i} & -0.0632+0.0049\mathrm{i} \\ 0.0005+0.0171\mathrm{i} & -0.0003-0.0022\mathrm{i} & 0.0064 & 0.0037+0.0078\mathrm{i} \\ 0.3788+0.0451\mathrm{i} & -0.0632-0.0049\mathrm{i} & 0.0037-0.0078\mathrm{i} & 0.2027 \end{pmatrix}. \tag{12}$$

Obviously, such a density matrix indicates the quantum state is a mixed one, as $\mathrm{Tr}(\hat{\rho}_t^2) = 0.965 < 1$.

Then, we use Eq. (12) to test Hardy's theorem based on the logic provided in Section 2. For the present quantum state $\hat{\rho}_t$, all the joint probabilities in Eqs. (4) and (5) can be calculated by using Eqs. (10) and (12).

Firstly, we run a mathematical program MATHEMATICA 5.0 routine FINDMINIMUM[20] to find the minimal value of H_1 in Eq. (4). This is a multidimensional-Powell-direction-set algorithm (as the arguments: α and β are freely changeable). Finally, the minimal value of H_1 is given as $(H_1)_{\min} = 0.026$, where the parameters of the corresponding measurement settings are

$$\alpha_t = 57.8^\circ,\ \beta_t = -66.7^\circ. \tag{13}$$

As a consequence, the values of Hardy's fraction and Hardy's function can be calculated as

$$P_t(\beta_{ts}, -\beta_{ti}) = 0.093,\ H_t = 0.067 > 0. \tag{14}$$

Equation (14) implies that the mixed state Eq. (12) generated in our experiments could be used to test Hardy's theorem; the successful probability could reach 9.3% and Hardy's function is 0.067.

Secondly, we perform the relevant experimental measurements to confirm the above mathematical routine. This is implemented under the measurement settings Eq. (13) by recording the coincidence counting rates: $n(\beta_{ts}, -\beta_{ti})$, $n(\beta_{ts}^{\perp}, -\beta_{ti}^{\perp})$, $n(\beta_{ts}, -\beta_{ti}^{\perp})$, and $n(\beta_{ts}^{\perp}, -\beta_{ti})$ respectively. Consequently, the normalized joint probabilities are obtained as:

$$P_{\exp}(\alpha_{ts}, -\alpha_{ti}) = 0.022 \pm 0.009, \tag{15a}$$

$$P_{\exp}(\beta_{ts}, -\alpha_{ti}^{\perp}) = 0.004 \pm 0.004, \tag{15b}$$

$$P_{\exp}(\alpha_{ts}^{\perp}, -\beta_{ti}) = 0.011 \pm 0.006, \tag{15c}$$

$$P_{\exp}(\beta_{ts}, -\beta_{ti}) = 0.112 \pm 0.019. \tag{15d}$$

Here, the uncertainty of each probability is calculated by assuming each coincidence counts take the usual Poissonian statistical distribution.[21] Consequently, the measured Hardy fraction and Hardy's function are $P_{\exp}(\beta_{ts}, -\beta_{ti}) = 0.112 \pm 0.019$ and $H_{\exp} = 0.075 \pm 0.022$, which mean an accuracy of about 3.4 standard deviations to test the HNP, and agree well with the above mathematical routine.

It was worth noticing that, the present experimental mixed state approaches practically to the mixed polarization entangled state Eq. (9) with $p = 0.977$ and $r = 0.47$. We choose such p and r values just because it has the same degree of purity with the generated entangled photon pairs in such condition. Indeed, the fidelity[22] between them can be calculated

as

$$F = \mathrm{Tr}\left(\sqrt{\hat{\rho}_t}\hat{\rho}\sqrt{\hat{\rho}_t}\right) = 0.984. \tag{16}$$

Such high value of fidelity implies that the criterion (11) demonstrated in Section 2 for successfully testing HNP with mixed entangled state should work for the present mixed entangled photon pairs as $\mathrm{Tr}(\rho_t^2) = 0.965 > 0.789$.

About the S parameter, describing the robustness of the testing can be easily calculated from our experimental results as $S_{\mathrm{exp}} = 3.03$, which is at the same order of magnitude as the pure-state experiment reported in Ref. [18], wherein $S_{\mathrm{ref}} = 6.39$. These two values are marked in Fig. 3 as a red solid diamond and black solid circular.

4. Conclusions and discussions

In conclusion, we investigated Hardy's theorem with the experimental mixed entangled states. Beginning with the mixed-state logic, we delivered the generic requirement to verify the existence of quantum nonlocality without inequality. The argument proposed in the present paper reduces naturally to the logic used successfully for the pure-state entanglement, and can be applied to the mixed-state photon entanglements generated by the standard SPDC process. As an experimental demonstration of our generic arguments, we performed the experimental test of Hardy's theorem using the mixed polarization entangled photon pairs. Based on the reconstructed density matrix of the mixed entangled state generated experimentally by the SPDC process, we demonstrated how to select the measurements settings to perform the test. The experimental results confirmed these selections and verified the existence of nonlocality.

Note that the numerical Hardy fraction (i.e., $p_t = 0.093$) is a bit less than that obtained by the experimental measurements (i.e, $P_{\mathrm{exp}} = 0.112 \pm 0.019$). Probably, the main reason is, due to the additional devices, e.g., a pair of additional quarter wave plates, which are introduced for the density matrix construction,[19] the degree of the purity of the reconstructed density matrix is practically less than that of the original mixed entangled state generated by the SPDC process. As a consequence, the mixed entangled photon pairs used directly for testing Hardy's theorem in experiments is slightly different from the reconstructed one, for measuring the degree of purity and for optimizing the measurement settings.

Appendix A: Analysis of the criterion for testing Hardy's theorem

In this appendix, we analyze the criterion for testing Hardy's theorem based on the logic proposed in Ref. [13].

The basic idea in this paper is, for a given mixed entangled state $\hat{\rho}$ one can select a pure entangled state

$$|\Phi\rangle = \cos\theta|H\rangle_s|H\rangle_i + \sin\theta|V\rangle_s|V\rangle_i, \tag{A1}$$

where $\theta \in (0, \pi)$. If the condition

$$D = b - \frac{a}{6} < 0 \tag{A2}$$

is satisfied, then the mixed entangled state $\hat{\rho}$ could be utilized to test Hardy's theorem, i.e., the existence of nonlocality could be verified with a non-zero probability. Above, the parameters a and b are defined as

$$a = \frac{\cos^2\theta\sin^2\theta(\cos\theta - \sin\theta)^2}{(1 - \sin\theta\cos\theta)^2}, \tag{A3a}$$

$$b = \frac{1}{2}\mathrm{Tr}|\hat{\rho} - |\Phi\rangle\langle\Phi||, \tag{A3b}$$

with $|A| = \sqrt{A^+A}$.

Applying the above logic to the mixed entangled state Eq. (9) with $r = 0.47$, we show how the D parameter changes with the p-parameter in Fig. S1. It is seen clearly that, only $p > 0.980$ or $\mathrm{Tr}(\rho^2) > 0.970$, then the mixed entangled state Eq. (9) could be used to test Hardy's theorem. Obviously, this condition is too rigorous compared with the criterion proposed in our proposal, wherein once $p > 0.848$ or $\mathrm{Tr}(\rho^2) > 0.789$, the mixed entangled state Eq. (9) could be used to test Hardy's theorem. This indicates that the arguments of our proposal are more suitable from the experimental point of view.

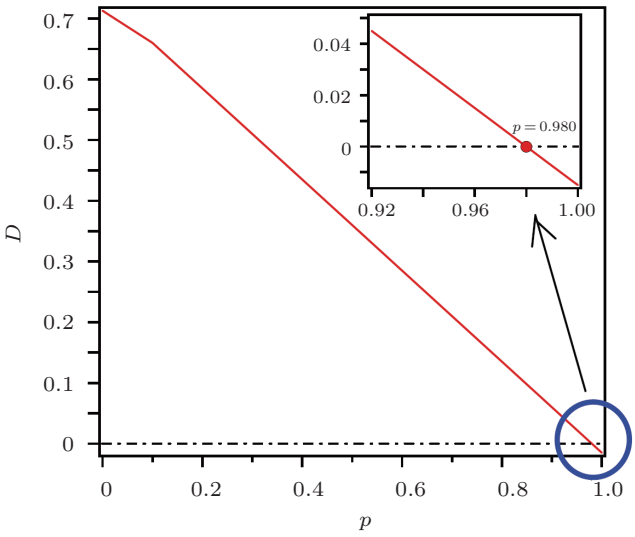


Fig. S1. (color online) Testing Hardy's theorem for mixed entangled state with different value of p using the criterion proposed in Ref. [13]. The horizontal dash-dotted line represents the critical value of $D = 0$. The insert figure is a zoom section from $p = 0.92$ to $p = 1$.

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